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SOME PROPERTIES OF CERTAIN CLASSES OF MEROMORPHICALLY p-VALENT FUNCTIONS INVOLVING EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. The authors investigate interesting properties of certain subclasses of meromorphically multivalent functions which are defined by means of extended multiplier transformations.

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1. INTRODUCTION

Let $\sum_p$ be the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and p-valent on the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}$. For a function $f \in \sum_p$ given by (1) and a function $g \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}), \tag{2}$$

one introduces the Hadamard product (or convolution) of f and g as the function $f * g$ defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{3}$$

We define now a linear operator $I_p^m(\lambda, \ell)$ (where $\lambda \geq 0,\ \ell > 0,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, 3, \cdots\}$) which acts as described below on a function $f \in \sum_p$ given by (1)

$$I_p^m(\lambda, \ell) f(z) = z^{-p} + \sum_{k=1}^{\infty} \left[\frac{\lambda k + \ell}{\ell} \right]^m a_{k-p} z^{k-p}. \tag{4}$$

We can write (4) also as

$$I_p^m(\lambda, \ell) f(z) = (\Phi_{\lambda,\ell}^{p,m} * f)(z),$$

where

$$\Phi^{p,m}_{\lambda,\ell}(z) = z^{-p} + \sum_{k=1}^{\infty} \left[\frac{\lambda k + \ell}{\ell}\right]^m z^{k-p}. \tag{5}$$

It follows easily from (4) that

$$\lambda z(I^m_p(\lambda,\ell)f(z))' = \ell I^{m+1}_p(\lambda,\ell)f(z) - (\lambda p + \ell)I^m_p(\lambda,\ell)f(z) \quad (\lambda > 0). \tag{6}$$

We also note that

$$I^0_p(\lambda,\ell)f(z) = f(z)$$

and

$$I^1_p(1,1)f(z) = \frac{(z^{p+1}f(z))'}{z^p} = (p+1)f(z) + zf'(z).$$

By specializing the parameters λ, ℓ, m, and p one obtains the following operators studied by various authors:

(i) $I^m_p(1,1) = D^m_p$ (see Aouf and Hossen [1], Liu and Owa [7], Liu and Srivastava [8], and Srivastava and Patel [11]);
(ii) $I^m_1(1,\ell) = D^m_\ell$ (see Cho et al. [4, 5]);
(iii) $I^m_1(1,1) = I^m$ (see Uralegaddi and Somanatha [12]);
(iv) $I^m_p(1,\ell) = I_p(m,\ell)$, where $I_p(m,\ell)$ is defined by

$$I_p(m,\ell)f(z) = z^{-p} + \sum_{k=1}^{\infty} \left[\frac{k+\ell}{\ell}\right]^m a_{k-p}z^{k-p};$$

(v) $I^m_p(\lambda,1) = D^m_{p,\lambda}$, where $D^m_{p,\lambda}f(z)$ is defined by

$$D^m_{p,\lambda}f(z) = z^{-p} + \sum_{k=1}^{\infty} [\lambda k + 1]^m a_{k-p}z^{k-p}. \tag{7}$$

We denote by $\sum^m_{p,\lambda,\ell}(\alpha,\delta,\mu,\gamma)$ the class of all functions $f \in \sum_p$ such that

$$\operatorname{Re}\left\{(1-\gamma)\left(\frac{I^m_p(\lambda,\ell)f(z)}{I^m_p(\lambda,\ell)g(z)}\right)^{\mu} + \gamma\frac{I^{m+1}_p(\lambda,\ell)f(z)}{I^{m+1}_p(\lambda,\ell)g(z)}\left(\frac{I^m_p(\lambda,\ell)f(z)}{I^m_p(\lambda,\ell)g(z)}\right)^{\mu-1}\right\} > \alpha, \tag{8}$$

where $g \in \sum_p$ satisfies the condition

$$\operatorname{Re}\left\{\frac{I^m_p(\lambda,\ell)g(z)}{I^{m+1}_p(\lambda,\ell)g(z)}\right\} > \delta \quad (0 \le \delta < 1,\ z \in U), \tag{9}$$

where α and μ are real numbers such that $0 \le \alpha < 1$, $\mu > 0$, $p \in \mathbb{N}$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\{\gamma\} > 0$.

