MATHEMATICA, Tome 53 (76), N° 2, 2011, pp. 181–188

POSINORMAL FACTORABLE MATRICES WHOSE INTERRUPTER IS DIAGONAL

H. CRAWFORD RHALY, JR.

Dedicated to Thomas L. Kriete, III

Abstract. First we determine sufficient conditions for a lower triangular factorable matrix to be a posinormal operator on ℓ^2. Then we compute the interrupter and determine when it will be a diagonal matrix. This leads us to a large collection of hyponormal factorable matrices.

MSC 2010. 47B99.
Key words. Posinormal operator, hyponormal operator, factorable matrix.

1. INTRODUCTION

Throughout this paper we assume that M is a lower triangular infinite matrix acting through multiplication to give a bounded linear operator on ℓ^2. If $\{a_n\}$ and $\{c_n\}$ are sequences of real or complex numbers, then $M :\equiv M(\{a_n\},\{c_n\})$ is said to be *factorable* if its nonzero entries m_{ij} satisfy $m_{ij} = a_i c_j$, where a_i depends only on i (for $i = 0, 1, 2, \dots$) and c_j depends only on j (for $j = 0, 1, 2, \dots$); a factorable matrix M is *terraced* if $c_j = 1$ for all j. The operator M is *hyponormal* if it satisfies $\langle [M^*, M]f, f\rangle \equiv \langle (M^*M - MM^*)f, f\rangle \geq 0$ for all f in ℓ^2.

Initially we consider the Cesàro matrix C, the factorable matrix that occurs when $a_i = \frac{1}{i+1}$ and $c_j = 1$ for all i,j, and we let D denote the diagonal matrix with diagonal $\{\frac{1}{2}, \frac{2}{3}, ..., \frac{n+1}{n+2}, \dots\}$. It can be verified that $CC^* = C^*DC$ and hence

$$\langle [C^*, C]f, f\rangle \equiv \langle (C^*C - CC^*)f, f\rangle = \langle (I - D)Cf, Cf\rangle \geq 0$$

for all f in ℓ^2, so C is easily seen to be a hyponormal operator on ℓ^2; for a different proof, as well as a proof that C is a bounded operator on ℓ^2, see [1]. This example provided the original motivation for the introduction of posinormal operators in [20]. M is *posinormal* if there is a bounded, positive operator P on ℓ^2 satisfying $MM^* = M^*PM$, and the operator P is referred to as an *interrupter* for M. We note that posinormal operators have also been studied in a more general setting in [2, 3, 5, 6, 9, 10, 12, 13, 15, 22, 23].

Using the fact that the interrupter for C is a diagonal matrix, we present the following adaptation of [20, Theorem 2.5].

The author wishes to express his gratitude to Billy Rhoades, who suggested the expansion of the study of the operators in [21] to include factorable matrices.

Proposition 1.1. *If q_n is chosen from the interval $\left[\frac{n+1}{n+2}, 1\right]$ for each n, then the factorable matrix $T = [t_{ij}]$, where $t_{ij} = a_i c_j$ with $a_i = \frac{\sqrt{q_i}}{i+1}$ and $c_j = \sqrt{q_j}$, is hyponormal.*

Proof. If Q is the diagonal matrix with diagonal $\{q_0, q_1, \ldots, q_n, \ldots\}$, then $I \geq Q \geq D$ and $T = \sqrt{Q}C\sqrt{Q}$, so that

$$\begin{aligned}[T^*, T] &= \sqrt{Q}C^*QC\sqrt{Q} - \sqrt{Q}C^*DC\sqrt{Q} + \sqrt{Q}C^*DC\sqrt{Q} - \sqrt{Q}CQC^*\sqrt{Q} \\ &= \sqrt{Q}C^*(Q-D)C\sqrt{Q} + \sqrt{Q}C(I-Q)C^*\sqrt{Q}.\end{aligned}$$

Therefore

$$\langle [T^*, T]f, f\rangle = \langle (Q-D)C\sqrt{Q}f, C\sqrt{Q}f\rangle + \langle (I-Q)C^*\sqrt{Q}f, C^*\sqrt{Q}f\rangle \geq 0$$

for all f in ℓ^2, so T is hyponormal. □

Seeing what happened for the Cesàro matrix, we now set out in search of other posinormal lower triangular factorable matrices having a diagonal matrix for interrupter, in hopes that this will once again result in manageable arithmetic that will uncover some more examples of hyponormal factorable matrices. As a first step, we obtain sufficient conditions for the posinormality of a lower triangular factorable matrix.

