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ALMOST STARLIKENESS OF COMPLEX ORDER λ ASSOCIATED WITH EXTENSION OPERATORS FOR BIHOLOMORPHIC MAPPINGS

CAMELIA MĂDĂLINA BĂLĂEŢI

Abstract. In this paper we continue the study of the Pfaltzgraff and Suffridge extension operator that provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$. Using the Loewner chains metod, we prove that if f is an almost starlike mapping of complex order λ on B^n then F is also almost starlike mapping of complex order λ on B^{n+1}. Certain consequences of this result will be also presented.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$. The unit ball in $\mathbb{C}^n$ is denoted by B^n. In the case of one variable B^1 is denoted by U. The ball of radius $r > 0$ in $\mathbb{C}^n$ with center at 0 will be denoted by B^n_r. Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm $\|A\| = \sup\{\|Az\| : \|z\| = 1\}$ and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^m)$. If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. A mapping $f \in H(B^n)$ is called normalized if $f(0) = 0$ and $Df(0) = I_n$. We say that $f \in H(B^n)$ is locally biholomorphic on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $J_f(z) = \det D(f)$ for $z \in B^n$.

Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n, and let $S(B^n)$ denote the set of normalized biholomorphic mappings on B^n. In the case of one variable, the set $S(B^1)$ is denoted by S, and $\mathcal{L}S(B^1)$ is denoted by $\mathcal{L}S$. A mapping $f \in S(B^n)$ is called starlike (respectively convex) if its image is a starlike domain with respect to the origin (respectively convex domain). The classes of normalized starlike (respectively convex) mappings on B^n will be denoted by $S^*(B^n)$ (respectively $K(B^n)$). In case of one variable, $S^*(B^1)$ (respectively $K(B^1)$) is denoted by S^* (respectively K).

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Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e., $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

Definition 1.1. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a *Loewner chain* if it satisfies the following conditions:

(i) $f(., t)$ is biholomorphic on $B^n, f(0, t) = 0$, $Df(0, t) = \mathrm{e}^t I_n$, $t \geq 0$;
(ii) $f(z, s) \prec f(z, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in B^n$.

We note that condition (ii) implies that there is a unique univalent Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that $f(z, s) = f(v(z, s, t), t)$, $z \in B^n$, $0 \leq s \leq t < \infty$, $z \in B^n$. Further, the normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = \mathrm{e}^{s-t} I_n$, $0 \leq s \leq t \leq \infty$, for the transition mapping.

We recall a compactness result for the set of Loewner chains.

Lemma 1.2. [5] *Every sequence of Loewner chains* $\{f_k(z, t)\}_{k \in \mathbb{N}}$*, such that* $\{\mathrm{e}^{-t} f_k(z, t)\}_{t \geq 0}$ *is a normal family on* B^n *for each* $k \in \mathbb{N}$*, contains a subsequence that converges locally uniformly on* B^n *to a Loewner chain* $f(z, t)$ *for each fixed* $t \geq 0$*, such that* $\{\mathrm{e}^{-t} f(z, t)\}_{t \geq 0}$ *is a normal family.*

The following class of holomorphic mappings of B^n plays the role of Carathéodory class in the n-dimensional case: $\mathcal{M} = \{h \in H(B^n) : h(0) = 0,\ Dh(0) = I_n,\ \mathrm{Re}\langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\}$.

Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. We recall the definition of a spirallike mapping of type α on B^n.

Definition 1.3. Suppose $\alpha \in (-\pi/2, \pi/2)$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *spirallike mapping of type* α if $\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\alpha}\langle [Df(z)]^{-1} f(z), z\rangle\right] > 0$, $z \in B^n \setminus \{0\}$.

This subclass of biholomorphic mappings can be characterized in terms of Loewner chains (see [7]).

Theorem 1.4. *Suppose* f *is a normalized locally biholomorphic mapping on* B^n, $\alpha \in (-\pi/2, \pi/2)$, $a = \tan \alpha$*. Then* f *is a spirallike mapping of type* α *if and only if* $F(z, t) = \mathrm{e}^{(1-\mathrm{i}a)t} f(\mathrm{e}^{\mathrm{i}at} z)$, $z \in B^n$, $t > 0$, *is a Loewner chain.*

The following definition was introduced by G. Kohr [8], [9] in the case $\alpha = 1/2$ and by Feng [2] in the case of $\alpha \in [0, 1)$ and on the unit ball in a complex Banach space X. For our purpose, we present this notion only in the Euclidean setting.

