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CERTAIN FAMILIES OF ANALYTIC UNIVALENT FUNCTIONS GENERATED BY HARMONIC UNIVALENT MAPPINGS

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Abstract. In the present paper we obtain some inclusion theorems and convolution characterizations for the classes of analytic univalent functions generated by harmonic univalent and sense-preserving mappings.

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1. INTRODUCTION

A continuous function f is said to be a complex-valued harmonic function in a simply connected domain D in complex plane $\mathbb{C}$ if both real and imaginary parts of f are real harmonic in D. Such functions can be expressed as $f = h+\bar{g}$, where h, g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D (see [6]).

Every harmonic function $f = h+\bar{g}$ is uniquely determined by the coefficients of power series expansions in the unit disk $U = \{z : |z| < 1\}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in U, |B_1| < 1, \tag{1.1}$$

where $A_n \in \mathbb{C}$ for $n = 2, 3, 4, \ldots$ and $B_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$. For further information about these mappings, one may refer to [4, 6, 7].

In 1984, Clunie and Sheil-Small [6] studied the family S_H of all univalent sense-preserving harmonic functions f of the form $f = h + \bar{g}$ in U, such that h and g are represented by (1.1). Note that S_H reduces to the well-known family S, the class of all normalized analytic univalent functions h given in (1.1), whenever the co-analytic part g of f is zero. Let K, K_H denote the respective subclasses of S, S_H where the images of $f(U)$ are convex.

In the last two decades, several researchers have defined various subclasses of S using subordination. A function h is said to be subordinate to F if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

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$h(z) = F(w(z))$ for all z in U. Using subordination, we define two subclasses of S as follows:

$$S^*[A,B,\alpha] = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}, z \in U \right\},$$

$$K[A,B,\alpha] = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}, z \in U \right\},$$

where $0 \le \alpha < 1, -1 \le B < B + (A-B)(1-\alpha) < A \le 1$. The condition $|B| \le 1$ implies that the function $[1 + (B + (A-B)(1-\alpha))z][1+Bz]^{-1}$ is convex and univalent in U. For different values of parameters A, B and α one can obtain several subclasses of S. For information about properties and subclasses of S, we refer to the survey article by the second author in [1].

Note that the convex domains are those domains that are convex in every direction. The following lemma will motivate us to construct certain analytic univalent function associated with $f \in S_H$.

Lemma 1.1 ([5, 6]). *A harmonic function $f = h + \bar{g}$ locally univalent in U is a univalent mapping of U and $f \in K_H$ if and only if $h - g$ is an analytic univalent mapping of U onto a domain convex in the direction of the real axis.*

For $f = h + \bar{g}$ in S_H, where h and g are given by (1.1), Lemma 1.1 led us to construct the function t with suitable normalization, given by

$$t(z) = \frac{h(z) - g(z)}{1 - B_1} = z + \sum_{n=2}^{\infty} \frac{A_n - B_n}{1 - B_1} z^n, \quad z \in U. \tag{1.2}$$

Since $f \in S_H$ is sense-preserving, it follows that $|B_1| < 1$. Hence the function t belongs to S. This observation has prompted us to define the following classes:

$$S_H[A,B,\alpha] := \{ f = h + \bar{g} \in S_H : t \in S^*[A,B,\alpha] \},$$

$$K_H[A,B,\alpha] := \{ f = h + \bar{g} \in S_H : t \in K[A,B,\alpha] \}.$$

In [2], the second author connected hypergeometric functions with harmonic mappings $f = h + \bar{g}$ by defining the convolution operator Ω by $\Omega(f) := f \tilde{*} (\phi_1 + \bar{\phi}_2) = h * \phi_1 + \overline{g * \phi_2}$, where $*$ denotes the convolution product of two power series and ϕ_1, ϕ_2 are defined by

$$\phi_1(z) = zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n,$$

$$\phi_2(z) = zF(a_2, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} z^n.$$