We note that

(i) For $\lambda = 1$ we have that $\sum_{p,\ell}^{m}(\alpha, \delta, \mu, \gamma)$ is the class of functions $f \in \sum_p$ satisfying the condition

(10)
$$\operatorname{Re}\left\{(1-\gamma)\left(\frac{I_p(m,\ell)f(z)}{I_p(m,\ell)g(z)}\right)^{\mu} + \gamma\frac{I_p(m+1,\ell)f(z)}{I_p(m+1,\ell)g(z)}\left(\frac{I_p(m,\ell)f(z)}{I_p(m,\ell)g(z)}\right)^{\mu-1}\right\} > \alpha,$$

where $g \in \sum_p$ is such that

(11)
$$\operatorname{Re}\left\{\frac{I_p(m,\ell)g(z)}{I_p(m+1,\ell)g(z)}\right\} > \delta \quad (0 \le \delta < 1,\ z \in U),$$

with $0 \le \alpha < 1$, $\mu > 0$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\{\gamma\} > 0$;

(ii) For $\ell = 1$ we have that $\sum_{p,\lambda}^{m}(\alpha, \delta, \mu, \gamma)$ is the class of functions $f \in \sum_p$ satisfying the condition

(12)
$$\operatorname{Re}\left\{(1-\gamma)\left(\frac{D_{p,\lambda}^{m}f(z)}{D_{p,\lambda}^{m}g(z)}\right)^{\mu} + \gamma\frac{D_{p}^{m+1}f(z)}{D_{p,\lambda}^{m+1}g(z)}\left(\frac{D_{p,\lambda}^{m}f(z)}{D_{p,\lambda}^{m}g(z)}\right)^{\mu-1}\right\} > \alpha,$$

where $g \in \sum_p$ is such that

(13)
$$\operatorname{Re}\left\{\frac{D_{p,\lambda}^{m}g(z)}{D_{p,\lambda}^{m+1}g(z)}\right\} > \delta \quad (0 \le \delta < 1,\ z \in U),$$

with $0 \le \alpha < 1$, $\mu > 0$, $\lambda > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\{\gamma\} > 0$;

(iii) For $\lambda = \ell = 1$ we have that $\sum_{p}^{m}(\alpha, \delta, \mu, \gamma)$ is the class of functions $f \in \sum_p$ satisfying the condition

(14)
$$\operatorname{Re}\left\{(1-\gamma)\left(\frac{D_{p}^{m}f(z)}{D_{p}^{m}g(z)}\right)^{\mu} + \gamma\frac{D_{p}^{m+1}f(z)}{D_{p}^{m+1}g(z)}\left(\frac{D_{p}^{m}f(z)}{D_{p}^{m}g(z)}\right)^{\mu-1}\right\} > \alpha,$$

where $g \in \sum_p$ is such that

(15)
$$\operatorname{Re}\left\{\frac{D_{p}^{m}g(z)}{D_{p}^{m+1}g(z)}\right\} > \delta \quad (0 \le \delta < 1,\ z \in U),$$

with $0 \le \delta < 1$, $\mu > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\{\gamma\} > 0$.

To establish our main results we need the following lemmas.

LEMMA 1. (see [9]) *Let Ω be a set in the complex plane $\mathbb{C}$ and let the function $\Psi\colon \mathbb{C}^2 \to \mathbb{C}$ satisfy the condition $\Psi(\mathrm{i}r_2, s_1) \notin \Omega$ for all reals $r_2, s_1 \le -\frac{1+r_2^2}{2}$. If q is analytic on U with $q(0) = 1$ and if $\Psi(q(z), zq^{'}(z)) \in \Omega$, for all $z \in U$, then $\operatorname{Re}\{q(z)\} > 0$ for all $z \in U$.*

LEMMA 2. (see [10]) *If q is analytic on U with $q(0) = 1$, and if $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ satisfies $\operatorname{Re}\{\lambda\} \ge 0$, then $\operatorname{Re}\{q(z) + \lambda zq^{'}(z)\} > \alpha$ $(0 \le \alpha < 1)$ implies*

$$\operatorname{Re}\{q(z)\} > \alpha + (1-\alpha)(2\gamma - 1),$$

where γ *is given by*

$$\gamma = \gamma(\mathrm{Re}\lambda) = \int_0^1 \left(1 + t^{\mathrm{Re}\{\lambda\}}\right)^{-1} \mathrm{d}t.$$

(*Note that* γ *is an increasing function of* $\mathrm{Re}\{\lambda\}$ *satisfying* $\frac{1}{2} \leq \gamma < 1$.) *The estimate is sharp in the sense that the bound cannot be improved.*

For real or complex numbers a, b, c $(c \notin \mathbb{Z}_0^-)$, the Gauss hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots .$$

Note that the above series converges absolutely for $z \in U$ and hence represents an analytic function on the unit disc U (see [13, chapter 14] for details).