Definition 1.5. Suppose $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be an *almost starlike mapping of order* α if $\mathrm{Re}\langle [Df(z)]^{-1} f(z), z\rangle > \alpha \|z\|^2$, $z \in B^n \setminus \{0\}$.

It is clear that if f is almost starlike of order α, then f is also starlike.

Q.H. Xu and T.S. Liu [14] proved the following characterization of almost starlikeness of order α in terms of Loewner chains.

THEOREM 1.6. *Suppose f is a normalized locally biholomorphic mapping in B^n, $0 \le \alpha < 1$. Then f is an almost starlike mapping of order α if and only if $F(z,t) = \mathrm{e}^{\frac{t}{1-\alpha}} f(\mathrm{e}^{\frac{\alpha}{\alpha-1}t} z)$, $z \in B^n$, $t \ge 0$, is a Loewner chain. In particular, f is a starlike mapping (i.e., $\alpha = 0$) if and only if $F(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain.*

Now recall the definition for almost starlikeness of complex order λ (see [1]).

DEFINITION 1.7. Let $\lambda \in \mathbb{C}^n$ be such that $\mathrm{Re}\lambda \le 0$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be an *almost starlike mapping of complex order* λ if $\mathrm{Re}\{(1-\lambda)\langle [Df(z)]^{-1} f(z), z\rangle\} > -\mathrm{Re}\lambda \|z\|^2$, $z \in B^n \setminus \{0\}$.

REMARK 1.8. In the case of $\lambda = \alpha/(\alpha-1)$, where $\alpha \in [0,1)$, in Definition 1.7, we obtain the usual notion of almost starlikeness of order α. In the case of $\lambda = -1$ we obtain the notion of almost starlikeness of order $1/2$. On the other hand, if $\lambda = \mathrm{i}\tan\alpha, \alpha \in (-\pi/2, \pi/2)$ we obtain the usual notion of spirallikenes of type α.

Let $S^*_\lambda(B^n)$ be the set of almost starlike mappings of complex order λ.

The following result provides a necessary and a sufficient condition for almost starlikeness of complex order λ, in terms of Loewner chains.

THEOREM 1.9. [1] *Suppose f is a normalized holomorphic mapping on $B^n, \lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < 0$. Then f is almost starlike mapping of order λ if and only if $F(z,t) = \mathrm{e}^{(1-\lambda)t} f(\mathrm{e}^{\lambda t} z)$, $z \in B^n$, $t \ge 0$ is a Loewner chain. In particular, f is a starlike mapping (i.e., $\lambda = 0$) if and only if $F(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain.*

As a consequence of Theorem 1.9, we recall the growth result for the almost starlike mapping of complex order λ.

COROLLARY 1.10. [1] *Let $f : B^n \to \mathbb{C}^n$ be an almost starlike mapping of complex order λ. Then $\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}$, $z \in B^n$.*

We next recall the notion of parametric representation on the unit ball B^n (see [5],[3],[11],[12]).

DEFINITION 1.11. We say that a normalized mapping $f \in H(B^n)$ *has parametric representation* (and denote by $f \in S^0(B^n)$) if there exists a mapping $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ such that

(i) $h(\cdot,t) \in \mathcal{M}$ for $t \ge 0$;
(ii) $h(z,.)$ is measurable on $[0,\infty)$ for $z \in B^n$,

and $f(z) = \lim_{t\to\infty} \mathrm{e}^t v(z,t)$ locally uniformly on B^n, where $v = v(z,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \quad \text{a.e.} \quad t \geq 0,\ v(z,0) = z,$$

for all $z \in B^n$.

It is known that a mapping f belongs to $S^0(B^n)$ if and only if there exists a Loewner chain $f(z,t)$ such that $\{\mathrm{e}^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f = f(\cdot,0)$ (see [5] and [11, 12]).

For $n \geq 1$, set $z' = (z_1,\ldots,z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$

Definition 1.12. [10] The Pfaltzgraff-Suffridge extension operator $\Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is given by $\Phi_n(f)(z) = F(z) = (f(z'), z_{n+1})[J_f(z')]^{\frac{1}{n+1}}$, $z = (z', z_{n+1}) \in B^{n+1}$.

We choose the branch of the power function such that $[J_f(z')]^{\frac{1}{n+1}} |_{z'=0} = 1$. Obviously, $F = \Phi_n(f) \in \mathcal{L}S_{n+1}$ whenever $f \in \mathcal{L}S_n$. It is easy to see that if $f \in S(B^n)$ then $F = \Phi_n(f) \in S(B^{n+1})$.

