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ON COMMUTATIVITY OF 2-TORSION FREE $*$-PRIME RINGS WITH GENERALIZED DERIVATIONS

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Abstract. Let $(R,*)$ be a 2-torsion free $*$-prime ring with involution $*$, $I \neq 0$ a $*$-ideal of R. An additive mapping F: $R \to R$ is called a generalized derivation on R if there exists a derivation d: $R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper, we prove the commutativity of a $*$-prime ring R admitting generalized derivations satisfying several conditions, but associated with a derivation commuting with $*$.

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1. INTRODUCTION

Throughout the present paper R will denote an associative ring with involution $*$ and center $Z(R)$. For each $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the skew-commutator $xy + yx$. A left (resp. right, two sided) ideal I of R is called a left (resp. right, two sided) $*$-ideal if $*(I) = I$. An ideal P of R is called $*$-prime ideal if $P(\neq R)$ is a $*$-ideal and for $*$-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. An ideal Q of R is called $*$-semiprime ideal if for any $*$-ideal I, $I^2 \subseteq Q$ implies that $I \subseteq Q$. A ring R equipped with an involution $*$ is said to be a $*$-prime ring if for any $a, b \in R$, $aRb = aR * (b) = \{0\}$ implies $a = 0$ or $b = 0$. Obviously, every prime ring equipped with involution $*$ is $*$-prime. The converse need not be true in general. Such a counterexample due to L. Oukhtite is as following: Let R be a prime ring, $S = R \times R^{\circ}$ where R° is the opposite ring of R, define $\sigma(x, y) = (y, x)$. From $(0, x)S(x, 0) = 0$, it follows that S is not prime. For the σ-primeness of S, we suppose that $(a, b)S(x, y) = 0$ and $(a, b)S\sigma((x, y)) = 0$, then we get $aRx \times yRb = 0$ and $aRy \times xRb = 0$, and hence $aRx = yRb = aRy = xRb = 0$, or equivalently $(a, b) = 0$ or $(x, y) = 0$. In all that follows, we set $S_{a_*}(R) = \{x \in R \mid *(x) = \pm x\}$, where $*$ is an involution of R.

An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation.

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An additive function $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition, an additive mapping F: $R \to R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d a derivation of R, is a generalized derivation; and if R has multiplicative identity 1, then all generalized derivations have this form. Over the last four decades, several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms, derivations or generalized derivations which are centralizing or commuting on appropriate subset of R (see [1], [4], [6], [7] and [14], for partial bibliography). In this paper, we invesigate the commutativity of $*$-prime ring R admittings a generalized derivations F and G satisfying any one of the following properties: (i) $F([x, y]) = (x \circ y)$, (ii) $F(x \circ y) = [x, y]$, (iii) $[F(x), y] = (F(x) \circ y)$, (iv) $F([x, y]) = [F(x), y]$, (v) $F(x \circ y) = (F(x) \circ y)$, (vi) $F(x)x = xG(x)$, (vii) $F(x^2) = x^2$, $(viii)$ $[F(x), y] = [x, G(y)]$, (ix) $F([x, y]) = [F(x), y] + [d(y), x]$ and (x) $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in I$.

2. PRELIMINARY RESULTS

We shall use without explicit mention the following basic identities, that hold for any $x, y, z \in R$:

- $[xy, z] = x[y, z] + [x, z]y$,
- $[x, yz] = y[x, z] + [x, y]z$,
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

We begin our discussion with the following results.

Lemma 2.1. [13, Lemma 3.1] *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If $a, b \in R$ such that $aIb = aI * (b) = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2. [16, Lemma 2.3] *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a derivation d commuting with $*$ such that $d^2(I) = 0$, then $d = 0$.*

Lemma 2.3. [15, Lemma 2.3] *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d commuting with $*$ such that $[d(x), x] = 0$ for all $x \in I$, then R is commutative.*

Lemma 2.4. *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d commuting with $*$ such that $[x, y]Id(x) = 0$ for all $x, y \in I$, then R is commutative.*

