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# ORBIT DECOMPOSITION OF SKEW GROUP ALGEBRAS

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**Abstract.** We describe the basic algebra Morita equivalent to the skew group algebra $\Lambda G$, where $\Lambda$ is the path algebra of a finite, connected, acyclic quiver and $G$ is a finite cyclic group. We give a structure theorem for the above case, based on combinatorial techniques. We prove that in this case $\Lambda G$ is isomorphic to a direct product of certain matrix algebras, which are described in detail.

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**Key words.** Artinian rings, skew group algebras, basic algebra.

## 1. PRELIMINARIES

In this paper we describe the basic algebra Morita equivalent to the skew group algebra $\Lambda G$, where $\Lambda$ is a path algebra of a finite, connected quiver, and $G$ is a finite group.

1.1. Let $\Lambda$ be an Artinian ring with $\Lambda/\operatorname{rad}(\Lambda) \simeq \bigoplus \operatorname{Mat}_{n_i}(\Delta_i)$, and let $P_i$ be the corresponding projective indecomposables such that ${}_\Lambda\Lambda = \bigoplus n_i P_i$. If we let $P := \bigoplus m_i P_i$, with $m_i \geq 1$, then $P$ is a progenerator of $\operatorname{Mod}\Lambda$, and $\Gamma$ and $\Lambda$ are Morita equivalent. Let $\Gamma := \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. Then we have $\Gamma/(\operatorname{rad}\Gamma) \simeq \bigoplus \operatorname{Mat}_{m_i}(\Delta_i)$. The smallest possibility for $P$ is that if we choose each $m_i = 1$. In this case $\Gamma$ is called the basic algebra of $\Lambda$. This means that we have to determine $\Lambda/\operatorname{rad}(\Lambda) \simeq \bigoplus \operatorname{Mat}_n(\Delta_i)$ and the corresponding projective indecomposables.

We recall the Wedderburn-Artin theorem and its proof cf. [3], because we shall need the notations.

THEOREM 1.2. *Let $\Lambda$ be a semisimple Artinian ring. Then $\Lambda = \bigoplus_{i=1}^r \Lambda_i$, where $\Lambda_i \simeq \operatorname{Mat}_{n_i}(\Delta_i)$, $\Delta_i$ is a division ring and the $\Lambda_i$ are uniquely determined. The ring $\Lambda$ has exactly $r$ isomorphism classes of irreducible modules $M_i$, and moreover $\operatorname{End}_\Lambda(M_i) \simeq \Delta_i^{op}$, and $\dim_{\Delta_i^{op}}(M_i) = n_i$.*

*Proof.* If $\Lambda$ is semisimple, then $\Lambda \simeq \bigoplus_{i=1}^r M_i$, where $M_i$ is semisimple, $M_i = M_{i,1} \bigoplus \cdots \bigoplus M_{i,n_i}$, where $M_{i,j} \simeq S_i$ is simple for all $i \in \{1, \ldots, n_i\}$, and $S_i \not\simeq S_j$ for $i \neq j$. In this case let $\Delta_i = \operatorname{End}_\Lambda(S_i)$, which is a division ring. We have that $\operatorname{End}_\Lambda(M_i) \simeq \operatorname{Mat}_{n_i}(\Delta_i)$; this implies $\operatorname{End}_\Lambda(M) = \bigoplus \operatorname{End}_\Lambda(M_i) \simeq \bigoplus \operatorname{Mat}_{n_i}(\Delta_i)$. On the other hand, $\Lambda \simeq \operatorname{End}_\Lambda({}_\Lambda\Lambda)^{\mathrm{op}}$, so it follows that $\Lambda \simeq \bigoplus_{i=1}^r \operatorname{Mat}_{n_i}(\Delta_i^{\mathrm{op}})$. □

1.3. We fix the following notation.

- $r$ is the number of isomorphism classes of simple $\Lambda$-modules;

- $\Delta_i := \mathrm{End}_\Lambda(S_i)$, where $S_i$ is simple;
- $n_i$ is the number of summands isomorphic to $S_i$.

Recall also from [3] the following theorem on lifting idempotents.