For $n \geq 2$, let $\widetilde{z} = (z_2,\ldots,z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \widetilde{z}) \in \mathbb{C}^n$. If $n = 1$ then Φ_1 reduces to the well-known Roper-Suffridge extension operator, and for $n \geq 2$ we have:

Definition 1.13. [13] The Roper-Suffridge extension operator $\Psi_n : \mathcal{L}S \to \mathcal{L}S_n$ is defined by $\Psi_n(f)(z) = (f(z_1), \widetilde{z}\sqrt{f'(z_1)})$, $z = (z_1, \widetilde{z}) \in B^n$.

We choose the branch of the power function such that $\sqrt{f'(z_1)}\, |_{z_1=0} = 1$.

I. Graham, G. Kohr and J.A. Pfaltzgraff [6] proved that if $f \in S(B^n)$ can in imbedded in a Loewner chain $f(z',t)$, then $\Phi_n(f)$ can be imbedded in a Loewner chain $F(z,t)$. Further, if $f \in S^0(B^n)$ then $\Phi_n(f) \in S^0(B^{n+1})$. Moreover, they proved that if $f \in S^*(B^n)$ then $\Phi_n(f) \in S^*(B^{n+1})$.

In this paper, we continue the study of the Pfaltzgraff-Suffridge extension operator and we obtain the connection with the notion of almost starlikeness of a complex order λ.

2. MAIN RESULTS

We begin this section with the compactness result of the set $S^*_\lambda(B^n)$ that will be useful in the proof of Theorem 2.7.

Theorem 2.1. *$S^*_\lambda(B^n)$ is a compact subset of $H(B^n)$.*

Proof. In view of Corollary 1.10, the set $S^*_\lambda(B^n)$ is a locally uniformly bounded set. Therefore, it suffice to show that $S^*_\lambda(B^n)$ is closed in $H(B^n)$. For this purpose, let $\{f_k\}$ a sequence of almost starlike mappings of complex order λ be such that $f_k \to f$ locally uniformly on B^n as $k \to \infty$. Then for each $k \in \mathbb{N}$, there is a Loewner chain $f_k(z,t)$ such that $f_k(z,0) = f_k(z)$, $z \in B^n$, and $\{e^{-t}f_k(z,t)\}_{t\geq 0}$ is a normal family. From Lemma 1.2 there is a subsequence $\{f_{k_p}(z,t)\}_{p\in\mathbb{N}}$ such that $f_{k_p}(z,t)\to f(z,t)$ locally uniformly on B^n for each $t\geq 0$, $f(z,t)$ is a Loewner chain and $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family. It is obvious that $f(z)=f(z,0)$, $z\in B^n$, and in view of the fact that every almost starlike mapping of complex order λ can embedded as the first element of a Loewner chain one deduces that $f\in S^*_\lambda(B^n)$. This completes the proof. □

Next, we prove that the Pfaltzgraff-Suffridge extension operator preserves the notion of almost starlikeness of complex order λ from dimension n into dimension $n+1$.

Theorem 2.2. *Assume f is an almost starlike mapping of complex order λ on B^n. Then $F=\Phi_n(f)$ is also an almost starlike mapping of complex order λ on B^{n+1}.*

Proof. Let $F: B^{n+1}\times[0,\infty)\to\mathbb{C}^{n+1}$ be given by

$$F(z,t)=\Big(f(z',t), z_{n+1}e^{\frac{t}{n+1}}[J_{f_t}(z')]^{\frac{1}{n+1}}\Big) \tag{1}$$

for $z=(z',z_{n+1})\in B^{n+1}$ and $t\geq 0$. We choose the branch of the power function such that $[J_{f_t}(z')]^{\frac{1}{n+1}}_{z'=0}=e^{\frac{nt}{n+1}}$. Using a Schwarz-type lemma for the Jacobian determinant of a holomorphic mapping from B^n into B^n and the structure of the automorphisms of B^n, Graham, Kohr and Pfaltzgraff proved in Theorem 2.1 that $F(z,t)$ is a Loewner chain (see [6, Theorem 2.1]). The fact that f is almost starlike of complex order λ is equivalent to the statement that $f(z',t)=e^{(1-\lambda)t}f(e^{\lambda t}z)$, $z\in B^n$, $t\geq 0$ is a Loewner chain. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (1) is a Loewner chain. On the other hand a simple computation yields that

$$F(z,t)=e^{(1-\lambda)t}\Big(f(e^{\lambda t}z), z_{n+1}e^{\lambda t}[J_{f_t}(e^{\lambda t}z)]^{\frac{1}{n+1}}\Big)=e^{(1-\lambda)t}F(e^{\lambda t}z)$$

for $z\in B^{n+1}$ and $t\geq 0$. Thus $F=F(\cdot,0)$ is an almost starlike mapping of complex order λ. This completes the proof. □