Each of the identities asserted by Lemma 3 below is fairly well known (for instance, cf. [13, chapter 14]).

LEMMA 3. *Let* a, b, c $(c \notin \mathbb{Z}_0^-)$ *be real or complex parameters. Then the following equalities hold true*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z) \tag{16}$$

(*if* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$),

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z), \tag{17}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{18}$$

and

$${}_2F_1\left(1, 1; 2; \frac{1}{2}\right) = 2\ell n2. \tag{19}$$

The methods we will use to obtain our main results are similar to those of Kwon et al. [6], El-Ashwah [3], and Aouf and Mostafa [2].

2. MAIN RESULTS

We will assume throughout the paper that the powers are understood as principle values.

THEOREM 4. *Let* $f \in \sum_{p,\lambda,\ell}^m(\alpha, \delta, \mu, \gamma)$, $\lambda, \ell > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$ *and* $\gamma \geq 0$. *Then*

$$\mathrm{Re}\left\{\left(\frac{I_p^m(\lambda, \ell)f(z)}{I_p^m(\lambda, \ell)g(z)}\right)^{\mu}\right\} > \frac{2\ell\alpha\mu + \delta\gamma\lambda}{2\ell\mu + \delta\gamma\lambda} \quad (0 \leq \alpha < 1,\ \mu > 0,\ z \in U), \tag{20}$$

where the function $g \in \sum_p$ *satisfies condition* (9).

Proof. Let $\beta = \dfrac{2\ell\alpha\mu + \delta\gamma\lambda}{2\ell\mu + \delta\gamma\lambda}$ and define the function q by

$$q(z) = \frac{1}{(1-\beta)} \left\{ \left(\frac{I_p^m(\lambda,\ell) f(z)}{I_p^m(\lambda,\ell) g(z)} \right)^{\mu} - \beta \right\}. \tag{21}$$

The function q is analytic on U and $q(0) = 1$. If we set

$$h(z) = \frac{I_p^m(\lambda,\ell) g(z)}{I_p^{m+1}(\lambda,\ell) g(z)}, \tag{22}$$

then, by hypothesis, $\mathrm{Re}\{h(z)\} > \delta$. Differentiating (21) and using the identity (6), we get

$$(1-\gamma)\left(\frac{I_p^m(\lambda,\ell) f(z)}{I_p^m(\lambda,\ell) g(z)} \right)^{\mu} + \gamma \frac{I_p^{m+1}(\lambda,\ell) f(z)}{I_p^{m+1}(\lambda,\ell) g(z)} \left(\frac{I_p^m(\lambda,\ell) f(z)}{I_p^m(\lambda,\ell) g(z)} \right)^{\mu-1}$$

$$= [\beta + (1-\beta) q(z)] + \frac{\lambda\gamma(1-\beta)}{\mu\ell} h(z) z q^{'}(z). \tag{23}$$

Define the function Ψ by

$$\Psi(r,s) = \beta + (1-\beta) r + \frac{\lambda\gamma(1-\beta)}{\mu\ell} h(z) s. \tag{24}$$

Using (24) and the fact that $f \in \sum_{p,\lambda,\ell}^{m}(\alpha,\delta,\mu,\gamma)$, we obtain

$$\left\{ \Psi(q(z), zq^{'}(z)) : z \in U \right\} \subset \Omega = \{ w \in \mathbb{C} : \mathrm{Re}\{w\} > \alpha \}.$$

The following relations hold for all reals $r_2, s_1 \le -\dfrac{1+r_2^2}{2}$

$$\begin{aligned} \mathrm{Re}\{\Psi(\mathrm{i}r_2, s_1)\} &= \beta + \frac{\lambda\gamma(1-\beta)s_1}{\mu\ell} \mathrm{Re}\{h(z)\} \\ &\le \beta - \frac{\lambda\gamma(1-\beta)\delta(1+r_2^2)}{2\mu\ell} \\ &\le \beta - \frac{\lambda\gamma(1-\beta)\delta}{2\mu\ell} = \alpha. \end{aligned}$$

Hence $\Psi(\mathrm{i}r_2, s_1) \notin \Omega$ for each $z \in U$. Applying now Lemma 1, we get $\mathrm{Re}\{q(z)\} > 0$, for $z \in U$, hence

$$\mathrm{Re}\left\{ \left(\frac{I_p^m(\lambda,\ell) f(z)}{I_p^m(\lambda,\ell) g(z)} \right)^{\mu} \right\} > \beta \quad (z \in U).$$

This finishes the proof. □

For $\ell = 1$ in Theorem 4 we obtain the following result.

