MATHEMATICA, Tome 53 (76), N° 2, 2011, pp. 157–164

APPLICATION OF A THREE CRITICAL POINTS THEOREM FOR A CLASS OF INCLUSION PROBLEMS

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Abstract. In this paper we prove the existence of at least three solutions for a differential inclusion problem involving the p-Laplacian with nonhomogeneous and nonsmooth Neumann boundary conditions. We use a three critical points theorem for locally Lipschitz functions given by A. Kristály, W. Marzantowicz, Cs. Varga [7].

MSC 2010. 34A60, 47J20, 49J52.

Key words. Nonhomogeneous Neumann boundary condition, elliptic differential inclusions, critical points.

1. INTRODUCTION

In this paper we prove *the existence of at least three solutions* for the following inclusion problem with nonhomogeneous and nonsmooth Neumann boundary conditions: For $\lambda > 0$, $\mu \geq 0$ we consider the problem $(P_{\lambda,\mu})$ of finding $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) \in \lambda \partial F(x, u(x)) \text{ a.e. } x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} \in \mu \partial G(x, u(x))) \text{ a.e. } x \in \Gamma, \end{cases}$$

where $\dfrac{\partial}{\partial n}$ denotes the unit outward normal on $\Gamma := \partial\Omega$, $\partial F(x,\eta)$, $\partial G(x,\eta)$, denote the generalized gradient (in the sense of Clarke) of $F(x,\cdot)$ and $G(x,\cdot)$ at $\eta \in \mathbb{R}$.

The main tool used in this paper is a multiplicity theorem of A. Kristály, W. Marzantowicz, Cs. Varga [7] for locally Lipschitz functions, which is an extension of a three critical points theorem of B. Ricceri [12]. For this, we impose conditions on the behavior around zero and close to infinity for the locally Lipschitz function F and only a growth condition for the locally Lipschitz perturbation G.

Different types of differential inclusion problems involving the p-Laplacian were studied by S. Carl and S. Heikkilä [1], G. Dai and W. Liu [5] (with homogeneous Dirichlet boundary conditions), G. Dai [4], [3] (with zero Neumann-type conditions), A. Kristály [6] (on the whole space $\mathbb{R}^N$). Such investigations

Both authors were supported in their research by MEdC-ANCS, Grant PN II ID PCE 2008 Nr. 501/ ID 2162/ 2009.

mainly use non-smooth Ricceri-type variational principles developed by S. Marano and D. Motreanu [10], [9].

The present paper completes the results of H. Lisei and Cs. Varga [8], where the authors used a mountain pass theorem with Cerami type conditions for locally Lipschitz functions to prove *the existence of at least one nontrivial solution* for a differential inclusion problem of the type $(P_{\lambda,\mu})$.

The paper has the following structure: Section 2 contains results from critical point theory for locally Lipschitz functions. In Section 3 the assumptions for our problem are given and the main result is formulated. In Section 4 auxiliary results are detailed and the proof of the main result (Theorem 7) of the paper is given.

2. PRELIMINARIES – BASIC NOTIONS AND RESULTS OF NONSMOOTH CALCULUS

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. Let $f : X \to \mathbb{R}$ be a *locally Lipschitz* function, i.e. each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|f(u_1) - f(u_2)| \leq L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$.

The *generalized directional derivative* of f at the point $u \in X$ in the direction $z \in X$ is

$$f^\circ(u; z) = \limsup_{w \to u, s \to 0^+} \frac{f(w + sz) - f(w)}{s}.$$

The *generalized gradient* (in the sense of Clarke [2]) of f at $u \in X$ is defined by $\partial f(u) = \{x^* \in X^* : \langle x^*, x\rangle \leq f^\circ(u; x),\ \forall x \in X\}$, where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. We say that $u \in X$ is a *critical point* of f, if $0 \in \partial f(u)$.