THEOREM 1.4. *If $\Lambda$ is an Artin algebra, then:*

(i) $\operatorname{rad}\Lambda$ *is a nilpotent ideal of $\Lambda$;*

(ii) *If $1 = e_1+\cdots+e_n$ is a decomposition into primitive orthogonal idempotents in $\Lambda/\operatorname{rad}\Lambda$, then there is the decomposition $1 = f_1+\cdots+f_n$ into primitive orthogonal idempotents in $\Lambda$ such that $\overline{f_i} := f_i + \operatorname{rad}\Lambda = e_i$.*

1.5. Since $\Lambda/\operatorname{rad}\Lambda$ is semisimple, we can apply the Wedderburn-Artin theorem, and we get $\Lambda/\operatorname{rad}\Lambda = \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(\Delta_i)$. Let $S_i$ be the simple $\Lambda$-module corresponding to the $i^{\text{th}}$ matrix algebra factor. This means that $\Lambda/\operatorname{rad}\Lambda \simeq \bigoplus_{i=1}^r n_i S_i$.

Now we choose an orthogonal $1 = e_1 + e_2 + \cdots + e_n$ in $\Lambda/\operatorname{rad}\Lambda$. By the above theorem we get the orthogonal decomposition $1 = f_1 + f_2 + \cdots + f_n$ in $\Lambda$. With the notations of Theorem 1.2, this implies ${}_\Lambda\Lambda = \bigoplus_{i=1}^r n_i P_i$, where $P_i/J(\Lambda)P_i \simeq S_i$. From here we can construct the progenerator $P = \bigoplus_{i=1}^r P_i$, and the corresponding basic algebra $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ associated to $\Lambda$.

Therefore, if we want to study the basic algebra of the skew group algebra $\Lambda G$, then we need information about $\Lambda G/\operatorname{rad}\Lambda G$, and also about the semisimplicity of $\Lambda G$. We only consider the case when the order $|G|$ of $G$ is invertible in $\Lambda$. We recall the following result from [4].

THEOREM 1.6. *If $|G|$ is invertible in $\Lambda$, then $\Lambda G/\operatorname{rad}\Lambda G = (\Lambda/\operatorname{rad}\Lambda)G$.*

## 2. PATH ALGEBRAS AND THE DECOMPOSITION OF $(\Lambda/\operatorname{rad}\Lambda)G$

In this section we discuss the decomposition of $\Lambda/\operatorname{rad}\Lambda$ and of $(\Lambda/\operatorname{rad}\Lambda)G$, where $\Lambda = KQ$ is a path algebra of a connected, finite quiver, and $G$ is a finite group such that the characteristic of the field $K$ does not divide the order of $G$.

Let $Q = (Q_0, Q_1)$ be a quiver and let $\Lambda = KQ$ be it's path algebra. The *arrow ideal* of $KQ$ is the ideal generated by all paths of $Q$. We denote by $\varepsilon_a = (a||a)$ the trivial path of $Q$, for $a \in Q_0$. In this case we have the following result from [1].

THEOREM 2.1. *Let $Q = (Q_0, Q_1)$ be a finite, connected quiver, $K$ an algebraically closed field and $R$ the arrow ideal of $KQ$.*

(1) *The set $\{\overline{\varepsilon_a} = \varepsilon_a + R\}$ is complete set of primitive orthogonal idempotents for $KQ/R$.*

(2) *If $Q$ is acyclic then* $\operatorname{rad} KQ = R$ *and $KQ$ is finite dimensional basic algebra.*

2.2. From now on we consider only finite, connected, acyclic quivers. By Theorem 2.1, we have the decomposition $\Lambda/\operatorname{rad}\Lambda \simeq Ke_1 \times \cdots \times Ke_n$, where $Q_0 = \{e_1, \ldots e_n\}$. We also have that $(\Lambda/\operatorname{rad}\Lambda)G \simeq (Ke_1 \times \cdots \times Ke_n)G$.

Consider first the following example from [2].

Example 2.3. Let $K$ be a field of characteristic not equal to 2, and let $Q$ be the following quiver

$$3 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 1 \xrightarrow{\alpha'} 2' \xrightarrow{\beta'} 3' \, .$$

Let $G = \langle \sigma \rangle$ of order 2, and let $\sigma e_1 = e_1$, $\sigma e_2 = e_{2'}$, $\sigma e_3 = e_{3'}$, $\sigma\alpha = \alpha'$ and $\sigma\beta = \beta\prime$. Then there is only one way of extending $\sigma$ to be a $K$-algebra automorphism of $\Lambda$. We obtain the decomposition

$$(\Lambda / \operatorname{rad} \Lambda)G = (Ke_1)G \times (Ke_2 \times Ke_{2'})G \times (Ke_3 \times Ke_{3'})G.$$

This example suggests that if we have the decomposition $\Lambda/\operatorname{rad}\Lambda = Ke_1 \times \cdots \times Ke_n$, then $(\Lambda/\operatorname{rad}\Lambda)G \simeq \bigoplus_{j\in J} H_j G$, where $H_j = Ke_{j,1} \times \cdots \times Ke_{j,n_j}$, such that $\{e_{j_1}, \ldots, e_{n_j}\}$ is a maximal $G$-invariant subset of $Q_0$, and for all $N$, the subset $N$ of $H_j$, is not $G$-invariant. The precise statement is as follows.