Here $F(a,b;c;z)$ is a well-known hypergeometric function and a's, b's, c's are complex parameters with $c \neq 0, -1, -2, \ldots$. Corresponding to any function

$f = h + \bar{g}$ given by (1.1), we have $\Omega(f) = H + \bar{G}$, where

$$(1.3)\ \ H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n,\ G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n,$$

$|B_1| < 1$. We shall frequently use the well-known Gauss summation formula

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},\ \mathrm{Re}(c-a-b) > 0.$$

In the present paper, we study certain connections of the mappings $f = h+\bar{g}$ in S_H with the corresponding analytic functions in the classes $S^*[A, B, \alpha]$ and $K[A, B, \alpha]$. More precisely, we obtain some inclusion theorems and convolution characterization theorems for the classes $S^*_H[A, B, \alpha]$ and $K_H[A, B, \alpha]$.

2. LEMMAS

Lemma 2.1. *A function h defined by the first part of* (1.1) *is in* $S^*[A, B, \alpha]$ *if* $\sum_{n=2}^{\infty}\{(n-1)(1+|B|) + (A-B)(1-\alpha)\}|A_n| \leq (A-B)(1-\alpha)$.

Proof. In view of the definition of $S^*[A, B, \alpha]$, $h \in S^*[A, B, \alpha]$ if and only if there exists an analytic function w such that $\frac{zh'(z)}{h(z)} = \frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}$ with $w(0) = 0$ and $|w(z)| < |z|$. Since $|w(z)| < 1$, the above equation is equivalent to

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{[B + (A-B)(1-\alpha)] - B\frac{zh'(z)}{h(z)}} \right| < 1.$$

On the other hand, on $|z| = 1$ we have

$$|zh'(z) - h(z)| - |[B + (A-B)(1-\alpha)]h(z) - Bzh'(z)| = \left| \sum_{n=2}^{\infty} (n-1)A_n z^n \right|$$
$$- \left| (A-B)(1-\alpha)z - \sum_{n=2}^{\infty} [(n-1)B - (A-B)(1-\alpha)]A_n z^n \right|$$
$$\leq \sum_{n=2}^{\infty} [(n-1)(1+|B|) + (A-B)(1-\alpha)]|A_n| - (A-B)(1-\alpha) \leq 0,$$

provided the given condition holds. Hence by maximum modulus Theorem it follows that $h \in S^*[A, B, \alpha]$. □

Lemma 2.2. *A function h defined by the first part in* (1.1) *is in* $K[A, B, \alpha]$ *if* $\sum_{n=2}^{\infty} n\{(n-1)(1+|B|) + (A-B)(1-\alpha)\}|A_n| \leq (A-B)(1-\alpha)$.

Proof. From the definition of $K[A, B, \alpha]$ it follows that $h \in K[A, B, \alpha]$ if and only if there exists an analytic function w such that

$$\frac{(zh'(z))'}{h'(z)} = \frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1 + Bw(z)}$$

with $w(0) = 0$ and $|w(z)| < |z| < 1$. This equality is equivalent to

$$\left| \frac{\frac{(zh'(z))'}{h'(z)} - 1}{[B + (A - B)(1 - \alpha)] - B\frac{(zh'(z))'}{h'(z)}} \right| < 1.$$

The remaining steps of the proof are similar to the proof of Lemma 2.1. □

LEMMA 2.3 ([2]). *Let $f = h + \bar{g}$ where h and g are analytic functions of the form* (1.1). *If $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ and the hypergeometric inequalities*

(i) $\sum_{n=2}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n| \leq 1, |B_1| < 1,$

(ii) $\sum_{j=1}^{2} \left(\frac{|a_j b_j|}{c_j - |a_j| - |b_j| - 1} + 1 \right) F(|a_j|, |b_j|; c_j; 1) \leq 2$

are satisfied, then $\Omega(f)$ is sense-preserving harmonic and univalent in U; and so $\Omega(f) \in S_H$.