Theorem 1. [2, Proposition 2.3.3], [2, Theorem 2.3.7] *Let $f, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then the following assertions hold:*

(a) $(f + g)^\circ(x; y) \leq f^\circ(x; y) + g^\circ(x; y)$, *for every $x, y \in X$.*
(b) (Lebourg's Mean Value Theorem) *For every $x, y \in X$ there exist an element u on the open line segment joining x and y, and $z \in \partial f(u)$ such that $f(y) - f(x) = \langle z, y - x\rangle$.*

Definition 2. The locally Lipschitz function $f : X \to \mathbb{R}$ is said to *satisfy the Palais-Smale condition at level $c \in \mathbb{R}$* shortly, $(PS)_c$), if every sequence $\{u_n\}$ in X satisfying $f(u_n) \to c$ and $f^\circ(u_n; v - u_n) + \varepsilon_n\|v - u_n\| \geq 0, \forall v \in X$, for a sequence $\{\varepsilon_n\} \subset [0, \infty[$ with $\varepsilon_n \to 0$, contains a convergent subsequence.

For every $\tau \geq 0$, we introduce the following class of functions:

$(\boldsymbol{\mathcal{G}_\tau})$: $g \in C^1(\mathbb{R}, \mathbb{R})$ is bounded, and $g(t) = t$ for any $t \in [-\tau, \tau]$.

The main tool we use in our paper is the following multiplicity theorem of A. Kristály, W. Marzantowicz, Cs. Varga [7] for locally Lipschitz functions.

THEOREM 3. ([7, Theorem 3.3.1]) *Let* $(X, \|\cdot\|)$ *be a real reflexive Banach space and* $\tilde{X}_i$ $(i = 1, 2)$ *be two Banach spaces such that the embeddings* $X \hookrightarrow \tilde{X}_i$ *are compact. Let* Λ *be a real interval,* $h : X \to [0, \infty)$ *be a convex and continuous function, and let* $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ *be two locally Lipschitz functions such that* $E_{\lambda,\mu} = h + \lambda\Phi_1 + \mu g \circ \Phi_2$ *restricted to* X *satisfies the* $(PS)_c$*-condition for every* $c \in \mathbb{R}$, $\lambda \in \Lambda$, $\mu \in [0, |\lambda| + 1]$ *and* $g \in \mathcal{G}_\tau$, $\tau \geq 0$. *Assume that* $h+\lambda\Phi_1$ *is coercive on* X *for all* $\lambda \in \Lambda$ *and that there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(x) + \lambda(\Phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(x) + \lambda(\Phi_1(x) + \rho)].$$

Then, there exist a non-empty open set $A \subset \Lambda$ *and* $\sigma > 0$ (*both independent of the perturbation* Φ_2) *with the property that for every* $\lambda \in A$ *there exists* $\mu_0 \in]0, |\lambda| + 1]$ *such that, for each* $\mu \in [0, \mu_0]$ *the functional* $h + \lambda\Phi_1 + \mu\Phi_2$ *has at least three critical points in* X *whose norms are less than* σ.

3. ASSUMPTIONS AND THE MAIN THEOREM

Let $\Omega \subset \mathbb{R}^N, N > 1$, be a bounded domain with smooth boundary Γ, let $p \in (1, N)$, $p^\star = \dfrac{Np}{N-p}, \bar{p}^\star = \dfrac{(N-1)p}{N-p}$, and we denote by $W = W^{1,p}(\Omega)$ the space endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x\right)^{1/p}.$$

Assumptions on the function F*:*

(F1) Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, such that $F(x, \cdot)$ is regular and locally Lipschitz for all $x \in \Omega$, and $F(x, 0) = 0$ for all $x \in \Omega$ and there exist $k_1 > 0$ and $r \in (p, p^*)$ such that

$$|\xi| \leq k_1(1 + |u|^{r-1}) \text{ for all } u \in \mathbb{R}, \xi \in \partial F(x, u), \text{ a.e. } x \in \Omega;$$

(F2) uniformly for a.e. $x \in \Omega$ we have $\displaystyle\lim_{|u|\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x, u)\}}{|u|^{p-1}} = 0$;

(F3) $\displaystyle\limsup_{|u|\to\infty} \frac{\operatorname{esssup}_{x\in\Omega} F(x, u)}{|u|^p} \leq 0$, $\displaystyle\max_{|u|\leq M} F(\cdot, u) \in L^1(\Omega)$ for all $M > 0$;

(F4) there exists $u_0 \in \mathbb{R}$ such that $F(x, u_0) > 0$ for a.e. $x \in \Omega$.