Corollary 5. *Let $f \in \sum_{p,\lambda}^{m}(\alpha,\delta,\mu,\gamma)$, $\lambda > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\gamma \geq 0$. Then*

$$\mathrm{Re}\left\{\left(\frac{D_{p,\lambda}^{m} f(z) f(z)}{D_{p,\lambda}^{m} f(z) g(z)}\right)^{\mu}\right\} > \frac{2\alpha\mu + \delta\gamma\lambda}{2\mu + \delta\gamma\lambda} \quad (0 \leq \alpha < 1,\ \mu > 0,\ z \in U),$$

where the function $g \in \sum_p$ satisfies condition (9) *with $\ell = 1$.*

Corollary 6. *Let the functions f and g be in $\sum_p$ and let g satisfy condition* (9)*. If $\lambda, \ell > 0$, $\gamma \geq 1$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and*

$$\mathrm{Re}\left\{(1-\gamma)\left(\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right) + \gamma\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \alpha \tag{25}$$

$$(0 \leq \alpha < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U),$$

then

$$\mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \beta = \frac{\alpha(2\ell+\delta\lambda)+\delta\lambda(\gamma-1)}{2\ell+\delta\gamma\lambda} \quad (z \in U). \tag{26}$$

Proof. We have

$$\gamma\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)} = \left\{(1-\gamma)\left(\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right) + \gamma\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} + (\gamma-1)\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)} \quad (z \in U).$$

Since $\gamma \geq 1$, using (25) and (20) (for $\mu = 1$), we deduce that

$$\mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \beta = \frac{\alpha(2\ell+\delta\lambda)+\delta\lambda(\gamma-1)}{2\ell+\delta\gamma\lambda}.$$

□

Corollary 7. *Let $\gamma \in \mathbb{C}^*$ with $\mathrm{Re}\{\gamma\} \geq 0$ and $\lambda, \ell > 0$. If $f \in \sum_p$ satisfies the following condition*

$$\mathrm{Re}\left\{(1-\gamma)(z^p I_p^m(\lambda,\ell)f(z))^{\mu} + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)(z^p I_p^m(\lambda,\ell)f(z))^{\mu-1}\right\} > \alpha$$

$$(0 \leq \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U),$$

then

$$\mathrm{Re}\left\{\left(z^p I_p^m(\lambda,\ell)f(z)\right)^{\mu}\right\} > \frac{2\alpha\ell\mu + \lambda\mathrm{Re}(\gamma)}{2\ell\mu + \lambda\mathrm{Re}(\gamma)} \quad (z \in U). \tag{27}$$

Moreover, if $\gamma \geq 1$, $\lambda, \ell > 0$, and $f \in \sum_p$ satisfy

$$\mathrm{Re}\left\{(1-\gamma)z^p I_p^m(\lambda,\ell)f(z) + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)\right\} > \alpha \quad (z \in U),$$

then

$$\mathrm{Re}\left\{z^p I_p^{m+1}(\lambda,\ell)f(z)\right\} > \frac{\alpha(2\ell+\lambda)+\lambda(\gamma-1)}{2\ell+\gamma\lambda} \tag{28}$$

$$(0 \leq \alpha < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

Proof. The relations (27) and (28) follow by considering $g(z) = \frac{1}{z^p}$ in Theorem 4 and Corollary 6, respectively. □

REMARK 8. Choosing γ, δ, ℓ, μ, λ, and m appropriately in Corollary 7, we obtain the following results.

(i) For $\gamma = \lambda = \ell = 1$ and $m = 0$ in Corollary 7, we have that

$$\operatorname{Re}\left\{\left(1 + p + \frac{zf'(z)}{f(z)}\right)(z^p f(z))^\mu\right\} > \alpha \tag{29}$$
$$(0 \leq \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ z \in U)$$

implies

$$\operatorname{Re}\{(z^p f(z))^\mu\} > \frac{2\mu\alpha + 1}{2\mu + 1} \quad (z \in U).$$

