MATHEMATICA, Tome 54 (77), N° Spécial, 2012, pp. 147–154

SOME APPLICATIONS OF THE GENERALIZED BERNARDI-LIBERA-LIVINGSTON INTEGRAL OPERATOR ON UNIVALENT FUNCTIONS

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Abstract. Using the generalized Bernardi-Libera-Livingston integral operator, we introduce and study some new subclasses of univalent functions. We also investigate the relations between these classes and the classes which are studied by Jin-Lin Liu.

MSC 2010. 30C45, 30C50.

Key words. Starlike, convex, close-to-convex, quasi-convex, strongly starlike, strongly convex functions.

1. INTRODUCTION

Let A be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic on the unit disk $U = \{z : |z| < 1\}$, also let S denote the subclass of A consisting of all univalent functions on U. Suppose that λ is a real number with $0 \leq \lambda < 1$. A function $f \in S$ is said to be *starlike of order* λ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda, \text{ for all } z \in U.$$

A function $f \in S$ is said to be *convex of order* λ if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \lambda, \text{ for all } z \in U.$$

We denote by $S^*(\lambda)$ and $C(\lambda)$ the classes of starlike, respectively, of convex functions of order λ. It is well known that $f \in C(\lambda)$ if and only if $zf' \in S^*(\lambda)$. Let $f \in A$ and $0 \leq \beta < 1$. The function f is called a *close-to-convex function of order* β *and type* λ if there exists a function $g \in S^*(\lambda)$ such that

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > \beta, \text{ for all } z \in U.$$

We denote by $K(\beta, \lambda)$ the class of close-to-convex functions of order β and type λ. A function $f \in A$ is called *quasi-convex of order* β *and type* λ if there exists a function $g \in C(\lambda)$ such that

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \beta, \text{ for all } z \in U.$$

We denote this class by $K^*(\beta, \lambda)$ (see [11]). It is easy to see that $f \in K^*(\beta, \gamma)$ if and only if $zf' \in K(\beta, \gamma)$ (cf. [10]). Let $f \in A$. If for some λ $(0 \leq \lambda < 1)$

and η $(0 < \eta \leq 1)$ we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| < \frac{\pi}{2}\,\eta, \text{ for all } z \in U, \tag{1}$$

then f is said to be *strongly starlike of order* η *and type* λ *in* U. The class of these functions is denoted by $S^*(\eta, \lambda)$. If $f \in A$ satisfies the condition

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \lambda\right)\right| < \frac{\pi}{2}\,\eta, \text{ for all } z \in U, \tag{2}$$

for some λ and η as above, then we say that f is *strongly convex of order* η *and type* λ *in* U. The class of these functions is denoted by $C(\eta, \lambda)$. Clearly, $f \in C(\eta, \lambda)$ if and only if $zf' \in S^*(\eta, \lambda)$. Also, $S^*(1, \lambda) = S^*(\lambda)$ and $C(1, \lambda) = C(\lambda)$.

For $c > -1$ and $f \in A$, the *generalized Bernardi-Libera-Livingston integral operator* $\mathrm{L}_c f$ is defined as follows

$$\mathrm{L}_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\mathrm{d}t. \tag{3}$$

This operator was studied for $c \in \mathbb{N} = \{1, 2, 3, \cdots\}$ by Bernardi in [1], and for $c = 1$ by Libera in [5] (see also [9]). The classes $ST_c(\eta, \lambda)$ and $CV_c(\eta, \lambda)$ have been introduced by Liu in [8] as follows

$$\begin{aligned} ST_c(\eta, \lambda) &= \left\{ f \in A : \mathrm{L}_c f \in S^*(\eta, \lambda),\ \frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)} \neq \lambda,\ z \in U \right\}, \\ CV_c(\eta, \lambda) &= \left\{ f \in A : \mathrm{L}_c f \in C(\eta, \lambda),\ \frac{(z(\mathrm{L}_c f(z))')'}{(\mathrm{L}_c f(z))'} \neq \lambda,\ z \in U \right\}. \end{aligned}$$

Using the operator given by (3), we introduce now the following classes

$$\begin{aligned} S_c^*(\lambda) &= \{ f \in A : \mathrm{L}_c f \in S^*(\lambda) \}, \\ C_c(\lambda) &= \{ f \in A : \mathrm{L}_c f \in C(\lambda) \}. \end{aligned}$$