Assumptions on the function G*:*

(G1) Let $G : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, such that $G(x, \cdot)$ is regular and locally Lipschitz for all $x \in \Gamma$, $G(\cdot, u)$ is measurable with respect to the $(N-1)$-dimensional Hausdorff measure on Γ for every $u \in \mathbb{R}$ and there exist $k_2 > 0$ and $\bar{r} \in [p, \bar{p}^*)$ such that

$$|\xi| \leq k_2(1 + |u|^{\bar{r}-1}) \text{ for all } u \in \mathbb{R}, \xi \in \partial G(x, u), \text{ a.e. } x \in \Gamma.$$

EXAMPLE 4. Let $N = 2$, $p \in (1,2)$, $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 : \|(x_1,x_2)\|_{\mathbb{R}^2} < 1\}$ and $\Gamma := \{(x_1,x_2) \in \mathbb{R}^2 : \|(x_1,x_2)\|_{\mathbb{R}^2} = 1\}$. We define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \Gamma \times \mathbb{R} \to \mathbb{R}$ by

$$F(x_1,x_2,u) = |x_1| \cdot |x_2| \cdot \max\{0, \ln(|u| + 1/2)\},$$
$$G(x_1,x_2,u) = (|x_1| - |x_2|) \cdot |u|^p - |u|.$$

The function F satisfies the assumptions (F1), (F2), (F3) and (F4) and G satisfies (G1).

We define $J, \mathcal{F}, \mathcal{G}, \mathcal{E}_{\lambda,\mu} : W \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x, \quad \mathcal{F}(u) = -\int_\Omega F(x,u(x))\mathrm{d}x,$$
$$\mathcal{G}(u) = -\int_\Gamma G(x,u(x))\mathrm{d}\Gamma, \quad \mathcal{E}_{\lambda,\mu}(u) = J(u) + \lambda\mathcal{F}(u) + \mu\mathcal{G}(u).$$

DEFINITION 5. We say that $u \in W$ is a *weak solution* of problem $(P_{\lambda,\mu})$, if it is a critical point of $\mathcal{E}_{\lambda,\mu}$, i.e. $0 \in \partial\mathcal{E}_{\lambda,\mu}(u)$.

Note, that $\mathcal{E}_{\lambda,\mu} = J + \lambda\mathcal{F} + \mu\mathcal{G}$ is a locally Lipschitz function (by (F1) and (G1) and Theorem 1.3 in [11]) and for $u \in W$ we have by Theorem 1 that the generalized directional derivative of $\mathcal{E}$ at the point $u \in W$ in the direction $w \in W$ is

$$\mathcal{E}^\circ_{\lambda,\mu}(u;w) \le \langle J'(u), w\rangle + \lambda\mathcal{F}^\circ(u;w) + \mu\mathcal{G}^\circ(u;w),$$

where

$$\langle J'(u), w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w \mathrm{d}x + \int_\Omega |u|^{p-2} u w \mathrm{d}x.$$

REMARK 6. Assume that the assumptions (F1) and (G1) hold. Let $u \in W$ be a critical point of $\mathcal{E}_{\lambda,\mu}$. From the definition of $\mathcal{E}_{\lambda,\mu}$, by Corollary 2 in Section 2.3 and Theorem 2.7.5 in [2] we have $\partial\mathcal{E}_{\lambda,\mu}(u) \subset \{J'(u)\} + \lambda\partial\mathcal{F}(u) + \mu\partial\mathcal{G}(u)$ and there exist $\eta(x) \in \partial F(x,u(x))$ for a.e. $x \in \Omega$, $\theta(x) \in \partial G(x,u(x))$ for a.e. $x \in \Gamma$ such that the following equality holds

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla y \mathrm{d}x + \int_\Omega |u|^{p-2} u y \mathrm{d}x = \lambda \int_\Omega \eta y \mathrm{d}x + \mu \int_\Gamma \theta y \mathrm{d}\Gamma \ \forall y \in W.$$

In some papers the function $u \in W$ satisfying the above equality is considered to be the definition of a weak solution of problem $(P_{\lambda,\mu})$.