2. SUFFICIENT CONDITIONS FOR A FACTORABLE MATRIX TO BE POSINORMAL

In [20] it was observed that the set of all posinormal operators on any Hilbert space H is an enormous collection that includes every invertible operator and all the hyponormal operators. Here we are concerned with $H = \ell^2$ and we employ the techniques of [21] to obtain an alternative route for identifying posinormal, and potentially hyponormal, lower triangular factorable matrices in cases when the matrices may not be invertible. Our first theorem presents sufficient conditions.

Theorem 2.1. *Suppose $M = M(\{a_n\}, \{c_n\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*
(1) $\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to 0;
(2) $\{c_n\}$ and $\left\{(n+1)\left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\frac{a_{n+1}}{a_n}\right)\right\}$ are bounded sequences.
Then M is posinormal.

Proof. We will display an operator B on ℓ^2 that satisfies $M^* = BM$; consequently, $M = M^*B^*$ also, and it will follow from [20, Theorem 2.1] that M is posinormal.

We define $B = [b_{mn}]$ by

$$b_{mn} = \begin{cases} c_m\left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\frac{a_{n+1}}{a_n}\right) & \text{if} \quad m \leq n; \\ -\frac{a_{n+1}}{a_n} & \text{if} \quad m = n+1; \\ 0 & \text{if} \quad m > n+1. \end{cases}$$

Condition (2) will help us show that B is a bounded operator on ℓ^2. Let $R = M(s,1)$, where $s = \left\{ \frac{1}{c_n} - \frac{1}{c_{n+1}} \frac{a_{n+1}}{a_n} : n = 0,1,2,\dots \right\}$, so R is a terraced matrix with all of its entries nonnegative. The diagonal matrix D_1 with diagonal $\left\{ (n+1)\left(\frac{1}{c_n} - \frac{1}{c_{n+1}} \frac{a_{n+1}}{a_n} \right) \right\}$ is positive and bounded, and hence $R = D_1 C$ is bounded. The diagonal matrix D_2 with diagonal $\{c_n\}$ is bounded, so RD_2 is bounded. We observe that $(B^* - RD_2)$ is the adjoint of a unilateral weighted shift; since $\{a_n\}$ is positive and decreasing, $(B^* - RD_2)$ is bounded. Therefore $B^* = RD_2 + (B^* - RD_2)$ is a bounded operator, and hence B is bounded also. A direct computation using condition (1) shows that $M^* = BM$, as needed. □

Corollary 2.2. *Suppose $M = M(\{a_n\},\{c_n\})$ acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *$\{a_n\}$ and $\{\frac{a_n}{c_n}\}$ are positive decreasing sequences that converge to 0;*

(2) *$\{c_n\}$ is a decreasing sequence such that $\lim_{n\to\infty} c_n > 0$;*

(3) *$\left\{ (n+1)\left(1 - \frac{a_{n+1}}{a_n}\right) \right\}$ is a bounded sequence.*

Then M is posinormal.

Example 2.3. Consider the case where $a_i = \frac{1}{i+1}$ and $c_j = \frac{1}{2} + \left(\frac{1}{10}\right)^{j+1}$ for each i, j. This example satisfies all parts of the hypothesis of Corollary 1, so the associated factorable matrix gives a posinormal operator on ℓ^2. We note that boundedness follows from the fact that all of the entries are nonnegative and are dominated by the corresponding entries of C.