Proof. We have

$$[x, y]Id(x) = 0 \text{ for all } x, y \in I. \tag{1}$$

Since I is a $*$-ideal and $d* = *d$, for all $x \in I \cap S_{a_*}(R)$, thus by Lemma 2.1, we have either $[x, z] = 0$ or $d(x) = 0$. Using the fact that $x - *(x) \in I \cap S_{a_*}(R)$ for all $x \in I$, then $[x - *(x), z] = 0$ or $d(x - *(x)) = 0$ for all $z \in I$. If $d(x - *(x)) = 0$, then $d(x) = *(d(x))$ and hence (1) gives that either $[x, z] = 0$ or $d(x) = 0$. On the other hand if $[x - *(x), z] = 0$, then $[x, z] = [*(x), z]$ for all $z \in I$. As I is $*$-ideal, it follows from (1) that $[x, z]yd(x) = *([x, z])yd(x) = 0$ and hence by Lemma 2.1, we get either $[x, z] = 0$ or $d(x) = 0$. Now let $A = \{x \in I \mid [x, z] = 0 \text{ } for \text{ } all \text{ } z \in I\}$ and $B = \{x \in I \mid d(x) = 0\}$. Then A and B are both additive subgroups of I and $A \cup B = I$. But $(I, +)$ is not union of two its proper subgroups shows that either $A = I$ or $B = I$. If $I = B$, then $d(x) = 0$ for all $x \in I$. For any $r \in R$, replace x by xr to get $xd(r) = 0$ and hence $Id(r) = 0$ for all $r \in R$. In particular $1Id(R) = 0 = *(1)Id(R)$. Thus, by Lemma 2.1 we get $d = 0$, a contradiction. If $I = A$, then $[x, y] = 0$ for all $x, y \in I$ and hence using the same technique as used in the proof of Theorem 1.1 of [14], we get the required result. □

3. $*$-IDEALS AND GENERALIZED DERIVATIONS

Theorem 3.1. *Let R be a 2-torsion free $*$-prime ring with involution $*$, I a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d commuting with $*$ such that $F(x) = 0$ for all $x \in I$, then R is commutative.*

Proof. For any $x \in I$, we have $F(x) = 0$. For any $r \in R$ replacing x by $[x, r]$, we get $F(x)r + xd(r) - F(r)x - rd(x) = 0$ and hence by the hypothesis we find that

$$xd(r) - F(r)x - rd(x) = 0. \tag{2}$$

Replace r by rx in (2) and use (2) to get $[x, r]d(x) = 0$ for all $x \in I, r \in R$. Again replacing r by yr, we get

$$[x, y]Rd(x) = \{0\} \text{ for all } x, y \in I. \tag{3}$$

For all $x \in I \cap S_{a_*}(R)$, relation (3) yields that $[x, y]Rd(x) = 0 = [x, y]R*(d(x))$. Since R is $*$-prime ring and hence we obtain either $[x, y] = 0$ or $d(x) = 0$.

Now for any $x \in I$, using the fact $x - *(x) \in I \cap S_{a_*}(R)$, then $[x - *(x), y] = 0$ or $d(x - *(x)) = 0$. If $d(x - *(x)) = 0$, then $d(x) = d(*(x)) = *(d(x))$ and hence from (3) since I is $*$-ideal either $[x, y] = 0$ or $d(x) = 0$. Suppose $[x - *(x), y] = 0$ for all $y \in I$. Since $x + *(x) \in I \cap S_{a_*}(R)$, then $d(x + *(x)) = 0$

or $[x + *(x), y] = 0$ for all $y \in I$. If $[x + *(x), y] = 0$, then $2[x, y] = 0$ that is, $[x, y] = 0$. If $d(x + *(x)) = 0$, then $d(x) = - * (d(x))$ again since I is $*$-ideal and by (3) we get either $[x, y] = 0$ or $d(x) = 0$. Consequently, for all $x \in I$, either $[x, y] = 0$ or $d(x) = 0$. Now using similar arguments as used in the last paragraph of the proof of Lemma 2.4, we get the required result. □

THEOREM 3.2. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Suppose that R admits a generalized derivation F associated with a derivation d commuting with $*$ such that*

(i) $F([x, y]) = (x \circ y)$, *for all $x, y \in I$, or*
(ii) $F([x, y]) + (x \circ y) = 0$, *for all $x, y \in I$, or*
(iii) $[d(x), F(y)] = [x, y]$, *for all $x, y \in I$, or*
(iv) $[d(x), F(y)] + [x, y] = 0$, *for all $x, y \in I$, or*
(v) $F(x \circ y) = [x, y]$, *for all $x, y \in I$, or*
(vi) $F(x \circ y) = -[x, y]$, *for all $x, y \in I$.*

If $F = 0$ or $d \neq 0$, then R is commutative.