The *main result* of our paper is the following theorem, in which we prove the existence and multiplicity of the weak solutions of $(P_{\lambda,\mu})$.

THEOREM 7. *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}, G : \Gamma \times \mathbb{R} \to \mathbb{R}$ be functions satisfying the conditions* **(F1)**, **(F2)**, **(F3)**, **(F4)** *and* **(G1)**. *Then, there exist a non-degenerate compact interval $[a,b] \subset (0,\infty)$ and a number $\sigma > 0$ (both independent of the perturbation G), such that for every $\lambda \in [a,b]$ there exists $\mu_0 \in (0, \lambda + 1]$ such that for each $\mu \in [0,\mu_0]$ problem $(P_{\lambda,\mu})$ has at least three distinct weak solutions with W-norms less than σ.*

4. PROOF OF THE MAIN THEOREM

It is well known that for $s \in [1, p^\star), \nu \in [p, \bar{p}^\star)$ the following embeddings

$$W \hookrightarrow L^s(\Omega) \text{ and } W \hookrightarrow L^\nu(\Gamma) \tag{1}$$

are compact. By (1) it follows that there exist $c_s, \hat{c}_\nu > 0$ such that

$$|u|_{L^s(\Omega)} \le c_s \|u\| \text{ and } |u|_{L^\nu(\Gamma)} \le \hat{c}_\nu \|u\| \ \forall\ u \in W.$$

Lemma 8. *It holds:*

$$\lim_{t \to 0^+} \frac{1}{t} \inf \left\{ \mathcal{F}(u) : \ u \in W, \ \frac{1}{p}\|u\|^p < t \right\} = 0.$$

Proof. We apply Lebourg's Mean Value Theorem (see Theorem 1) and the assumption **(F1)** for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$

$$|F(x,u)| = |F(x,u) - F(x,0)| \le k_1 \left(|u| + |u|^r\right).$$

By Lebourg's Mean Value Theorem and by **(F2)** it follows that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $u \in \mathbb{R}$ with $|u| < \delta_\varepsilon$, and a.e. $x \in \Omega$ it holds

$$|F(x,u)| \le \varepsilon |u|^p.$$

Finally, we have that there exist $K_{1\varepsilon} > 0$ such that for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$

$$|F(x,u)| \le \varepsilon |u|^p + K_{1\varepsilon}|u|^r, \tag{2}$$

where $r \in (p, \bar{p}^*)$. Taking into account (2) and the continuous embeddings $W \hookrightarrow L^s(\Omega)$, for $s = p$ and $s = r$ respectively, we have for $u \in W$

$$\mathcal{F}(u) \ge -\varepsilon c_p^p \|u\|^p - K_{1\varepsilon} c_r^r \|u\|^r.$$

Then, we have

$$0 \ge \frac{1}{t} \inf \left\{ \mathcal{F}(u) : \ u \in W, \ \frac{1}{p}\|u\|^p < t \right\} \ge -p c_p^p \varepsilon - p^{r/p} K_{1\varepsilon} c_r^r t^{\frac{r}{p}-1}.$$

Since $p < r$, $\varepsilon > 0$ is arbitrary, we get the desired limit for $t \to 0^+$. □

Lemma 9. *There exists $\rho_0 > 0$ such that the function $\varphi : W \times [0,\infty) \to \mathbb{R}$ defined by*

$$\varphi(u, v, \lambda) = J(u) + \lambda \mathcal{F}(u) + \lambda \rho_0,$$

satisfies the inequality

$$\sup_{\lambda \ge 0} \inf_{u \in W} \varphi(u, \lambda) < \inf_{u \in W} \sup_{\lambda \ge 0} \varphi(u, \lambda). \tag{3}$$