Proposition 2.4. *Let $\Lambda$ be the path algebra of a finite, connected and acyclic quiver $Q = (Q_0, Q_1)$, and let $G$ be a group of automorphisms of $\Lambda$ of order $s$. Then we have*

$$(\Lambda / \operatorname{rad} \Lambda)G \simeq \bigoplus_{j\in J} \widetilde{\operatorname{Orb}(e_j)}G,$$

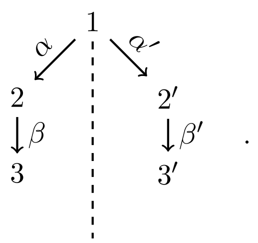
*where $\operatorname{Orb}(e_j)$ is the orbit of $e_j$ and $\widetilde{\operatorname{Orb}(e_j)} = Ke_{j_1} \times \cdots \times Ke_{j_{n_j}}$ for $e_{j_k} \in \operatorname{Orb}(e_j)$.*

*Proof.* It is clear that $(\Lambda/\operatorname{rad}\Lambda)G \simeq \bigoplus_{j\in J} H_j G$, where $H_j$ is the $G$-invariant part of $Ke_1 \times \cdots \times Ke_n$. We prove that $H_j = \operatorname{Orb}(e_r)$ for some $r \in \{1, \ldots, n\}$. It is also clear that $\operatorname{Orb}(e_r)$ is $G$-invariant.

Conversely, if $H_j$ is $G$-invariant, then let $e_r \in (Q_0 \cap H_j)$. Then we have the implications: $g(e_r) \in H_j \Rightarrow \cdots \Rightarrow g^s(e_r) \in H_j$. Now suppose that there exists $h \in H_j$, $h \neq 0$ such that $0 \neq h \in H_j \setminus \{e_r, \ldots, g^s(e_r)\}$. This means that $\{e_r, \ldots, g^s(e_r)\}$ is a $G$-invariant subset of $H_j$, which is a contradiction. $\square$

Next we examine the decomposition of $\widetilde{\operatorname{Orb}(e_i)}G$. Before this, we examine some examples.

Example 2.5. Let $K$ be a field of characteristic not equal to 2 and let $Q$ be the following quiver.



Let $G = \langle\sigma\rangle$ be of order 2, and let $\sigma e_1 = e_1$, $\sigma e_2 = e_{2'}$, $\sigma e_3 = e_{3'}$, $\sigma\alpha = \alpha'$ and $\sigma\beta = \beta\prime$. Then there is a unique way to extend $\sigma$ to b a $K$-algebra automorphism of $\Lambda$. (In a more general case we have a quiver with symmetry axis $T$ and $\sigma \in \mathrm{Aut}(\Lambda)$ inverts the elements from the two sides of $T$.) Observe that $\mathrm{Orb}(e_1) = \{e_1\}, \mathrm{Orb}(e_2) = \{e_2, e_{2'}\}$ and $\mathrm{Orb}(e_3) = \{e_3, e_{3'}\}$.

EXAMPLE 2.6. Let $\sigma \in \mathrm{Aut}(\Lambda)$ be such that $\sigma e_i = e_{i+1}$ and $\sigma e_n = e_1$. In this case we have $\Lambda/\operatorname{rad}\Lambda = \widetilde{\mathrm{Orb}(e_1)}$.

THEOREM 2.7. *Assume that*

$$(\Lambda/\operatorname{rad}\Lambda)G \simeq \bigoplus_{i\in I_1}(Ke_i)G \oplus \bigoplus_{j\in I_2}(Ke_j \times Ke_{j'})G.$$

*Then:*

(i) $(Ke_i)G$ *has a set* $\{\widetilde{e_i}, \widetilde{\widetilde{e_i}}\}$ *of primitive orthogonal idempotents;*

(ii) $(Ke_i \times Ke_{i'})G \simeq \begin{pmatrix} K & K \\ K & K \end{pmatrix}$.