(ii) For $\gamma \in \mathbb{C}^*$ with $\operatorname{Re}\{\gamma\} \geq 0$, $\mu = \lambda = \ell = 1$, and $m = 0$ in Corollary 7, we have that

$$\operatorname{Re}\left\{(1 + \gamma p)z^p f(z) + \gamma z^{p+1} f'(z)\right\} > \alpha$$
$$(0 \leq \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ z \in U)$$

implies

$$\operatorname{Re}\{z^p f(z)\} > \frac{2\alpha + \operatorname{Re}\{\gamma\}}{2 + \operatorname{Re}\{\gamma\}} \quad (z \in U).$$

(iii) Replacing $f(z)$ by $-\frac{zf'(z)}{p}$ in (ii), we have that

$$-\operatorname{Re}\left\{(1 + \gamma + \gamma p)\frac{z^{p+1} f'(z)}{p} + \frac{\gamma}{p} z^{p+2} f''(z)\right\} > \alpha$$
$$(0 \leq \alpha < 1,\ p \in \mathbb{N},\ z \in U)$$

implies

$$-\operatorname{Re}\left\{\frac{z^{p+1}}{p} f'(z)\right\} > \frac{2\alpha + \operatorname{Re}\{\gamma\}}{2 + \operatorname{Re}\{\gamma\}} \quad (z \in U).$$

(iv) For $\gamma \in \mathbb{R}$ with $\gamma \geq 1$, $\mu = \lambda = \ell = 1$, and $m = 0$ in Corollary 7, we have that

$$\operatorname{Re}\left\{(1 + \gamma p)z^p f(z) + \gamma z^{p+1} f'(z)\right\} > \alpha$$
$$(0 \leq \alpha < 1,\ p \in \mathbb{N},\ z \in U)$$

implies

$$\operatorname{Re}\{z^p f(z)\} > \frac{3\alpha + \gamma - 1}{2 + \gamma} \quad (z \in U).$$

(v) For $\gamma = \lambda = 1$ in Corollary 7 we have that

$$\mathrm{Re}\left\{z^p I_p(m+1,\ell)f(z)(z^p I_p(m,\ell)f(z))^{\mu-1}\right\} > \alpha$$

$$(0 \le \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U)$$

implies

$$\mathrm{Re}\left\{(z^p I_p(m,\ell)f(z))^{\mu}\right\} > \frac{2\ell\mu\alpha + 1}{2\ell\mu + 1} \quad (z \in U).$$

(vi) For $\gamma \in \mathbb{C}^*$ with $\mathrm{Re}\{\gamma\} \ge 0$, $\mu = \lambda = 1$ in Corollary 7, we have that

$$\mathrm{Re}\left\{(1-\gamma)z^p I_p(m,\ell)f(z) + \gamma z^p I_p(m+1,\ell)f(z)\right\} > \alpha$$

$$(0 \le \alpha < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U)$$

implies

$$\mathrm{Re}\left\{z^p I_p(m,\ell)f(z)\right\} > \frac{2\ell\alpha + \mathrm{Re}\{\gamma\}}{2\ell + \mathrm{Re}\{\gamma\}} \quad (z \in U).$$

(vii) For $\gamma = \lambda = \ell = 1$, in Corollary 7 we have that

$$\mathrm{Re}\left\{z^p D_p^{m+1} f(z)(z^p D_p^m f(z))^{\mu-1}\right\} > \alpha$$

$$(0 \le \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U)$$

implies

$$\mathrm{Re}\left\{(z^p D_p^m f(z))^{\mu}\right\} > \frac{2\mu\alpha + 1}{2\mu + 1} \quad (z \in U).$$

(viii) For $\mu = \lambda = \ell = 1$, in Corollary 7 we have that

$$\mathrm{Re}\left\{(1-\gamma)(z^p D_p^m f(z)) + \gamma z^p D_p^{m+1} f(z)\right\} > \alpha$$

$$(0 \le \alpha < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U)$$

implies

$$\mathrm{Re}\left\{z^p D_p^m f(z)\right\} > \frac{2\alpha + \mathrm{Re}\{\gamma\}}{2 + \mathrm{Re}\{\gamma\}} \quad (z \in U).$$

THEOREM 9. *Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}\{\gamma\} > 0$ and $\lambda, \ell > 0$. Assume that $f \in \sum_p$ satisfies the following condition*

(30)
$$\mathrm{Re}\left\{(1-\gamma)(z^p I_p^m(\lambda,\ell)f(z))^{\mu} + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)(z^p I_p^m(\lambda,\ell)f(z))^{\mu-1}\right\} > \alpha$$

$$(0 \le \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

Then

(31)
$$\mathrm{Re}\left\{(z^p I_p^m(\lambda,\ell)f(z))^{\mu}\right\} > \alpha + (1-\alpha)(2\rho - 1),$$

where

(32)
$$\rho = \frac{1}{2}\ {}_2F_1\left(1, 1; \frac{\mu\ell}{\lambda\mathrm{Re}\{\gamma\}} + 1; \frac{1}{2}\right).$$