Corollary 2.4. *If $a = \{a_n\}$ is a positive decreasing sequence that converges to 0 and $\{na_n\}$ is an increasing sequence that converges to $L < +\infty$, then the terraced matrix $M = M(a,1)$ is posinormal.*

Remark 2.5. We note that the sufficient conditions of Theorem 2.1 are not necessary for the posinormality of a factorable matrix M. Consider, for example, the discrete generalized Cesàro matrices (see [16, 17]), occurring when $a_i = \frac{\alpha^i}{i+1}$ and $c_j = \frac{1}{\alpha^j}$ for all i,j and $0 < \alpha < 1$. Since $\{c_j\}$ is not bounded, condition (2) of the theorem is not satisfied, although these matrices were shown to be posinormal in [20, Theorem 4.1], where a different approach was used to prove that B is a bounded operator on ℓ^2.

In view of Remark 2.5, we include the following modification of Theorem 2.1.

Theorem 2.6. *Suppose $M = M(\{a_n\},\{c_n\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *$\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to 0;*

(2) *the matrix B from the proof of Theorem 2.1 is a bounded operator on ℓ^2.*

Then M is posinormal.

The next result gives sufficient conditions for the posinormality of the adjoint of a lower triangular factorable matrix.

THEOREM 2.7. *Suppose $M = M(\{a_n\}, \{c_n\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*
(1) *$\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to 0,*
(2) *$\left\{\frac{c_{n-1}}{c_n}\right\}$ and $\left\{\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}\right\}$ are bounded sequences for $n \geq 1$.*
Then M^* is posinormal.

Proof. We define $T = [t_{mn}]$ by

$$t_{mn} = \begin{cases} \frac{a_m}{a_n} & \text{if} \quad n = 0; \\ a_m(\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}) & \text{if} \quad 0 < n \leq m; \\ -\frac{c_{n-1}}{c_n} & \text{if} \quad n = m+1; \\ 0 & \text{if} \quad n > m+1. \end{cases}$$

Since the sequence $\left\{\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}\right\}$ is bounded for $n \geq 1$, the diagonal matrix D with diagonal $\left\{\frac{1}{a_0}, \frac{1}{a_1} - \frac{c_0}{c_1}\frac{1}{a_0}, \frac{1}{a_2} - \frac{c_1}{c_2}\frac{1}{a_1}, \frac{1}{a_3} - \frac{c_2}{c_3}\frac{1}{a_2}, \dots\right\}$ is bounded, so MD is bounded. Also, the weighted shift W with weight sequence $\left\{\frac{c_{n-1}}{c_n}\right\}$ is bounded for $n \geq 1$. Therefore $T = MD - W^*$ is a bounded operator. A routine computation shows that $M = TM^*$. By [20, Theorem 2.1], M^* is posinormal. □

COROLLARY 2.8. *Suppose $M = M(\{a_n\}, \{c_n\})$ acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*
(1) *$\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to 0;*
(2) *$\{c_n\}$ is a decreasing sequence such that $\lim_{n\to\infty} c_n > 0$;*
(3) *$\{na_n\}$ is an increasing sequence that converges to $L < +\infty$.*
Then M^* is posinormal.

COROLLARY 2.9. *If $a = \{a_n\}$ is a positive decreasing sequence that converges to 0 and $\{(n+1)a_n\}$ is an increasing sequence that converges to $L < +\infty$, then $M^* = M(a, 1)^*$ is posinormal.*

We note that Example 2.3 satisfies all the conditions in the hypothesis of Corollary 2.8, so the associated factorable matrix is both posinormal and coposinormal. In fact, any factorable matrix that satisfies all of the conditions in the hypothesis of Corollary 2.8 will be both posinormal (see Corollary 2.2) and coposinormal.

We close this section with a modified version of Theorem 2.7.

THEOREM 2.10. *Suppose $M = M(\{a_n\}, \{c_n\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*
(1) *$\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to 0;*

(2) *the matrix T from the proof of Theorem* 2.7 *is a bounded operator on* ℓ^2.

Then M^* is posinormal.