Obviously, $f \in CV_c(\eta, \lambda)$ if and only if $zf' \in ST_c(\eta, \lambda)$. In [6] and [7], J. L. Liu introduced and investigated the classes $S_\sigma^*(\lambda)$, $C_\sigma(\lambda)$, $K_\sigma(\beta, \lambda)$, $K_\sigma^*(\beta, \lambda)$, $ST_\sigma(\eta, \lambda)$, and $CV_\sigma(\eta, \lambda)$, by making use of the integral operator I^σ given by

$$\mathrm{I}^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t)\mathrm{d}t, \quad \sigma > 0,\ f \in A. \tag{4}$$

The operator I^σ was introduced by Jung, Kim and Srivastava in [3], and then it was investigated by Uralogaddi and Somanatha in [14], Li in [4], and Liu in [6]. The following relations can be easily verified for the integral operators given by (3) and (4).

$$\mathrm{I}^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n, \tag{5}$$

$$\mathrm{L}_c f(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{n+c} a_n z^n, \tag{6}$$

$$z(\mathrm{I}^{\sigma}\mathrm{L}_c f(z))' = (c+1)\mathrm{I}^{\sigma} f(z) - c\mathrm{I}^{\sigma}\mathrm{L}_c f(z), \tag{7}$$

$$z(\mathrm{L}_c\mathrm{I}^{\sigma} f(z))' = (c+1)\mathrm{I}^{\sigma} f(z) - c\mathrm{L}_c\mathrm{I}^{\sigma} f(z). \tag{8}$$

It follows from (5) that one can define the operator I^{σ} for any real number σ. In this paper we investigate the properties of the classes $S_c^*(\lambda)$, $C_c(\lambda)$, $K_c(\beta, \lambda)$, $K_c^*(\beta, \lambda)$, $ST_c(\eta, \lambda)$, and $CV_c(\eta, \lambda)$. We also study the relations between these classes and the classes introduced by Liu in [6] and [7]. For our purposes we need the following lemmas.

Lemma 1. ([10]) *Let* $u = u_1 + \mathrm{i}u_2$, $v = v_1 + \mathrm{i}v_2$, *and let* ψ *be a complex function* $\psi : D \subset \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. *Suppose that* ψ *satisfies the following conditions*

(i) ψ *is continuous on* D,
(ii) $(1, 0) \in D$ *and* $\mathrm{Re}\{\psi(1, 0)\} > 0$,
(iii) $\mathrm{Re}\{\psi(\mathrm{i}u_2, v_1)\} \le 0$ *for all* $(\mathrm{i}u_2, v_1) \in D$ *with* $v_1 \le -\frac{1+u_2^2}{2}$.

Let $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ *be analytic on* U *so that* $(p(z), zp'(z)) \in D$ *for all* $z \in U$. *If*

$$\mathrm{Re}\{\psi(p(z), zp'(z))\} > 0, \text{ for all } z \in U,$$

then $\mathrm{Re}\{p(z)\} > 0$, *for all* $z \in U$.

Lemma 2. ([12]) *Assume that the function* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *is analytic on* U *and that* $p(z) \neq 0$, *for all* $z \in U$. *Let* $0 < \eta \le 1$. *If there exists a point* $z_0 \in U$ *such that* $|\arg p(z_0)| = \frac{\pi}{2}\eta$ *and*

$$|\arg(p(z))| < \frac{\pi}{2}\eta \text{ for } |z| < |z_0|,$$

then $\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k\eta$ *with* $k \ge \frac{1}{2}(r + \frac{1}{r})$ *when* $\arg p(z_0) = \frac{\pi}{2}\eta$, *and with* $k \le \frac{-1}{2}(r + \frac{1}{r})$ *when* $\arg p(z_0) = \frac{-\pi}{2}\eta$, *where* $p(z_0)^{1/\eta} = \pm \mathrm{i}r$ $(r > 0)$.

2. MAIN RESULTS

In this section we obtain some inclusion theorems, using the methods developed in [13].

Theorem 3. *For* $f \in A$ *the following hold hold true.*

(i) *If* $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)}\right\} > 0$, *then* $S_c^*(\lambda) \subset S_{c+1}^*(\lambda)$.
(ii) *If* $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(\mathrm{L}_{c+1} f(z))'}{\mathrm{L}_{c+1} f(z)}\right\} > 0$, *then* $S_{c+1}^*(\lambda) \subset S_c^*(\lambda)$.