Proof. Let

$$q(z) = (z^p I_p^m(\lambda,\ell)f(z))^\mu. \tag{33}$$

Then q is analytic on U and $q(0) = 1$. Differentiating (33) with respect to z and using relation (6), we obtain

$$(1-\gamma)(z^p I_p^m(\lambda,\ell)f(z))^\mu + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)(z^p I_p^m(\lambda,\ell)f(z))^{\mu-1}$$
$$= q(z) + \frac{\gamma\lambda z q'(z)}{\ell\mu}.$$

Hence (30) yields

$$\mathrm{Re}\left\{q(z) + \frac{\gamma\lambda z q'(z)}{\ell\mu}\right\} > \alpha \quad (z \in U).$$

In view of Lemma 2 this implies that

$$\mathrm{Re}\{q(z)\} > \alpha + (1-\alpha)(2\rho - 1),$$

where

$$\rho = \rho(\mathrm{Re}\{\gamma\}) = \int_0^1 \left(1 + t^{\frac{\lambda \mathrm{Re}\{\gamma\}}{\ell\mu}}\right)^{-1} \mathrm{d}t.$$

Putting $\mathrm{Re}\{\gamma\} = \gamma_1 > 0$, we have

$$\rho = \int_0^1 \left(1 + t^{\frac{\lambda\gamma_1}{\ell\mu}}\right)^{-1} \mathrm{d}t = \frac{\ell\mu}{\lambda\gamma_1}\int_0^1 u^{\frac{\ell\mu}{\lambda\gamma_1}-1}(1+u)^{-1}\mathrm{d}u.$$

Using (16), (17), (18), and (19), we obtain

$$\rho = {}_2F_1(1, \frac{\ell\mu}{\lambda\gamma_1}; \frac{\ell\mu}{\lambda\gamma_1} + 1; -1)$$
$$= \frac{1}{2}\,{}_2F_1(1, 1; \frac{\ell\mu}{\lambda\gamma_1} + 1; \frac{1}{2}).$$

This finishes the proof. □

Choosing $\ell = 1$ in Theorem 9, we obtain the next result.

Corollary 10. *Let* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\{\gamma\} > 0$ *and* $\lambda > 0$. *Assume that* $f \in \sum_p$ *satisfies the condition*

$$\mathrm{Re}\left\{(1-\gamma)(z^p D_{p,\lambda}^m f(z))^\mu + \gamma z^p D_{p,\lambda}^m f(z)(z^p D_{p,\lambda}^m f(z))^{\mu-1}\right\} > \alpha$$
$$(0 \le \alpha < 1,\ \mu > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

Then

$$\mathrm{Re}\left\{z^p D_{p,\lambda}^m f(z) f(z)\right\}^\mu > \alpha + (1-\alpha)(2\rho - 1),$$

where

$$\rho = \frac{1}{2}\ {}_2F_1\left(1, 1; \frac{\mu}{\lambda \mathrm{Re}\{\gamma\}} + 1; \frac{1}{2}\right).$$

Corollary 11. *Let $\gamma \in \mathbb{R}$ with $\gamma \geq 1$. If $f \in \sum_p$ satisfies*

$$\operatorname{Re}\left\{(1-\gamma)z^p I_p^m(\lambda,\ell)f(z) + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)\right\} > \alpha \tag{34}$$
$$(0 \leq \alpha < 1,\ \lambda,\ \ell > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U),$$

then

$$\operatorname{Re}\{z^p I_p^{m+1}(\lambda,\ell)f(z)\} > \alpha + (1-\alpha)(2\rho^* - 1)(1-\gamma^{-1}) \quad (z \in U),$$

where

$$\rho^* = \frac{1}{2}\ {}_2F_1\left(1,1;\frac{\ell}{\gamma\lambda}+1;\frac{1}{2}\right).$$

Proof. The assertion follows by using the identity

$$\begin{aligned}\gamma z^p I_p^{m+1}(\lambda,\ell)f(z) = &\left[(1-\gamma)z^p I_p^m(\lambda,\ell)f(z) + \gamma z^p I_p^{m+1}(\lambda,\ell)f(z)\right]\\ &+ (\gamma-1)z^p I_p^m(\lambda,\ell)f(z).\end{aligned} \tag{35}$$