3. POSINORMAL FACTORABLE MATRICES WITH A DIAGONAL INTERRUPTER

In order to obtain the interrupter for M, we will use the matrix B mentioned in Theorem 2.6 (and displayed in the proof of Theorem 2.1). For B bounded and satisfying $M^*= BM$, we now compute the interrupter $P = B^*B$; the entries of $P = [p_{mn}]$ are given by

$$p_{mn} = \begin{cases} \frac{c_n^2 c_{n+1}^2 a_{n+1}^2 + (\sum_{k=0}^n c_k^2)(c_{n+1}a_n - c_n a_{n+1})^2}{c_n^2 c_{n+1}^2 a_n^2} & \text{if} \quad m = n; \\ \frac{(c_m a_{m+1} - c_{m+1} a_m)[c_n(\sum_{k=0}^{n+1} c_k^2) a_{n+1} - c_{n+1}(\sum_{k=0}^{n} c_k^2) a_n]}{c_m c_{m+1} c_n c_{n+1} a_m a_n} & \text{if} \quad m > n; \\ \frac{(c_n a_{n+1} - c_{n+1} a_n)[c_m(\sum_{k=0}^{m+1} c_k^2) a_{m+1} - c_{m+1}(\sum_{k=0}^{m} c_k^2) a_m]}{c_m c_{m+1} c_n c_{n+1} a_m a_n} & \text{if} \quad m < n. \end{cases}$$

Inspection of the entries reveals that P will be a diagonal matrix when the sequence $\left\{ \left(\sum_{k=0}^n c_k^2\right) \frac{a_n}{c_n} \right\}$ is constant or when the sequence $\left\{ \frac{a_n}{c_n} \right\}$ is constant; the latter possibility will not be useful because of condition (1) in Theorem 2.6, so we proceed with consideration of the former, and that leads to the following results.

THEOREM 3.1. *For fixed $\alpha \geq 1$, take $a_i = \sqrt{\frac{\Gamma(i+1)}{\Gamma(i+\alpha)} \frac{1}{i+\alpha}}$ and $c_j = \sqrt{\frac{\Gamma(j+\alpha)}{\Gamma(j+1)}}$ for each i, j. Then $M = M(\{a_n\}, \{c_n\})$ is a hyponormal bounded operator on ℓ^2.*

Proof. We note that M is bounded for $\alpha \geq 1$ since all of its entries are nonnegative and dominated by the corresponding entries of C. Next we need to show that the hypothesis of Theorem 2.6 is satisfied. It is straightforward to verify that $\left\{ \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha)}} \frac{1}{n+\alpha} \right\}$ and $\left\{ \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \frac{1}{n+\alpha} \right\}$ are positive decreasing sequences that converge to 0, so we leave that to the reader. To assist in showing that B is bounded, we let W denote the weighted shift with weights $\left\{ \frac{a_{n+1}}{a_n} \right\}$; W is bounded since $lim_{n->\infty} \frac{a_{n+1}}{a_n} = 1$. Next we observe that all of the entries of $B+W$ are nonnegative and dominated by the corresponding entries of αC^*, so $B+W$ is bounded. Therefore $B = (B+W)-W$ is bounded, and it follows that $P = B^*B$ is bounded also.

It can be shown by induction that $\sum_{k=0}^n c_k^2 = \frac{1}{\alpha} \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}$ for all n. Therefore $\left(\sum_{k=0}^n c_k^2\right) \frac{a_n}{c_n} = \frac{1}{\alpha}$ for all n. It is easily verified that $p_{nn} = \frac{n+\alpha}{n+1+\alpha}$ for each n. It follows that $MM^* = M^*PM$ and hence

$$\langle [M^*, M]f, f \rangle \equiv \langle (M^*M - MM^*)f, f \rangle = \langle (I - P)Mf, Mf \rangle \geq 0$$

for all f in ℓ^2, since $I - P$ is a diagonal matrix with diagonal $\left\{ \frac{1}{n+1+\alpha} \right\}$; therefore the factorable matrix M is a hyponormal operator on ℓ^2. $\square$

We observe that when $\alpha = 1$, M is the Cesàro matrix C. Next we consider the matrix M that occurs when $\alpha = 2$; that is, when $a_i = \frac{1}{\sqrt{i+1}(i+2)}$ and $c_j = \sqrt{j+1}$ for all i,j. For this example, the interrupter P is the diagonal matrix with diagonal $\left\{\frac{2}{3}, \frac{3}{4}, ..., \frac{n+2}{n+3}, \dots\right\}$ satisfying $MM^* = M^*PM$, as required, and, since $I - P \geq 0$, this factorable matrix M is a hyponormal bounded operator on ℓ^2. It is worth noting that the adjoint of this matrix was studied in [18], and the techniques used there can be adapted to the more general situation here to give a proof of the following theorem.