*Proof.* Let $\tilde{e_1} = \frac{e_1+e_1\sigma}{2}$, $\tilde{\tilde{e_1}} = \frac{e_1-e_1\sigma}{2}$. In this way we get a set of primitive orthogonal idempotents of $(Ke_i)G$. To prove the second part, we let

$$e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_{2'} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e_2\sigma \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{2'}\sigma \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and we are done. □

Now by Proposition 2.4 and Theorem 2.7, we expect that $\widetilde{\mathrm{Orb}\, e_i}G \simeq \mathrm{Mat}_l(K)$, where $l = \mathrm{card}(\mathrm{Orb}\, e_i)$ if $l > 1$, and $\widetilde{\mathrm{Orb}(e_i)}G \simeq K\times K$ if $l = 1$. Before proving this we look at the following example.

EXAMPLE 2.8. Consider $\widetilde{\mathrm{Orb}(e_i)}G$, with $|G| = 3$. This means that we have a $\sigma$-cycle of length 3 in $\widetilde{\mathrm{Orb}(e_i)}G$ (that is, $\sigma^3(e_i) = e_i$). We have $3 \times 3$ basis elements in $\widetilde{\mathrm{Orb}(e_i)}G$:

$$\{e_i, \sigma(e_i), \sigma^2(e_i), e_i\sigma, \sigma(e_i)\sigma, \sigma^2(e_i)\sigma, e_i\sigma^2, \sigma(e_i)\sigma^2, \sigma^2(e_i)\sigma^2\}.$$

We want to map each basis element to one of the $\delta_{i,j}$-s, where $\delta_{i,j}$ is a matrix with 1 at $(i, j)$ and 0-s elsewhere. If we can do this, it means that $\widetilde{\mathrm{Orb}(e_i)}G \simeq \mathrm{Mat}_3(K)$.

We have to arrange the basis elements in a $3 \times 3$ matrix $M$ such that $m_{i,j} = m_{i,k} \times m_{k,j}$, for all $k \in \{1, 2, 3\}$. A suitable arrangement is

$$\begin{pmatrix} e & e\sigma^2 & e\sigma \\ \sigma(e)\sigma & \sigma(e) & \sigma(e)\sigma^2 \\ \sigma^2(e)\sigma^2 & \sigma^2(e)\sigma & \sigma^2(e) \end{pmatrix}.$$

By generalizing this idea, we get the required matrix, and we can prove our result.

Theorem 2.9. *Let $Q = (Q_0, Q_1)$ be a finite, connected, acyclic quiver, and let $K$ be an algebraically close field. Let $\Lambda = KQ$ and assume that $G = \langle\sigma\rangle$ has order invertible in $\Lambda$. Then $\widetilde{\mathrm{Orb}(e)}G \simeq \mathrm{Mat}_l(K)$, where $l$ is the cardinality of* $\mathrm{Orb}(e)$ *and $l > 1$.*

*Proof.* We observe that we have exactly $l^2$ basis elements, namely $\sigma^i(e)\sigma^j$, for $i \in \{0, \dots, l-1\}$ and $j \in \{0, \dots, l-1\}$. We want to arrange them in an $l \times l$ matrix $M = (m_{i,j})$ such way that $m_{i,j} = m_{i,k} \times m_{k,j}$, for all $k \in \{1, \dots, l\}$. We denote the basis elements by $a_{i,j} := \sigma^{i-1}(e)\sigma^{j-1}$, where $i, j \in \{1, \dots, l\}$, and we arrange them as follows.

$$M = \begin{pmatrix} a_{1,1} & a_{1,l} & \cdots & a_{1,2} \\ a_{2,2} & a_{2,1} & \cdots & a_{2,3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l-1,l-1} & a_{l-1,l-2} & \cdots & a_{l-1,l} \\ a_{l,l} & a_{l,l-1} & \cdots & a_{l,1} \end{pmatrix}.$$

Consequently, we have

$$m_{i,j} = \begin{cases} a_{i,i-j+1}, & \text{if } j \leq i; \\ a_{i,l-(j-i)+1}, & \text{if } i < j. \end{cases}$$

Observe that $a_{i,l-(j-i)+1} = \sigma^{i-1}(e)\sigma^{l-(j-i)} = \sigma^{i-1}(e)\sigma^{i-j} = a_{i,i-j+1}$, so we have that $m_{i,j} = a_{i,i-j+1}$, for all $i, j \in \{1, \dots, l\}$.

We now to check that $m_{i,j} = m_{i,k} \times m_{k,j}$, for all $i, j, k \in \{1, \dots, l\}$. We have

$$\begin{aligned} m_{i,k} \times m_{k,j} &= a_{i,i-k+1} \times a_{k,k-j+1} = \sigma^{i-1}(e)\sigma^{i-k}\sigma^{k-1}(e)\sigma^{k-j} \\ &= \sigma^{i-1}(e)\sigma^{i-k+k-1}(e)\sigma^{i-k}\sigma^{k-j} = \sigma^{i-1}(e)\sigma^{i-j} = m_{i,j}. \end{aligned}$$

This means that $\widetilde{\mathrm{Orb}(e)}G \simeq \mathrm{Mat}_l(K)$. □

Our main result now follows by Theorems 1.6, 2.4 and 2.9.