□

Remark 12. (i) Note that if $\gamma = \mu > 0$, $\lambda = \ell = 1$, and $m = 0$ in Corollary 7, that is,

$$\begin{aligned}&\operatorname{Re}\left\{(1+\gamma p)(z^p f(z))^\gamma + \gamma z^{p+1} f'(z)(z^p f(z))^{\gamma-1}\right\} > \alpha\\ &(0 \leq \alpha < 1,\ p \in \mathbb{N},\ z \in U),\end{aligned} \tag{36}$$

then (27) implies that

$$\operatorname{Re}\left\{(z^p f(z))^\gamma\right\} > \frac{2\alpha+1}{3} \quad (z \in U). \tag{37}$$

On the other hand, if $f \in \sum_p$ satisfies condition (36) then, by Theorem 9, we get

$$\operatorname{Re}\left\{(z^p f(z))^\gamma\right\} > 2(1-\ell n2)\alpha + (2\ell n2 - 1) \quad (z \in U),$$

which is better than (37).

(ii) We observe that if $\gamma \in \mathbb{R}$ satisfies $\gamma > 0$ and

$$k(z) = \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)} + \left(\frac{1}{\gamma} - 1\right)\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)} \quad (z \in U),$$

then Theorem 4, applied for $\mu = 1$, yields that

$$\operatorname{Re}\{k(z)\} > \frac{\alpha}{\gamma}$$

implies

$$\operatorname{Re}\left\{\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right\} > \frac{2\ell\alpha + \delta\gamma\lambda}{2\ell + \delta\gamma\lambda}, \tag{38}$$

whenever

$$\operatorname{Re}\left\{\frac{I_p^m(\lambda,\ell)g(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \delta \quad (0 \leq \delta < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0, z \in U).$$

Let $\gamma \to +\infty$. Then it follows from (38) that

$$\mathrm{Re}\{k(z)\} \geq 0 \ (z \in U)$$

implies

$$\mathrm{Re}\left\{\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right\} \geq 1 \ (z \in U),$$

whenever

$$\mathrm{Re}\left\{\frac{I_p^m(\lambda,\ell)g(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \delta \quad (0 \leq \delta < 1,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

We will extend in the following theorem the above results.

THEOREM 13. *Suppose that the functions f and g are in $\sum_p$ and suppose that g satisfies condition* (9). *If*

$$\mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)} - \frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right\} > -\frac{(1-\alpha)\delta\lambda}{2\ell} \tag{39}$$

$$(0 \leq \alpha < 1,\ 0 \leq \delta < 1,\ \lambda,\ell > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U),$$

then

$$\mathrm{Re}\left\{\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)}\right\} > \alpha \quad (z \in U) \tag{40}$$

and

$$\mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} > \frac{(2\ell+\lambda\delta)\alpha-\lambda\delta}{2\ell} \tag{41}$$

$$(0 \leq \alpha < 1,\ 0 \leq \delta < 1,\ \lambda,\ell > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

Proof. Let

$$q(z) = \frac{1}{(1-\alpha)}\left\{\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)} - \alpha\right\}. \tag{42}$$

Then q is analytic on U and $q(0) = 1$. For

$$\phi(z) = \frac{I_p^m(\lambda,\ell)g(z)}{I_p^{m+1}(\lambda,\ell)g(z)} \quad (z \in U) \tag{43}$$

we observe that, by hypothesis, $\mathrm{Re}\{\phi(z)\} > \delta$ $(0 \leq \delta < 1)$ for $z \in U$. A simple computation shows that

$$\begin{aligned}\frac{\lambda(1-\alpha)zq'(z)\phi(z)}{\ell} &= \frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)} - \frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)g(z)} \\ &= \Psi(q(z), zq'(z)),\end{aligned}$$

where

$$\Psi(r,s) = \frac{\lambda(1-\alpha)\phi(z)s}{\ell} \quad (\ell \in \mathbb{R}\backslash\{0\}).$$

Using (39), we obtain

$$\left\{\Psi(q(z), zq^{'}(z); z \in U\right\} \subset \Omega = \left\{w \in \mathbb{C} : \mathrm{Re}\{w\} > -\frac{\lambda\delta(1-\alpha)}{2\ell}\right\}.$$