Proof. (*i*) If $F = 0$, then $x \circ y = 0$ for all $x, y \in I$. Replacing y by yz, we get $y[x, z] = 0$ for all $x, y, z \in I$. In particular, $[x, z]I[x, z] = 0 = [x, z]I * ([x, z])$. Thus by Lemma 2.1, we get $[x, z] = 0$ and hence by [14, proof of theorem 1.1] we get the required result.

Therefore, we shall assume that $d \neq 0$. Given that F is a generalized derivation of R such that

$$F([x, y]) = (x \circ y) \text{ for all } x, y \in I. \tag{4}$$

This can be rewritten as $F(x)y + xd(y) - F(y)x - yd(x) = (x \circ y)$. Now, replacing y by yx in the above expression we find that

$$[x, y]d(x) = 0. \tag{5}$$

Again replace y by zy in (5) and use (5) to get $[x, z]yd(x) = 0$, that is,

$$[x, z]Id(x) = \{0\} \text{ for all } x, z \in I. \tag{6}$$

Hence by Lemma 2.4, we get the required result.

(*ii*) If F satisfies $F([x, y]) = -(x \circ y)$, then $(-F)$ satisfies $(-F)([x, y]) = (x \circ y)$ for all $x, y \in I$ and hence by part (*i*), our result follows.

(*iii*) If $F = 0$, then for any $x, y \in I$ we have $[x, y] = 0$. Thus using the same arguments as used in the proof of Theorem 1.1 of [14], we get the required result.

Henceforth, we shall assume that $d \neq 0$. For any $x, y \in I$, we have

$$[d(x), F(y)]) = [x, y]. \tag{7}$$

Replacing y by yz in (7), we get

$$F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = y[x, z]. \tag{8}$$

Again replace z by $zd(x)$ in (8) to get

$$y[d(x), z]d^2(x) + yz[d(x), d^2(x)] + [d(x), y]zd^2(x) = yz[x, d(x)]. \tag{9}$$

Now, replacing y by ty in (9), we obtain

$$[d(x),t]yzd^2(x)=0 \text{ for all } x,y,z,t\in I. \tag{10}$$

Let $x\in I\cap S_{a*}(R)$; since d commute with $*$, (10) yields that

$$[d(x),t]yId^2(x)=\{0\}=[d(x),t]yI*(d^2(x)) \text{ for all } y,t\in I.$$

Thus by Lemma 2.1, either $[d(x),t]y=0$ or $d^2(x)=0$. If $[d(x),t]y=0$ for all $y,t\in I$, then $[d(x),t]I=0$ so that $[d(x),t]=0$ by Lemma 2.1. Therefore, for each $x\in I\cap S_{a*}R$, we have $[d(x),t]=0$ or $d^2(x)=0$ for all $t\in I$. Now for any $x\in I$, using the fact that $x-*(x)\in I\cap S_{a*}R$ and hence $[d(x-*(x)),t]=0$ or $d^2(x-*(x))=0$. If $d^2(x-*(x))=0$, then $d^2(x)=*(d^2(x))$ and hence by in view of Lemma 2.1 either $[d(x),t]=0$ or $d^2(x)=0$. If $d^2(x-*(x))=0$ and using the fact that $x+*(x)\in I\cap S_{a*}$, then $[d(x+*(x)),t]=0$ or $d^2(x+*(x))=0$. If $[d(x+*(x)),t]=0$, then $2[d(x),t]=0$. Since R is 2-torsion free so that $[d(x),t]=0$ for all $t\in I$. If $d^2(x+*(x))=0$, then $d^2(x)=-*(d^2(x))$ and hence again by Lemma 2.1 either $[d(x),t]=0$ for all $t\in I$ or $d^2(x)=0$. Consequently, for all $x\in I$ we find that either $[d(x),t]=0$ or $d^2(x)=0$. This means that I is the union of two its additive subgroups $U=\{x\in I\mid [d(x),t]=0 \ \mathit{for\ all}\ t\in I\}$ and $V=\{x\in I\mid d^2(x)=0\}$. Since a group cannot be the union of its two proper subgroups and hence either $I=U$ or $I=V$. If $I=V$, then $d^2(x)=0$ for all $x\in I$ and hence by Lemma 2.2, we get a contradiction. On the other hand if $I=U$, then $[d(x),t]=0$ for all $x,t\in I$ and in particular $[d(x),x]=0$ for all $x\in I$ and hence by Lemma 2.3, R is commutative.