Theorem 2.10. *Let $Q = (Q_0, Q_1)$ be a finite, connected, acyclic quiver, and denote $\Lambda = KQ$, where $K$ is an algebraically closed field. Let $G = \langle\sigma\rangle$ be a cyclic group such that the characteristic of $K$ does not divide the order of $G$. Then*

$$\Lambda G/\operatorname{rad}(\Lambda G) \simeq |I_1|(K \times K) \bigoplus_{j>1} |I_j| \mathrm{Mat}_j(K),$$

*where $|I_j|$ is the number of $\sigma$-cycles of length $j$.*

2.11. Since full matrix algebras have isomorphic simple modules, we may now construct $e = \sum_{i=1}^{n_1}(\tilde{e_{1_i}} + \tilde{\tilde{e_{1_i}}}) + \sum_{j>1} e_j$, where the $\tilde{e_{1_i}} + \tilde{\tilde{e_{1_i}}}$ corresponds to a summand of the form $(K \times K)$, and every $e_j$ corresponds to a full matrix algebras. This describes the basic algebra $(\Lambda G)^b = e(\Lambda G)e$ Morita equivalent to $\Lambda G$.

We take a look again at our previous Example 2.3.

EXAMPLE 2.12. By using the calculations in Example 2.3, we have

$$(\Lambda/\operatorname{rad}\Lambda)G \simeq k \times k \times \begin{pmatrix} k & k \\ k & k \end{pmatrix} \times \begin{pmatrix} k & k \\ k & k \end{pmatrix}$$

Observe that $\begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$ are isomorphic as $\begin{pmatrix} k & k \\ k & k \end{pmatrix}$-modules. We may construct the basic algebra $(\Lambda G)^b = e(\Lambda G)e$, where $e = \tilde{e_1} + \tilde{\tilde{e_1}} + e_2 + e_3$ corresponds to the non-isomorphic components.

Further, we have that

$$\{e_1, e_2, e_3, e_{2'}, e_{3'}, \alpha, \beta, \alpha', \beta', \beta\alpha, \beta'\alpha'\, e_1\sigma, e_2\sigma, e_3\sigma,$$
$$e_{2'}\sigma, e_{3'}\sigma, \alpha\sigma, \beta\sigma, \alpha'\sigma, \beta'\sigma, \beta\alpha\sigma, \beta'\alpha'\sigma\}$$

is a $k$-basis of $\Lambda G$. By multiplying on the left and on the write we deduce that

$$\left\{ \tilde{e_1}, \tilde{\tilde{e_1}}, \frac{\alpha + \alpha\sigma}{2}, \frac{\alpha - \alpha\sigma}{2}\ \frac{\beta(\alpha + \alpha\sigma)}{2}, \frac{\beta(\alpha - \alpha\sigma)}{2} \right\}$$

is a basis of $e\Lambda Ge$.

Now letting $\tilde{\alpha} := \frac{\alpha+\alpha\sigma}{2}$ and $\tilde{\tilde{\alpha}} := \frac{\alpha+\alpha\sigma}{2}$, we obtain that $e\Lambda Ge$ is isomorphic to the path algebra of the quiver

$$\begin{array}{ccccc} \tilde{e_1} & \xrightarrow{\tilde{\alpha}} & e_2 & \xleftarrow{\tilde{\tilde{\alpha}}} & \tilde{\tilde{e_1}} \\ & & \downarrow{\scriptstyle \beta} & & \\ & & e_3 & & \end{array}$$

The determination, in general, of a basis to the basic algebra of $\Lambda G$ and the construction of a quiver whose path algebra is this basic algebra, are combinatorial questions which will be discussed in another paper.

## REFERENCES


[1] ASSEM, I., SIMSON, D. and SKOWRONSKI, A., *Elements of the Representation Theory of Associative Algebras*, Cambridge Univ. Press, New York, 2006.

[2] AUSLANDER, M., REITEN, I. and SMALØ, S.O., *Representation Theory of Artin Algebras*, Cambridge Univ. Press, New York, 1989.

[3] BENSON, D.J., *Representations and cohomology*, Vol. **1**, Cambridge Univ. Press, New York, 1995.

[4] REITEN, I. and RIEDTMANN, C., *Skew group algebras in the representation theory of artin algebras*, J. Algebra, **92** (1985), 224–282.



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