In particular, if $\lambda=\tan\alpha$ in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Assume f is a spirallike mapping of type α on B^n. Then $F=\Phi_n(f)$ is a spirallike mapping of type α on B^{n+1}.*

Next, if we consider $\lambda=\dfrac{\alpha}{\alpha-1}$ in the proof of Theorem 2.2, we obtain the following particular case.

Corollary 2.4. *Assume f is an almost starlike mapping of order α on B^n. Then $F=\Phi_n(f)$ is an almost starlike mapping of order α on B^{n+1}.*

Another consequence of Theorem 2.2 is given in the following result due to Graham, Kohr and Pfaltzgraff [6]. This result provides a positive answer to the question of Pfaltzgraff and Suffridge regarding the preservation of starlikeness under the operator Φ_n (see [10]).

COROLLARY 2.5. *Assume that* $f \in S^*(B^n)$. *Then* $F = \Phi_n(f) \in S^*(B^{n+1})$.

EXAMPLE 2.6. Let f_j, $j = 1, \ldots, n$, be almost starlike functions of complex order λ. It is not difficult to deduce that the mapping $f : B^n \to \mathbb{C}^n$ given by $f(z') = (f_1(z_1), \ldots, f_n(z_n))$ is almost starlike mapping of complex order λ on B^n. By Theorem 2.2 , we deduce that $F : B^{n+1} \to \mathbb{C}^{n+1}$ given by

$$F(z) = \Big(f_1(z_1), \ldots, f_n(z_n), z_{n+1} \prod_{j=1}^{n} [f_j'(z_j)]^{\frac{1}{n+1}}\Big),\ z = (z', z_{n+1}) \in B^n$$

is almost starlike of complex order λ on B^{n+1}. For example, the mapping

$$F(z) = \Big(z_1\Big(1 + \frac{1+\lambda}{1-\lambda} z_1\Big)^{-\frac{2}{1+\lambda}}, \ldots, z_n\Big(1 + \frac{1+\lambda}{1-\lambda} z_n\Big)^{-\frac{2}{1+\lambda}},$$
$$z_{n+1} \prod_{j=1}^{n} \Big[(1 - z_j)\Big(1 + \frac{1+\lambda}{1-\lambda} z_j\Big)\Big]^{-\frac{3+\lambda}{(1+n)(1+\lambda)}}\Big)$$

is almost starlike of complex order λ on B^{n+1}.

We close this section with the compactness result of the set $\Phi_n[S^*_\lambda(B^n)]$. This result is a direct consequence of Theorem 2.2 and Corollary 1.10.

THEOREM 2.7. *The set* $\Phi_n[S^*_\lambda(B^n)]$ *is compact.*

Proof. Since $\Phi_n[S^*_\lambda(B^n)]$ is a subset of $S^*_\lambda(B^{n+1})$, we deduce in view of Corollary 1.10 that $\Phi_n[S^*_\lambda(B^n)]$ is locally uniformly bounded on B^{n+1}. It remains to prove that $\Phi_n[S^*_\lambda(B^n)]$ is closed. To this end, let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $\Phi_n[S^*_\lambda(B^n)]$ which converges locally uniformly on B^{n+1} to a mapping F as $k \to \infty$. Also let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $S^*_\lambda(B^n)$ be such that $F_k = \Phi_n(f_k)$, $k \in \mathbb{N}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is a locally uniformly bounded sequence on B^n, there is a subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$ of $\{f_k\}_{k\in\mathbb{N}}$ which converges locally uniformly on B^n to a mapping f. Since $S^*_\lambda(B^n)$ is a compact set, we deduce that $f \in S^*_\lambda(B^n)$. Also it is easy to see that the subsequence $\{\Phi_n(f_{k_p})\}_{p\in\mathbb{N}}$ converges locally uniformly on B^{n+1} to $\Phi_n(f)$, and thus we must have $F = \Phi_n(f)$. Hence $F \in \Phi_n[S^*_\lambda(B^n)]$, as desired. This completes the proof. □

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University of Petroşani
Department of Mathematics
Str. Universităţii Nr. 20
33200 Petroşani, Romania
E-mail: madalina@upet.ro