Proof. (i) Suppose that $f \in S_c^*(\lambda)$ and set

$$\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)} - \lambda = (1-\lambda)p(z), \tag{9}$$

where $p(z) = 1 + \sum\limits_{n=2}^{\infty} c_n z^n$. An easy calculation shows that

$$\frac{\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}\left[2 + c + \frac{z(\mathrm{L}_{c+1}f(z))''}{(\mathrm{L}_{c+1}f(z))'}\right]}{\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)} + c + 1} = \frac{zf'(z)}{f(z)}. \tag{10}$$

Setting $H(z) = \frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}$, we get

$$1 + \frac{z(\mathrm{L}_{c+1}f(z))''}{(\mathrm{L}_{c+1}f(z))'} = H(z) + \frac{zH'(z)}{H(z)}. \tag{11}$$

Since $H(z) = \lambda + (1-\lambda)p(z)$, by (10) and (11), we obtain

$$(1-\lambda)p(z) + \frac{(1-\lambda)zp'(z)}{\lambda + c + 1 + (1-\lambda)p(z)} = \frac{zf'(z)}{f(z)} - \lambda. \tag{12}$$

Let

$$\psi(u,v) = (1-\lambda)u + \frac{(1-\lambda)v}{\lambda + c + 1 + (1-\lambda)u}.$$

Then ψ is a continuous function on $D = \left(\mathbb{C} \setminus \{\frac{\lambda+c+1}{\lambda-1}\}\right) \times \mathbb{C}$ and $(1,0) \in D$. Also, $\psi(1,0) > 0$ and for all $(\mathrm{i}u_2, v_1) \in D$ with $v_1 \le -\frac{1+u_2^2}{2}$ we have

$$\mathrm{Re}\,\psi(\mathrm{i}u_2, v_1) = \frac{(1-\lambda)(\lambda+c+1)v_1}{(1-\lambda)^2u_2^2 + (\lambda+c+1)^2} \le \frac{-(1-\lambda)(\lambda+c+1)(1+u_2^2)}{2[(1-\lambda)^2u_2^2 + (\lambda+c+1)^2]} < 0.$$

Therefore the function ψ satisfies the conditions of Lemma 1. Taking into account the hypothesis and (12), we have $\mathrm{Re}\{\psi(p(z), zp'(z))\} > 0$, hence Lemma 1 implies that $\mathrm{Re}p(z) > 0$, for $z \in U$, which finishes the proof.

(ii) This assertion can be proved by the same method as (i), using the formula obtained by replacing $c+1$ with c in (10) . □

Theorem 4. *For $f \in A$ the following assertions hold true.*

(i) *If* $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)}\right\} > 0$, *then* $C_c(\lambda) \subset C_{c+1}(\lambda)$.

(ii) *If* $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(\mathrm{L}_{c+1} f(z))'}{\mathrm{L}_{c+1} f(z)}\right\} > 0$, *then* $C_{c+1}(\lambda) \subset C_c(\lambda)$.

Proof. (i) In view of assertion (i) of Theorem 3 we have the following chain of equivalent relations $f \in C_c(\lambda) \Leftrightarrow \mathrm{L}_c f \in C(\lambda) \Leftrightarrow z(\mathrm{L}_c f)' \in S^*(\lambda) \Leftrightarrow \mathrm{L}_c zf' \in S^*(\lambda) \Leftrightarrow zf' \in S_c^*(\lambda) \to zf' \in S_{c+1}^*(\lambda) \Leftrightarrow \mathrm{L}_{c+1}zf' \in S^*(\lambda) \Leftrightarrow z(\mathrm{L}_{c+1}f)' \in S^*(\lambda) \Leftrightarrow \mathrm{L}_{c+1}f \in C(\lambda) \Leftrightarrow f \in C_{c+1}(\lambda)$.

Assertion (ii) can be proved using a similar method. □

Theorem 5. *If $c \ge -\lambda$, then $f \in S^*(\lambda)$ implies $f \in S_c^*(\lambda)$.*

Proof. Differentiating logarithmically both sides of (3) with respect to z, we obtain

$$\frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)} + c = \frac{(c+1)f(z)}{\mathrm{L}_c f(z)}. \tag{13}$$