(iv) Using the same techniques as used above with necessary variations we get the required result.

(v) If $F=0$, then we have $[x,y]=0$ for all $x,y\in I$ and hence using the same arguments as used in the proof of Theorem 1.1 of [14], we get the required result.

Therefore, we shall assume that $d\neq 0$. For any $x,y\in I$, we have $F(x\circ y)=[x,y]$. This can be rewritten as

$$F(x)y+xd(y)+F(y)x+yd(x)=[x,y]. \tag{11}$$

Replacing y by yx in (11), we get

$$(x\circ y)d(x)=0 \text{ for all } x,y\in I. \tag{12}$$

Now, replace y by zy in (12), to get $[x,z]yd(x)=0$ for all $x,y,z\in I$ and hence by Lemma 2.4 we get the required result.

(vi) Use similar arguments as above. □

Theorem 3.3. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Suppose that R admits a generalized derivation F associated with a nonzero derivation d commuting with $*$ such that*

(i) $[F(x),y]=(F(x)\circ y)$, *for all $x,y\in I$, or*

(ii) $[F(x),y]+(F(x)\circ y)=0$, *for all $x,y\in I$, or*

(iii) $F([x,y]) = [F(x), y]$, *for all* $x, y \in I$, *or*
(iv) $F([x,y]) + [F(x), y] = 0$, *for all* $x, y \in I$, *or*
(v) $F(x \circ y) = (F(x) \circ y)$, *for all* $x, y \in I$, *or*
(vi) $F(x \circ y) + (F(x) \circ y) = 0$, *for all* $x, y \in I$, *or*
(vii) $F(x^2) = x^2$, *for all* $x \in I$, *or*
(viii) $F(x^2) + x^2 = 0$, *for all* $x \in I$, *or*
(ix) $F([x,y]) = [F(x), y] + [d(y), x]$ *for all* $x, y \in I$, *or*
(x) $F(x \circ y) = F(x) \circ y - d(y) \circ x$ *for all* $x, y \in I$.

Then R *is commutative.*

Proof. (i) We have

$$[F(x), y] = (F(x) \circ y) \text{ for all } x, y \in I. \tag{13}$$

Replacing y by yx in (13) and using (13), we get $y[F(x), x] = 0$ for all $x, y \in I$ that is, $I[F(x), x] = \{0\}$ and hence

$$[F(x), x] = 0 \text{ for all } x \in I. \tag{14}$$

Linearizing (14), we get $[F(x), y] + [F(y), x] = 0$ for all $x, y \in I$. Now, replacing y by yx we find that $y[d(x), x] + [y, x]d(x) = 0$. Again replacing y by zy and simplifying we arrive at $[z, x]yd(x) = 0$ for all $x, y, z \in I$. Notice that the arguments given in the proof of the last paragraph of Theorem 3.2 are still valid in the present situation and hence repeating the same process we get the required result.

(ii) Using similar techniques with necessary variations, we get the required result.

(iii) We have $F([x, y] = [F(x), y]$ for all $x \in I$. This can be rewritten as

$$F(x)y + xd(y) - F(y)x - yd(x) = [F(x), y]. \tag{15}$$

Replacing y by yx in (15), we get $[x, y]d(x) = y[F(x), x]$ for all $x, y \in I$. Again replace y by zy, we find that $[x, z]yd(x) = 0$. The last expression is same as equation (6) and hence the result follows.