THEOREM 3.2. *If M is the matrix defined in Theorem* 3.1*, then M has norm $\|M\| = \frac{2}{\alpha}$ and spectrum $\sigma(M) = \{\lambda : \left|\lambda - \frac{1}{\alpha}\right| \leq \frac{1}{\alpha}\}$.*

We note that although the factors of the nonzero matrix entries $m_{ij} = a_i c_j$ in Theorem 3.1 are reminiscent of the factors in the operators of Kay, Soul, and Trutt [7], those operators – other than the Cesàro matrix – do not satisfy our requirement that the sequence $\left\{\left(\sum_{k=0}^n c_k^2\right)\frac{a_n}{c_n}\right\}$ be constant. For other related information, see [8, Section 4.1.3].

THEOREM 3.3. *Let M denote the matrix defined in Theorem* 3.1*. For fixed $\alpha \geq 1$ and for each $n \geq 0$, choose q_n in the interval $\left[\frac{n+\alpha}{n+1+\alpha}, 1\right)$. If Q is the diagonal matrix with diagonal $\{q_n\}$, then $T \equiv \sqrt{Q}M\sqrt{Q}$ is another hyponormal factorable matrix.*

Proof. To see that T is factorable, note that $T = [t_{ij}]$ where $t_{ij} = r_i s_j$ with $r_i = a_i\sqrt{q_i}$ depending only on i and $s_j = c_j\sqrt{q_j}$ depending only on j. To settle the question of hyponormality, we note that $I \geq Q \geq P$ and that reasoning similar to that used in Proposition 1.1 shows that

$$[T^*, T] = \sqrt{Q}M^*(Q - P)M\sqrt{Q} + \sqrt{Q}M(I - Q)M^*\sqrt{Q}.$$

Therefore

$$\langle [T^*, T]f, f\rangle = \langle (Q - P)M\sqrt{Q}f, M\sqrt{Q}f\rangle + \langle (I - Q)M^*\sqrt{Q}f, M^*\sqrt{Q}f\rangle \geq 0$$

for all f in ℓ^2, so T is hyponormal. □

We close with a theorem and corollary that will summarize the general situation encountered here. These results will be followed by some more examples, whose details are left to the interested reader.

THEOREM 3.4. *Suppose $M = M(\{a_n\}, \{c_n\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *$\{a_n\}$ and $\left\{\frac{a_n}{c_n}\right\}$ are positive decreasing sequences that converge to* 0*;*

(2) *the matrix $B = [b_{mn}]$ from the proof of Theorem 2.1 is a bounded operator on ℓ^2;*

(3) *the sequence $\{(\sum_{k=0}^n c_k^2)\frac{a_n}{c_n}\}$ is constant;*

(4) *$1 \geq p_{nn} \geq 0$ for all n, where $p_{nn} = \frac{c_n^2 c_{n+1}^2 a_{n+1}^2 + (\sum_{k=0}^n c_k^2)(c_{n+1}a_n - c_n a_{n+1})^2}{c_n^2 c_{n+1}^2 a_n^2}$.*

Then M is posinormal with a diagonal interrupter, and, furthermore, M is hyponormal.

Corollary 3.5. *Suppose M is a factorable matrix that acts as a bounded operator on ℓ^2 and satisfies conditions (1)–(4). If Q is the diagonal matrix with diagonal $\{q_n\}$ where $1 \geq q_n \geq p_{nn}$ for all n, then $T \equiv \sqrt{Q}M\sqrt{Q}$ is another hyponormal factorable matrix.*

Example 3.6. The factorable matrices $M = M(\{a_n\}, \{c_n\})$ defined below are bounded on ℓ^2 and can be shown to satisfy conditions (1)–(4) of Theorem 3.4.