LEMMA 2.4 ([2]). *If $a, b, c > 0$, then*

(i) $\sum_{n=1}^{\infty} (n - 1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c - a - b - 1}F(a, b; c; 1)$ *if* $c > a + b + 1$,

(ii) $\sum_{n=2}^{\infty} (n - 1)^2 \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left(\frac{(a)_2(b)_2}{(c - a - b - 2)_2} + \frac{ab}{c - a - b - 1} \right)$
$F(a, b; c; 1)$ *if* $c > a + b + 2$.

3. MAIN RESULTS

THEOREM 3.1. *Let $f = h + \bar{g}$ be of the form* (1.1), *and for $j = 1, 2$, suppose $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ and $\Omega(f) \in S_H$. If the coefficient conditions*

(i) $\sum_{n=2}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n| \leq 1,$

(ii) $\sum_{j=1}^{2} \left(\frac{(1 + |B|)}{(A - B)(1 - \alpha)} \frac{|a_j b_j|}{c_j - |a_j| - |b_j| - 1} + 1 \right) F(|a_j|, |b_j|; c_j; 1)$
$\leq (2 + |1 - B_1|) < 4$

are satisfied, then $\Omega(f) \in S_H[A, B, \alpha]$.

Proof. In order to prove that $\Omega(f) \in S_H[A, B, \alpha]$, it suffices to prove that

$$\begin{aligned} T(z) :&= \frac{H(z) - G(z)}{1 - B_1} \\ &= z + \sum_{n=2}^{\infty} \left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n \right] \frac{1}{1 - B_1} z^n \end{aligned} \tag{3.1}$$

is in $S^*[A,B,\alpha]$. Note that $|A_n| \le 1$ and $|B_n| \le 1$, by the condition (i). As an application of Lemma 2.1, $T \in S^*[A,B,\alpha]$ provided that $Q_1 \le 1$, where

$$\begin{aligned}
Q_1 &:= \sum_{n=2}^{\infty} \left[\frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)}\right] \\
&\times \left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\frac{A_n}{1-B_1} - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\frac{B_n}{1-B_1}\right| \\
&\le \sum_{n=2}^{\infty} \left[\frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)|1-B_1|}\right] \\
&\times \left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right) \\
&= \frac{(1+|B|)}{|1-B_1|(A-B)(1-\alpha)} \\
&\times \sum_{n=2}^{\infty}(n-1)\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right) \\
&+ \frac{1}{|1-B_1|}\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right) \\
&= \frac{(1+|B|)}{|1-B_1|(A-B)(1-\alpha)}\left(\frac{|a_1b_1|}{c_1-|a_1|-|b_1|-1}F(|a_1|,|b_1|;c_1;1)\right. \\
&\left.+\frac{|a_2b_2|}{c_2-|a_2|-|b_2|-1}F(|a_2|,|b_2|;c_2;1)\right) \\
&+ \frac{1}{|1-B_1|}(F(|a_1|,|b_1|;c_1;1)+F(|a_2|,|b_2|;c_2;1)-2)
\end{aligned}$$

by Lemma 2.3. Therefore, it follows that $T \in S^*[A,B,\alpha]$ if the inequality

$$\begin{aligned}
&\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left(\frac{(1+|B|)}{(A-B)(1-\alpha)}\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right) \\
&\qquad \times F(|a_j|,|b_j|;c_j;1) - \frac{2}{|1-B_1|} \le 1
\end{aligned}$$

holds. But this inequality is true because of the given condition (ii). □

THEOREM 3.2. *Let $f = h + \bar{g}$ given by* (1.1) *be in S_H. If the inequality*

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{(n-1)(1+|B|)+(A-B)(1-\alpha)\}|A_n| \\
&+\sum_{n=1}^{\infty}\{(n-1)(1+|B|)+(A-B)(1-\alpha)\}|B_n| \le (A-B)(1-\alpha)|1-B_1|
\end{aligned}$$

is satisfied, then $f \in S_H[A,B,\alpha]$.