Differentiating logarithmically both sides of (13), we have

$$p(z) + \frac{zp'(z)}{c+\lambda+p(z)} = \frac{zf'(z)}{f(z)} - \lambda \tag{14}$$

where $p(z) = \frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)} - \lambda$. Let

$$\psi(u,v) = u + \frac{v}{u+c+\lambda}.$$

Then ψ is a continuous function on $D = (\mathbb{C} \setminus \{-c-\lambda\}) \times \mathbb{C}$ and $(1,0) \in D$. Also, Re $\psi(1,0) > 0$. If $(\mathrm{i}u_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$, then

$$\mathrm{Re}\ \psi(\mathrm{i}u_2, v_1) = \frac{v_1(c+\lambda)}{u_2^2 + (c+\lambda)^2} \leq 0.$$

Since $f \in S^*(\lambda)$, relation (16) yields

$$\mathrm{Re}(\psi(p(z), zp'(z))) = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \lambda\right\} > 0.$$

We conclude from Lemma 1 that $\mathrm{Re}\{p(z)\} > 0$. □

Corollary 6. *If $c \geq \lambda$, then $f \in C(\lambda)$ implies $f \in C_c(\lambda)$.*

Proof. We have $f \in C(\lambda) \Leftrightarrow zf' \in S^*(\lambda) \Longrightarrow zf' \in S_c^*(\lambda) \Leftrightarrow \mathrm{L}_c zf' \in S^*(\lambda) \Leftrightarrow z(\mathrm{L}_c f)' \in S^*(\lambda) \Leftrightarrow \mathrm{L}_c f \in C(\lambda) \Leftrightarrow f \in C_c(\lambda)$. □

Theorem 7. *For $f \in A$ the following assertions hold true.*
(i) *If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|\arg\left(\frac{z(\mathrm{L}_c f(z))'}{\mathrm{L}_c f(z)} - \lambda\right)\right|, \ for\ z \in U,$$

then $ST_c(\eta,\lambda) \subset ST_{c+1}(\eta,\lambda)$, where $c > -1$.
(ii) *If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|\arg\left(\frac{z(\mathrm{L}_{c+1} f(z))'}{\mathrm{L}_{c+1} f(z)} - \lambda\right)\right|, \ for\ z \in U,$$

then $ST_{c+1}(\eta,\lambda) \subset ST_c(\eta,\lambda)$, where $c > -1$.

Proof. (i) Let $f \in ST_c(\eta,\lambda)$ and put

$$\frac{z(\mathrm{L}_{c+1} f(z))'}{\mathrm{L}_{c+1} f(z)} = \lambda + (1-\lambda)p(z), \tag{15}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic on U with $p(z) \neq 0$, for $z \in U$. It is easy to see that

$$z(\mathrm{L}_{c+1}f(z))' + (c+1)\mathrm{L}_{c+1}f(z) = (c+2)f(z). \tag{16}$$

Differentiating logarithmically with respect to z both sides of (16), we obtain

$$\frac{z\left(\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}\right)'}{\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)} + c + 1} + \frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)} = \frac{zf'(z)}{f(z)}. \tag{17}$$

Using (15) and (17), we get

$$\frac{(1-\lambda)zp'(z)}{\lambda + c + 1 + (1-\lambda)p(z)} + (1-\lambda)p(z) = \frac{zf'(z)}{f(z)} - \lambda. \tag{18}$$

Suppose that there exists $z_0 \in U$ such that $|\arg(p(z))| < \frac{\pi}{2}\,\eta$ for $|z| < |z_0|$ and $|\arg(p(z_0))| = \frac{\pi}{2}\,\eta$. Lemma 2 implies then that $\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k\eta$ and $p(z_0)^{1/\eta} = \pm \mathrm{i}r (r > 0)$, where $k \geq \frac{1}{2}(r + \frac{1}{r})$, when $\arg(p(z_0)) = \frac{\pi}{2}\,\eta$, and $k \leq \frac{-1}{2}(r + \frac{1}{r})$ when $\arg(p(z_0)) = \frac{-\pi}{2}\,\eta$. If $p(z_0)^{1/\eta} = \mathrm{i}r$, then $\arg(p(z_0)) = \frac{\pi}{2}\eta$, hence by considering (18), we have

$$\begin{aligned}
&\left|\arg\left(\frac{z_0(\mathrm{L}_c f(z_0))'}{\mathrm{L}_c f(z_0)} - \lambda\right)\right| \geq \arg\left(\frac{z_0 f'(z_0)}{f(z_0)} - \lambda\right) \\
&= \arg\left\{(1-\lambda)p(z_0)\left[1 + \frac{ik\eta}{\lambda + c + 1 + (1-\lambda)r^\eta \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\eta}}\right]\right\} \\
&= \frac{\pi}{2}\eta + \tan^{-1}\left\{\frac{P}{Q}\right\} \geq \frac{\pi}{2}\eta \quad (\text{because } k \geq \frac{1}{2}(r + \frac{1}{r}) \geq 1),
\end{aligned}$$