(a) M determined by $a_i = \frac{1}{(i+2)(2i+3)}$ and $c_j = j+1$ for all i, j;

(b) M determined by $a_i = \frac{i+1}{(i+2)(2i+3)(3i^2+9i+5)}$ and $c_j = (j+1)^2$ for all i, j;

(c) M determined by $c_j = \beta^j$ and $a_i = \frac{\beta^i}{\sum_{k=o}^{i} \beta^{2k}}$, where $\beta \geq 2$, for all i, j.

REFERENCES

[1] Brown, A., Halmos, P. R. and Shields, A. L., *Cesàro operators*, Acta Sci. Math. (Szeged), **26** (1965), 125–137.

[2] Cha, H., Lee, K. and Kim, J., *Superclasses of posinormal operator*, Int. Math. J., **2** (2002), 543–550.

[3] Duggal, B.P. and Kubrusly, C., *Weyl's theorem for posinormal operators*, J. Korean Math. Soc., **42** (2005), 529–541.

[4] Halmos, P.R., *A Hilbert space problem book*, Second Edition, Springer-Verlag, Berlin, 1982.

[5] Itoh, M. *Characterization of posinormal operators*, Nihonkai Math. J., **11** (2000), 97–101.

[6] Jeon, I.H., Kim, S.H., Ko, E. and Park, J.E., *On positive-normal operators*, Bull. Korean Math. Soc., **39** (2002), 33–41.

[7] Kay, E., Soul, H. and Trutt, D., *Some subnormal operators and hypergeometric kernel functions*, J. Math. Anal. Appl., **53** (1976), 237–242.

[8] Kriete, T.L., III, and Rhaly, H.C., Jr., *Translation semigroups on reproducing kernel Hilbert spaces*, J. Operator Theory, **17** (1987), 33–83.

[9] Kubrusly, C. and Duggal, B., *On posinormal operators*, Adv. Math. Sci. Appl., **17** (2007), 131–147.

[10] Kubrusly, C., *Tensor product of proper contractions, stable and posinormal operators*, Publ. Math. Debrecen, **71** (2007), 425–437.

[11] Leibowitz, G., *Rhaly matrices*, J. Math. Anal. Appl., **128** (1987), 272–286.

[12] Mecheri, S.. *Generalized Weyl's theorem for posinormal operators*, Math. Proc. R. Ir. Acad., **107** (2007), 81–89.

[13] Mecheri, S. and Seddik, M., *Weyl type theorems for posinormal operators*, Math. Proc. R. Ir. Acad., **108** (2008), 69–79.

[14] Olmsted, J.M.H., *Advanced calculus*, Appleton-Century-Crofts, New York, 1961.

[15] Panayappan, S. and Radharamani, A., *Posinormal composition and weighted composition operators*, Int. J. Contemp. Math. Sci., **4** (2009), 1261–1264.

[16] Rhaly, H.C., Jr.,, *Discrete generalized Cesàro operators*, Proc. Amer. Math. Soc., **86** (1982), 405–409.

[17] Rhaly, H.C., Jr., *Generalized Cesàro matrices*, Canad. Math. Bull., **27** (1984), 417–422.

[18] Rhaly, H.C., Jr., *An averaging operator on the Dirichlet space*, J. Math. Anal. Appl., **98** (1984), 555–561.

[19] Rhaly, H.C., Jr., *Terraced matrices*, Bull. Lond. Math. Soc., **21** (1989), 399–406.

[20] Rhaly, H.C., Jr., *Posinormal operators*, J. Math. Soc. Japan, **46** (1994), 587–605.

[21] Rhaly, H.C., Jr., *Posinormal terraced matrices*, Bull. Korean Math. Soc., **46** (2009), 117–123.

[22] Veluchamy, T. and Thulasimani, T., *Posinormal composition operators on weighted Hardy space*, Int. Math. Forum, **5** (2010), 1195–1205.

[23] Veluchamy, T. and Thulasimani, T., *Factorization of posinormal operator*, Int. J. Contemp. Math. Sci., **5** (2010), 1257–1261.

Received August 26, 2010
Accepted February 22, 2011

1081 Buckley Drive
Jackson, Mississippi 39206, U.S.A.
E-mail: `rhaly@alumni.virginia.edu`
`rhaly@member.ams.org`