For all reals $r_2, s_1 \leq -\dfrac{(1+r_2^2)}{2}$ we have that

$$\begin{aligned}\mathrm{Re}\left\{\Psi(\mathrm{i}r_2, s_1)\right\} &= \frac{\lambda s_1(1-\alpha)\mathrm{Re}\{\phi(z)\}}{\ell} \leq -\frac{\lambda\delta(1-\alpha)(1+r_2^2)}{2\ell} \\ &\leq -\frac{\lambda\delta(1-\alpha)}{2\ell}.\end{aligned}$$

This shows that $\Psi(\mathrm{i}r_2, s_1) \notin \Omega$ for each $z \in U$. Hence, by Lemma 1, we conclude that $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$. This proves (40). The proof of (41) follows by using (40) and (41) in the identity

$$\begin{aligned}\mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)}\right\} &= \mathrm{Re}\left\{\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)g(z)} - \frac{I_p^{m}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)g(z)}\right\} \\ &\quad + \mathrm{Re}\left\{\frac{I_p^{m}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)g(z)}\right\}.\end{aligned}$$

This finishes the proof. □

Putting $\ell = 1$ in Theorem 13, we obtain the next result.

Corollary 14. *Suppose that the functions f and g are in $\sum_p$ and suppose that g satisfies condition* (9) *with $\ell = 1$. If*

$$\mathrm{Re}\left\{\frac{D_{p,\lambda}^{m+1}f(z)}{D_{p,\lambda}^{m+1}g(z)} - \frac{D_{p,\lambda}^{m}f(z)}{D_{p,\lambda}^{m}g(z)}\right\} > -\frac{(1-\alpha)\delta\lambda}{2}$$

$$(0 \leq \alpha < 1,\ 0 \leq \delta < 1,\ \lambda > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U),$$

then

$$\mathrm{Re}\left\{\frac{D_{p,\lambda}^{m}f(z)}{D_{p,\lambda}^{m}g(z)}\right\} > \alpha \quad (z \in U)$$

and

$$\mathrm{Re}\left\{\frac{D_{p,\lambda}^{m+1}f(z)}{D_{p,\lambda}^{m+1}g(z)}\right\} > \frac{(2+\lambda\delta)\alpha - \lambda\delta}{2}$$

$$(0 \leq \alpha < 1,\ 0 \leq \delta < 1,\ \lambda > 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0,\ z \in U).$$

For $\lambda = 1$ in Theorem 13 we get the following result.

Corollary 15. *Suppose that the functions f and g are in $\sum_p$ and suppose that g satisfies*

$$\mathrm{Re}\left\{\frac{I_p(m,\ell)g(z)}{I_p(m+1,\ell)g(z)}\right\} > \delta \quad (0 \leq \delta < 1,\ z \in U).$$

If

$$\operatorname{Re}\left\{\frac{I_p(m+1,\ell)g(z)}{I_p(m+1,\ell)g(z)}-\frac{I_p(m,\ell)f(z)}{I_p(m,\ell)g(z)}\right\}>-\frac{(1-\alpha)\delta}{2\ell}$$

$$(0\le\alpha<1,\ 0\le\delta<1,\ \ell>0,\ p\in\mathbb{N},\ m\in\mathbb{N}_0,\ z\in U),$$

then

$$\operatorname{Re}\left\{\frac{I_p(m,\ell)f(z)}{I_p(m,\ell)g(z)}\right\}>\alpha\quad(z\in U)$$

and

$$\operatorname{Re}\left\{\frac{I_p(m+1,\ell)f(z)}{I_p(m+1,\ell)g(z)}\right\}>\frac{(2\ell+\delta)\alpha-\delta}{2\ell}$$

$$(0\le\alpha<,\ 0\le\delta<1,\ \ell>0,\ p\in\mathbb{N},\ m\in\mathbb{N}_0,\ z\in U).$$

REMARK 16. For $\delta=\lambda=\ell=1,\ m=0$, and $g(z)=\dfrac{1}{z^p}$ in Theorem 13 we get that

$$\operatorname{Re}\left\{z^pf(z)+\frac{z^{p+1}}{p}f'(z)\right\}>-\frac{(1-\alpha)}{2p}\quad(0\le\alpha<1,\ p\in\mathbb{N},\ z\in U)$$

implies

$$\operatorname{Re}\{z^pf(z)\}>\alpha\quad(0\le\alpha<1,\ p\in\mathbb{N},\ z\in U)$$

and

$$\operatorname{Re}\{(1+p)z^pf(z)+z^{p+1}f'(z)\}>\frac{3\alpha-1}{2}\quad(0\le\alpha<1,\ p\in\mathbb{N},\ z\in U).$$

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