Proof. For every $t > 0$ we define

$$\beta(t) = \inf \{\mathcal{F}(u) : \ u \in W, \ J(u) < t\}.$$

We have that $\beta(t) \le 0$, for $t > 0$, and by Lemma 8 it follows

$$\lim_{t \to 0^+} \frac{\beta(t)}{t} = 0. \tag{4}$$

By **(F4)** immediately follows that $u_0 \in W\setminus\{0\}$ (the constant function) satisfies $\mathcal{F}(u_0) < 0$. Therefore it is possible to choose a number $\eta > 0$ such that $0 < \eta < -\mathcal{F}(u_0)\,(J(u_0))^{-1}$. By (4) we get the existence of a number $t_0 \in (0, J(u_0))$ such that $-\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) > \mathcal{F}(u_0)\,(J(u_0))^{-1}\,t_0. \tag{5}$$

Due to the choice of t_0 and using (5), we conclude that there exists $\rho_0 > 0$ such that

$$-\beta(t_0) < \rho_0 < -\mathcal{F}(u_0)\,(J(u_0))^{-1}\,t_0 < -\mathcal{F}(u_0). \tag{6}$$

Now, we prove that the function φ satisfies the inequality (3). The function

$$\lambda \in [0,\infty) \mapsto \inf_{u\in W} \{J(u) + \lambda(\rho_0 + \mathcal{F}(u))\}$$

is obviously upper semicontinuous on $[0,\infty)$. It follows from (6) that

$$\lim_{\lambda\to\infty} \inf_{u\in W} \varphi(u,\lambda) \le \lim_{\lambda\to\infty} \{J(u_0) + \lambda(\rho_0 + \mathcal{F}(u_0))\} = -\infty.$$

Thus we find an element $\overline{\lambda} \in [0,\infty)$ such that

$$\sup_{\lambda\ge 0} \inf_{u\in W} \varphi(u,\lambda) = \inf_{u\in W} \left\{J(u) + \overline{\lambda}(\rho_0 + \mathcal{F}(u))\right\}. \tag{7}$$

Since $-\beta(t_0) < \rho_0$, it follows from the definition of β that for all $u \in W$ with $J(u) < t_0$ we have $-\mathcal{F}(u) < \rho_0$. Hence

$$t_0 \le \inf\{J(u) : u \in W,\ -\mathcal{F}(u) \ge \rho_0\}. \tag{8}$$

On the other hand,

$$\begin{aligned}
\inf_{u\in W} \sup_{\lambda\ge 0} \varphi(u,\lambda) &= \inf_{u\in W} \left\{ J(u) + \sup_{\lambda\ge 0} (\lambda(\rho_0 + \mathcal{F}(u))) \right\} \\
&= \inf\{J(u) : u \in W,\ -\mathcal{F}(u) \ge \rho_0\}.
\end{aligned}$$

Thus inequality (8) is equivalent to

$$t_0 \le \inf_{u\in W} \sup_{\lambda\ge 0} \varphi(u,\lambda). \tag{9}$$

We consider two cases: First, when $0 \le \overline{\lambda} < \frac{t_0}{\rho_0}$, then we have that

$$\inf_{u\in W} \left\{J(u) + \overline{\lambda}(\rho_0 + \mathcal{F}(u))\right\} \le \varphi(0,\overline{\lambda}) = \overline{\lambda}\rho_0 < t_0.$$

Combining this inequality with (7) and (9) we obtain (3).

Second, if $\frac{t_0}{\rho_0} \le \overline{\lambda}$, then from (6), it follows that

$$\begin{aligned}
&\inf_{u\in W} \left\{J(u) + \overline{\lambda}(\rho_0 + \mathcal{F}(u))\right\} \le J(u_0) + \overline{\lambda}(\rho_0 + \mathcal{F}(u_0)) \\
&\le J(u_0) + \frac{t_0}{\rho_0}(\rho_0 + \mathcal{F}(u_0)) < t_0.
\end{aligned}$$