Proof. From the definition of $S_H[A,B,\alpha]$, it suffices to prove that the function t given by (1.2) is in the class $S^*[A,B,\alpha]$. But, as an application of Lemma 2.1, we only need to show that $Q_2 \le 1$, where

$$\begin{aligned} Q_2 : &= \sum_{n=2}^{\infty} \frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)} \left| \frac{A_n - B_n}{1-B_1} \right| \\ &\le \sum_{n=2}^{\infty} \frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)} \left[\frac{|A_n|+|B_n|}{|1-B_1|} \right]. \end{aligned}$$

Thus $Q_2 \le 1$ holds because of the given condition. □

THEOREM 3.3. *Let $f = h + \bar{g}$ be of the form* (1.1) *and for $j = 1, 2$, suppose $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ such that $c_j > |a_j| + |b_j| + 2$ and $\Omega(f) \in S_H$. If the coefficient conditions*

(i) $\displaystyle\sum_{n=2}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n| \le 1,$

(ii) $$\sum_{j=1}^{2} \left\{ \frac{(1+B)}{(A-B)(1-\alpha)} \frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2} + \left(\frac{2(1+|B|)}{(A-B)(1-\alpha)} + 1 \right) \frac{|a_j b_j|}{c_j-|a_j|-|b_j|-1} + 1 \right\} F(|a_j|,|b_j|;c_j;1) \le 2 + |1-B_1| < 4$$

are satisfied, then $\Omega(f) \in K_H[A,B,\alpha]$.

Proof. In view of the definition of $K_H[A,B,\alpha]$ and the fact that $\Omega(f) \in S_H$, it suffices to prove that the function T given by (3.1) is in $K[A,B,\alpha]$. Note that $|A_n| \le 1$ and $|B_n| \le 1$, by the condition (i). In the view of Lemma (2.2), the function $T \in K[A,B,\alpha]$ provided that $Q_3 \le 1$, where

$$\begin{aligned} Q_3 &:= \sum_{n=2}^{\infty} n \left[\frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)} \right] \\ &\times \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{A_n}{1-B_1} - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{B_n}{1-B_1} \right| \\ &\le \sum_{n=2}^{\infty} n \left[\frac{(n-1)(1+|B|)+(A-B)(1-\alpha)}{(A-B)(1-\alpha)|1-B_1|} \right] \\ &\times \left[\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \right] \\ &= \frac{1+|B|}{|1-B_1|(A-B)(1-\alpha)} \sum_{n=2}^{\infty} [(n-1)^2 + (n-1)](D_1 + D_2) \end{aligned}$$

$$+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(n-1)(D_1+D_2)+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(D_1+D_2)$$
$$=\frac{1}{|1-B_1|}\left[\frac{1+|B|}{(A-B)(1-\alpha)}\sum_{n=2}^{\infty}(n-1)^2(D_1+D_2)\right.$$
$$\left.+\left(\frac{1+|B|}{(A-B)(1-\alpha)}+1\right)\sum_{n=2}^{\infty}(n-1)(D_1+D_2)+\sum_{n=2}^{\infty}(D_1+D_2)\right],$$

where $D_j=\frac{(|a_j|)_{n-1}(|b_j|)_{n-1}}{(c_j)_{n-1}(1)_{n-1}}$ for $j=1,2$. Using Lemma 2.4, we find that Q_3 is less than or equal to

$$\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left\{\frac{(1+|B|)}{(A-B)(1-\alpha)}\frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2}+\left(\frac{2(1+|B|)}{(A-B)(1-\alpha)}+1\right)\right.$$
$$\left.\times\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right\}F(|a_j|,|b_j|;c_j;1)-\frac{2}{|1-B_1|}.$$

This proves that $Q_3\le 1$ by the given condition (ii). □

Theorem 3.4. *Let $f=h+\bar{g}$ given by (1.1) be in S_H. If the inequality*

$$\sum_{n=2}^{\infty}n\{(n-1)((1+|B|)+(A-B)(1-\alpha)\}|A_n|$$
$$+\sum_{n=1}^{\infty}n\{(n-1)(1+|B|)+(A-B)(1-\alpha)\}|B_n|\le(A-B)(1-\alpha)$$

is satisfied, then $f\in K_H[A,B,\alpha]$.