(iv) Using the same techniques as (iii) with necessary variations, we get the required result.

(v) We have $F(x \circ y) = (F(x) \circ y)$ for all $x, y \in I$. This can be rewritten as

$$F(x)y + xd(y) + F(y)x + yd(x) = (F(x) \circ y). \tag{16}$$

Replacing y by yx in (16), we get $(x \circ y)d(x) = -y[F(x), x]$ for all $x, y \in I$. Again replace y by zy in the above expression we find that $[x, z]yd(x) = 0$ for all $x, y, z \in I$ and hence use the arguments as used in the last paragraph of Theorem 3.2 (v), we get the required result.

(vi) Using similar arguments as above.

(vii) We have

$$F(x^2) = x^2 \text{ for all } x \in I.$$

Replacing x by $x + y$ in the above relation, we get

$$F(x^2 + y^2 + xy + yx) = x^2 + y^2 + xy + yx \text{ for all } x, y \in I. \tag{17}$$

Using the given hypothesis in (17), we obtain $F(xy+yx) = xy+yx$ for all $x, y \in I$. This can be written as $F(x \circ y) - x \circ y = 0$ for all $x, y \in I$ and hence by Theorem 2.2 of [17], we get the required result.

$(viii)$ Using similar arguments as above.

(ix) For all $x, y \in I$, we have

$$F([x,y]) = [F(x), y] + [d(y), x]. \tag{18}$$

Replacing y by yx in (18) and using (18), we find that

$$2[x,y]d(x) = y[F(x), x] + y[d(x), x] \quad \text{for all } x, y \in I. \tag{19}$$

Now replace y by yz in (19), to get $2[x,y]zd(x) = 0$ for all $x, y, z \in I$. Since R is 2-torsion free, we get $[x,y]zd(x) = 0$ for all $x, y, z \in I$. Thus, result follows from Lemma 2.4.

(x) We have

$$F(x \circ y) = F(x) \circ y - d(y) \circ x \quad \text{for all } x, y \in I. \tag{20}$$

Replacing y by yx in (20) and using (20), we find that

$$(x \circ y)d(x) = -y[F(x), x] - y(d(x) \circ x) + [y, x]d(x). \tag{21}$$

Replace y by zy in (21) and use (21), to get $2[x,z]yd(x) = 0$ for all $x, y, z \in I$. Since R is 2-torsion free , we get $[x,z]yd(x) = 0$ for all $x, y, z \in I$. Hence we get the required result by Lemma 2.4. □

Theorem 3.4. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Suppose R admits a pair of generalized derivations F and G associated with derivation d and nonzero derivation g respectively commuting with $*$ such that*

(i) $F(x)x = xG(x)$, *for all $x, y \in I$, or*
(ii) $F(x)x + xG(x) = 0$, *for all $x, y \in I$, or*
(iii) $[F(x), y] = [x, G(y)]$, *for all $x \in I$, or*
(iv) $[F(x), y] + [x, G(y)] = 0$, *for all $x \in I$.*

Then R is commutative.

Proof. (i) By hypothesis, we have

$$F(x)x = xG(x) \quad \text{for all } x \in I.$$

On linearizing the above relation we find that

$$F(x)y + F(y)x = xG(y) + yG(x) \text{ for all } x, y \in I. \tag{22}$$

Replace x by xy in (22), to get

$$F(x)y^2 + xd(y)y + F(y)xy = xyG(y) + yG(x)y + yxg(y). \tag{23}$$

Right multiplication by y to the relation (22) yields that

$$F(x)y^2 + F(y)xy = xG(y)y + yG(x)y \text{ for all } x, y \in I. \tag{24}$$

Combining (23) and (24), we obtain

(25) $$xd(y)y = yxg(y) + x[y, G(y)] \text{ for all } x, y \in I.$$

Now, replacing x by zx in (25), we get

(26) $$zxd(y)y = yzxg(y) + zx[y, G(y)] \text{ for all } x, y, z \in I.$$

Left multiplying to (25) by z, we arrive at

(27) $$zxd(y)y = zyxg(y) + zx[y, G(y)] \text{ for all } x, y, z \in I.$$

From (26) and (27), we get $[y, z]xg(y) = 0$ and by Lemma 2.4 we get the required result.

