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ON PERFECTLY α-IRRESOLUTE FUNCTIONS

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Abstract. In this paper, some results concerning properties of perfectly α-irresolute functions and their relationships with other types of functions between topological spaces are obtained. Some new characterizations of connected spaces are given by using perfectly α-irresolute functions, and behaviour of some α-separation axioms under perfectly α-irresoluteness are investigated.

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Key words. α-open, perfectly α-irresolute function, α-irresolute function

1. INTRODUCTION

In 1965 Njastad [17] introduced a weak form of open sets called α-open sets in topological spaces. Since the advent of this notion, some strong and weak forms of continuity have been introduced during the last years. Two types of these continuities are α-continuity introduced by Mashhour et al. [13] and perfectly continuity introduced by Noiri [18]. On the other hand, in 1980, Maheshwari and Thakur [9] introduced and investigated the notion of α-irresoluteness of functions between topological spaces. After then some strong forms of this notions are introduced by Lo Faro [8], Navalagi [14] and recently Zorlutuna [22] as strongly α-irresoluteness, completely α-irresoluteness, and perfectly α-irresoluteness, respectively. This paper is devoted to the investigation of a class of functions called perfectly α-irresolute functions, which are stronger than α-irresolute functions. In section 3, characterizations and fundamental properties are given. In section 4, we use perfectly α-irresolute functions as a tool to set new characterizations of connectedness, and some separation axioms are investigated. Section 5 deals with graphs of perfectly α-irresolute functions. In the last section, the relationships with other types of functions are given.

2. PRELIMINARIES

Throughout the present paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated, and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f from a topological space (X, τ) into a topological space (Y, σ). Let A be a subset of a space X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A = \mathrm{Cl}(\mathrm{Int}(A))$). A subset A of a space X is called α-open [17] (resp. preopen [12]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$). The complement of an α-open set is said to be α-closed. The family of all α-open subsets of (X, τ) is denoted by τ^α. It is known that τ^α is a topology for X by Njastad [17]. For a subset of A of (X, τ), the closure of A with respect to τ^α is denoted by τ^α-Cl(A). A space X is said to be locally indiscrete [16] if every open subset of X is closed.

Let us recall the following definitions which we shall require later.

Definition 1. A function $f : X \to Y$ is called perfectly continuous [18] (resp. α-continuous [13]) if $f^{-1}(V)$ is clopen (resp. α-open) in X for every open set V of Y.

Definition 2. A function $f : X \to Y$ is called α-irresolute [9] (resp. contra α-irresolute [3], α-precontinuous [2] if $f^{-1}(V)$ is α-open (resp. α-closed, preopen) in X for every α-open set V of Y.

Definition 3. A function $f : X \to Y$ is called slightly α-continuous [4] $f^{-1}(V)$ is α-open in X for every clopen set V of Y.

3. FUNDAMENTAL PROPERTIES

Definition 4. A function $f : X \to Y$ is said to be perfectly α-irresolute if $f^{-1}(V)$ is clopen in X for every α-open set V of Y.

Theorem 1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the followings are equivalent:*

(1) *f is perfectly α-irresolute;*

(2) *for every α-closed subset F of Y, $f^{-1}(F)$ is clopen in X;*

(3) *$f : (X, \tau) \to (Y, \sigma^\alpha)$ is perfectly continuous.*

Proof. Obvious. □

Remark 1. It is easily shown that every α-open set in a locally indiscrete space is clopen.

Then we have the following theorem. Its proof is clear.

Theorem 2. *A space X is locally indiscrete if and only if the identity map of X is perfectly α-irresolute.*

Lemma 1. [6] *The following properties are equivalent for a subset A of a space X:*

(1) *A is clopen;*

(2) *A is α-closed and α-open;*

(3) *A is α-closed and preopen.*

Theorem 3. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is perfectly α-irresolute;*

(2) *f is contra-α-irresolute and α-irresolute;*

(3) *f is contra-α-irresolute and α-precontinuous.*

Proof. The proof follows immediately from Lemma 1. □

DEFINITION 5. A space X is called strongly α-regular [14] if for any α-closed set $F \subseteq X$ and any point $x \in X - F$, there exist disjoint α-open sets U and V such that $x \in U$ and $F \subseteq V$.

The following theorem gives a characterization of strongly α-regular spaces as an analogous to that in general topology, hence its proof is omitted.