We apply again (7) and (9), which implies (3). □

We fix $g \in \mathcal{G}_\tau$ ($\tau \geq 0$), $\lambda \in [0, \infty)$, $\mu \in [0, \lambda + 1]$, and $c \in \mathbb{R}$ and define the functional $E_{\lambda,\mu} : W \to \mathbb{R}$ by

$$E_{\lambda,\mu}(u) = J(u) + \lambda\mathcal{F}(u) + \mu(g \circ \mathcal{G})(u), \quad u \in W.$$

LEMMA 10. *The functional $E_{\lambda,\mu}$ is coercive on W and satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Observe that $u \in W \mapsto J(u) + \lambda\mathcal{F}(u)$ is coercive on W, due to **(F3)**; thus, the functional $E_{\lambda,\mu}$ is also coercive on W. Consequently, it is enough to consider a bounded sequence $\{u_n\} \subset W$ such that

$$E^\circ_{\lambda,\mu}(u_n; w - u_n) \geq -\varepsilon_n \|w - u_n\| \quad \text{for all} \quad w \in W, \tag{10}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n \to 0$.

Due to Theorem 1, we have for every $\bar{w}, w \in W$ that

$$E^\circ_{\lambda,\mu}(\bar{w}; w) \leq \langle J'(\bar{w}), w\rangle + \lambda\mathcal{F}^\circ(\bar{w}; w) + \mu(g \circ \mathcal{G})^\circ(\bar{w}; w). \tag{11}$$

Because the sequence $\{u_n\}$ is bounded, there exists $u \in W$ and a subsequence which we denote also by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in W, $u_n \to u$ strongly in $L^p(\Omega)$ and in $L^p(\Gamma)$ (since the embeddings $W \hookrightarrow L^p(\Omega)$ and $W \hookrightarrow L^p(\Gamma)$ are compact).

Using (10) for $w := u$ and applying relation (11) for the pairs $\bar{w} := u_n, w := u - u_n$ and $\bar{w} := u, w := u_n - u$, we have

$$\begin{aligned} \|u - u_n\|_p^p \leq\ & \varepsilon_n \|u - u_n\| - E^\circ_{\lambda,\mu}(u; u_n - u) \\ & + \lambda\mathcal{F}^\circ(u_n; u - u_n) + \lambda\mathcal{F}^\circ(u; u_n - u) \\ & + \mu(g \circ \mathcal{G})^\circ(u_n; u - u_n) + \mu(g \circ \mathcal{G})^\circ(u; u_n - u). \end{aligned} \tag{12}$$

Since $\{u_n\}$ is bounded in W, obviously we have $\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0$. Now, fix $w^* \in \partial E_{\lambda,\mu}(u)$; in particular, we have $\langle w^*, u_n - u\rangle \leq E^\circ_{\lambda,\mu}(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in W, it follows

$$\liminf_{n\to\infty} E^\circ_{\lambda,\mu}(u; u_n - u) \geq 0.$$

For the remaining four terms in the estimation (12) we use the fact that $\mathcal{F}^\circ(\cdot;\cdot)$ and $(g \circ \mathcal{G})^\circ(\cdot;\cdot)$ are upper semicontinuous functions (see Proposition 2.1.1 in [2]). Since $u_n \to u$ strongly in $L^p(\Omega)$, we have

$$\limsup_{n\to\infty} \mathcal{F}^\circ(u_n; u - u_n) \leq \mathcal{F}^\circ(u; 0) = 0;$$

the remaining terms in (12) are treated similarly (we use that $u_n \to u$ strongly in $L^p(\Gamma)$). Combining the above outcomes, we obtain $\limsup\limits_{n\to\infty} \|u - u_n\|^p \leq 0$, i.e., $u_n \to u$ strongly in W. □

Proof of Theorem 7. We apply Theorem 3 by choosing $X = W$, $\tilde{X}_1 = L^p(\Omega), \tilde{X}_2 = L^p(\Gamma)$, $\Lambda = [0, \infty)$, and consider $h : W \to [0, \infty)$ to be the following convex and continuous function $h(u) = \frac{\|u\|^p}{p}$. The hypotheses of Theorem 3 hold due to Lemma 9 and Lemma 10. □

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Received October 9, 2009
Accepted December 3, 2009

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