(*ii*) If $F(x)x + xG(x) = 0$ for all $x \in I$, then using the same techniques as used above with necessary variations we get the required result. This completes the proof of our theorem.

(*iii*) We have

$$[F(x), y] = [x, G(y)] \text{ for all } x, y \in I.$$

Replacing y by yx in the above expression, we obtain

(28) $$y[F(x), x] = [x, y]g(x) + y[x, g(x)] \text{ for all } x, y \in I.$$

Again replace y by zy in (28), to get $[x, z]yg(x) = 0$ for all $x, y, z \in I$. Hence, we get the required result by Lemma 2.4.

(*iv*) Further, if $[F(x), y] + [x, G(y)] = 0$ for all $x, y \in I$, then using the same techniques as used above with necessary variations we get the required result. □

The following is an immediate corollary of the above theorem.

Corollary 3.5. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Let d and g be derivations of R such that at least one of them is nonzero and commuting with $*$. If $d(x)x = xg(x)$ for all $x \in I$, then R is commutative.*

Proceeding on the same lines with necessary variations and taking $G = F$ or $G = -F$ in Theorem 3.4 (*i*) and (*ii*), we get the following:

Corollary 3.6. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with $*$ such that $[F(x), x] = 0$ for all $x \in I$ or if $F(x) \circ x = 0$ for all $x \in I$. Then R is commutative.*

Proceeding on the same lines with necessary variations and taking $G = F$ or $G = -F$ in Theorem 3.4 (*iii*) and (*iv*), one can prove the following result.

Corollary 3.7. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with $*$ such that*

(i) $[F(x), y] = [x, F(y)]$, *for all $x \in I$, or*

(ii) $[F(x), y] + [x, F(y)] = 0$, *for all $x \in I$.*

Then R is commutative.

Example 3.8. Let S be any ring and let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}$. Define $* : R \to R$ by $*\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$. and $d : R \to R$ by $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $F : R \to R$ by $F\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then R is a ring under usual operations, I is a $*$-ideal, and it is easy to see that d is a derivation of R and F is a generalized derivation of R, and d is commuting with $*$ such that satisfying any one of the following properties: (i) $F([x,y]) = (x \circ y)$, (ii) $[d(x), F(y)] = [x,y]$, (iii) $F(x \circ y) = [x,y]$, (iv) $[F(x), y] = (F(x) \circ y)$, (v) $F([x,y]) = [F(x), y]$, (vi) $F(x \circ y) = (F(x) \circ y)$, (vii) $F(x^2) = x^2$, $(viii)$ $F([x,y]) = [F(x), y] + [d(y), x]$, (ix) $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in I$, but R is not commutative. Hence, in Theorem 3.2 and Theorem 3.3, the hypothesis of primeness cannot be omitted.

Example 3.9. Let $\mathbb{Z}$ be the ring of integers in which $a^2 = 0$ for all $a \in \mathbb{Z}$ and let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and $I = \left\{ \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$. Define $* : R \to R$ by $*\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ -b & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in R$. and $d : R \to R$ by $d\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, $g : R \to R$ by $g\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a-b & 0 \end{pmatrix}$, and $F : R \to R$ by $F\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$, $G : R \to R$ by $G\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then R is a ring under usual operations, I is a $*$-ideal, and it is easy to see that d and g are derivations of R and F and G are generalized derivations of R, and d is commuting with $*$ such that satisfying any one of the following properties: (i) $F(x)x = xG(x)$, (ii) $F(x)x + xG(x) = 0$, (iii) $[F(x), y] = [x, G(y)]$, and (iv) $[F(x), y] + [x, G(y)] = 0$ for all $x, y \in I$, but R is not commutative. Hence, in Theorem 3.4, the hypothesis of primeness cannot be omitted.

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