Proof. This is similar to the proof of Theorem 3.2, hence it is omitted. □

The next two theorems give characterizations of functions in $S_H[A,B,\alpha]$ and $K_H[A,B,\alpha]$.

Theorem 3.5. *If $f(z)=h(z)+\overline{g(z)}\in S_H$ then $f\in S_H[A,B,\alpha]$ if and only if $\frac{1}{z}[(h(z)-g(z))*F_1(z)]\neq 0$ for all z in U and all ξ, such that $|\xi|=1$, where*

$$F_1(z):=\frac{z+\left(\frac{\xi-(B+(A-B)(1-\alpha))}{(A-B)(1-\alpha)}\right)z^2}{(1-z)^2}.$$

Proof. By the definition of $S_H[A,B,\alpha]$, it is clear that $f\in S_H[A,B,\alpha]$ if and only if $t(z)$ given by (1.3) is in $S^*[A,B,\alpha]$. But, $t\in S^*[A,B,\alpha]$ if and only if $\frac{zt'(z)}{t(z)}\prec\frac{1+(B+(A-B)(1-\alpha))z}{1+Bz}$, that is, $\frac{zt'(z)}{t(z)}\neq\frac{1+(B+(A-B)(1-\alpha))\varsigma}{1+B\varsigma}$ for $z\in U$ and $|\varsigma|=1$, which is equivalent to $\frac{1}{z}[(1+B\varsigma)zt'-(1+(B+(A-B)(1-\alpha))\varsigma)t]\neq 0$.

Since $zt' = t * \frac{z}{(1-z)^2}$ and $t = t * \frac{z}{1-z}$, the above inequality is equivalent to

$$\frac{1}{z}\left[t(z) * \left[\frac{-(A-B)(1-\alpha)\varsigma z + [1 + (B + (A-B)(1-\alpha))\varsigma]z^2}{(1-z)^2}\right]\right]$$
$$= \frac{-(A-B)(1-\alpha)\varsigma}{(1-B_1)z}\left[(h(z) - g(z)) * \left(\frac{z + \left(\frac{\xi - (B + (A-B)(1-\alpha))}{(A-B)(1-\alpha)}\right)z^2}{(1-z)^2}\right)\right] \neq 0,$$

where $|-1/\varsigma| = |\xi| = 1$, and the result follows. □

Corollary 3.6. *If $f(z) = h(z) + \overline{g(z)} \in S_H$, then $f \in K_H[A, B, \alpha]$ if and only if $\frac{1}{z}[(h(z) - g(z)) * F_2(z)] \neq 0$ for all $z \in U$ and all ξ with $|\xi| = 1$, where*

$$F_2(z) := \frac{1}{(1-z)^3}\left[z + \left(\frac{2\xi - (2B + (A-B)(1-\alpha))}{(A-B)(1-\alpha)}\right)z^2\right].$$

Proof. Note that $t \in K[A, B, \alpha]$ if and only if $zt'(z) \in S_H[A, B, \alpha]$. Setting $p(z) = \frac{z + \left(\frac{\xi - (B + (A-B)(1-\alpha))}{(A-B)(1-\alpha)}\right)z^2}{(1-z)^2}$, we have $zp'(z) = \frac{z + \left(\frac{2\xi - 2B - (A-B)(1-\alpha)}{(A-B)(1-\alpha)}\right)z^2}{(1-z)^3}$. Using the identity $zt' * p = t * zp'$, the result follows from Theorem 3.5. □

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