where

$$P = k\eta\left[\lambda + c + 1 + r^\eta(1-\lambda)\cos\frac{\pi}{2}\eta\right] \tag{19}$$

and

$$\begin{aligned}
Q = (\lambda + c + 1)^2 &+ r^{2\eta}(1-\lambda)^2 + (1-\lambda)(\lambda + c + 1)\cos\frac{\pi}{2}\eta \\
&+ k\eta r^\eta(1-\lambda)\sin\frac{\pi}{2}\eta.
\end{aligned}$$

This contradicts the fact that $f(z) \in ST_c(\eta, \lambda)$.

Now suppose that $p(z_0)^{1/\eta} = -\mathrm{i}r$. Then $\arg(p(z_0)) = \frac{-\pi}{2}\eta$ and we get

$$\begin{aligned}
&-\left|\arg\left(\frac{z_0(\mathrm{L}_c f(z_0))'}{\mathrm{L}_c f(z_0)} - \lambda\right)\right| \leq \arg\left(\frac{z_0 f'(z_0)}{f(z_0)} - \lambda\right) \\
&= \frac{-\pi}{2}\eta + \arg\left\{1 + \frac{ik\eta}{\lambda + c + 1 + (1-\lambda)r^\eta \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\eta}}\right\} \\
&= \frac{-\pi}{2}\eta + \tan^{-1}\left\{\frac{P}{R}\right\} \leq \frac{-\pi}{2}\eta \quad (\text{because } k \leq \frac{-1}{2}(r + \frac{1}{r}) \leq -1),
\end{aligned}$$

where P is given by (19) and

$$R=(\lambda+c+1)^2+r^{2\eta}(1-\lambda)^2+2r^{\eta}(1-\lambda)(\lambda+c+1)\cos\frac{\pi}{2}\eta$$
$$-k\eta r^{\eta}(1-\lambda)\sin\frac{\pi}{2}\eta.$$

This contradicts our assumption that $f\in ST_c(\eta,\lambda)$, therefore $|\arg(p(z))|<\frac{\pi}{2}$, for $z\in U$. Finally we get

$$\left|\arg\left(\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}-\lambda\right)\right|<\frac{\pi}{2}\eta,\ \text{for } z\in U.$$

Since for every λ $(0\le\lambda<1)$ we have

$$\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}\neq\lambda,$$

we conclude that $f\in ST_{c+1}(\eta,\lambda)$.

The proof of (ii) is similar to the proof of (i), therefore we omit it. □

COROLLARY 8. *For $f\in A$ the following assertions hold true.*

(i) *If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\lambda\right)\right|\le\left|\arg\left(\frac{z(\mathrm{L}_cf(z))'}{\mathrm{L}_cf(z)}-\lambda\right)\right|\ \ for\ z\in U,$$

then $CV_c(\eta,\lambda)\subset CV_{c+1}(\eta,\lambda)$.

(i) *If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\lambda\right)\right|\le\left|\arg\left(\frac{z(\mathrm{L}_{c+1}f(z))'}{\mathrm{L}_{c+1}f(z)}-\lambda\right)\right|,\ \ for\ z\in U,$$

then $CV_{c+1}(\eta,\lambda)\subset CV_c(\eta,\lambda)$.

Proof. We give only the proof of part (i) and for this we have $f\in CV_c(\eta,\lambda)\Leftrightarrow \mathrm{L}_cf\in C(\eta,\lambda)\Leftrightarrow z(\mathrm{L}_cf)'\in S^*(\eta,\lambda)\Leftrightarrow \mathrm{L}_czf'\in S^*(\eta,\lambda)\Leftrightarrow zf'\in ST_c(\eta,\lambda)\Longrightarrow zf'\in ST_{c+1}(\eta,\lambda)\Leftrightarrow \mathrm{L}_{c+1}zf'\in S^*(\eta,\lambda)\Leftrightarrow z(\mathrm{L}_{c+1}f)'\in S^*(\eta,\lambda)\Leftrightarrow \mathrm{L}_{c+1}f\in C(\eta,\lambda)\Leftrightarrow f\in CV_{c+1}(\eta,\lambda)$. □

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