THEOREM 4. *A space (X, τ) is strongly α-regular if and only if for every point x of X and every α-open set V containing x, there exists an α-open set U such that $x \in U \subseteq \tau^{\alpha}\text{-Cl}(U) \subseteq V$.*

THEOREM 5. *Let (Y, σ) be a strongly α-regular space. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(1) *f is perfectly α-irresolute;*
(2) *for every α-open subset V of Y, $f^{-1}(V)$ is regular closed in X;*
(3) *for every α-open subset V of Y, $f^{-1}(V)$ is closed in X;*
(4) *f is contra-α-irresolute.*

Proof. The following implications are obvious: (1)⇒(2)⇒(3) ⇒(4). We show the implication (4)⇒(1). Let x be an arbitrary point of X and V an α-open set of (Y, σ) containing $f(x)$. Since (Y, σ) is strongly α-regular, there exists an α-open set W in (Y, σ) containing $f(x)$ such that $\sigma^{\alpha}\text{-Cl}(W) \subseteq V$. Since f is contra-α -irresolute, there exists an α-open set U containing x such that $f(U) \subseteq \text{Cl}(W)$. Then $f(U) \subseteq \sigma^{\alpha}\text{-Cl}(W) \subseteq V$. Hence, f is α-irresolute. Since f is contra-α -irresolute and α-irresolute, by Theorem 3, f is perfectly α-irresolute. □

THEOREM 6. *A function $f : X \to Y$ is perfectly α-irresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is perfectly α-irresolute.*

Proof. Let V be any α-open set of Y. Then $X \times V$ is an α-open set of $X \times Y$. Since g is perfectly α-irresolute, $f^{-1}(V) = g^{-1}(X \times V)$ is clopen in X. Thus f is perfectly α-irresolute. □

THEOREM 7. *Let A be any subset of X. If $f : X \to Y$ is perfectly α-irresolute, then $f\,|_A\colon A \to Y$ is perfectly α-irresolute.*

Proof. Let V be a α-open subset of Y. Then, $(f\,|_A)^{-1}(V) = f^{-1}(V) \cap A$. Since $f^{-1}(V)$ is clopen, $(f\,|_A)^{-1}(V)$ is clopen in the relative topology of A. □

THEOREM 8. *The following properties hold for functions $f : X \to Y$ and $g : Y \to Z$:*
(a) *If $f : X \to Y$ is perfectly α-irresolute and $g : Y \to Z$ is α-irresolute, then $g \circ f : X \to Z$ is perfectly α-irresolute.*
(b) *If $f : X \to Y$ is perfectly α-irresolute and $g : Y \to Z$ is α-continuous, then $g \circ f : X \to Z$ is perfectly continuous.*

(c) *If $f : X \to Y$ is slightly α-continuous and $g : Y \to Z$ is perfectly α-irresolute, then $g \circ f : X \to Z$ is α-irresolute.*

(d) *If $f : X \to Y$ is perfectly α-irresolute and $g : Y \to Z$ is contra α-irresolute, then $g \circ f : X \to Z$ is perfectly α-irresolute.*

Proof. They follow from definitions. □

THEOREM 9. *If $f : X \to Y$ is a surjective open and closed function and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is perfectly α-irresolute function, then g is perfectly α-irresolute function.*

Proof. Let V be any α-open set in Z. Since $g \circ f$ is perfectly α-irresolute, $(g \circ f)^{-1}(V)$ is clopen in X. Since f is surjective open and closed, $f((g \circ f)^{-1}(V)) = f((f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is clopen in Y. Therefore, g is perfectly α-irresolute. □

Let $\{X_\lambda : \lambda \in \Lambda\}$ and $\{Y_\lambda : \lambda \in \Lambda\}$ be any two families of spaces with same index set Λ. For each $\lambda \in \Lambda$, let, $f_\lambda : X_\lambda \to Y_\lambda$ be a function. The product space $\Pi\{X_\lambda : \lambda \in \Lambda\}$ is denoted by ΠX_λ and the product function $\Pi f_\lambda : \Pi X_\lambda \to \Pi Y_\lambda$ is simply denoted by $f : \Pi X_\lambda \to \Pi Y_\lambda$.

THEOREM 10. *Let $\{Y_\lambda : \lambda \in \Lambda\}$ be a family of spaces. If a function $f : X \to \Pi Y_\lambda$ is perfectly α-irresolute, then $P_\lambda \circ f : X \to Y_\lambda$ is perfectly α-irresolute for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.*

Proof. This follows from Theorem 8 because every open continuous surjection P_λ is α-irresolute. □

THEOREM 11. *If the function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is perfectly α-irresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is perfectly α-irresolute for each $\lambda \in \Lambda$.*

Proof. Let $\lambda_0 \in \Lambda$ be an arbitrary fixed index and let V_{λ_0} be any α-open set of Y_{λ_0}. Then $\Pi Y_\mu \times V_{\lambda_0}$ is α-open in ΠY_λ, where $\lambda_0 \neq \mu \in \Lambda$. Since f is perfectly α-irresolute, then $f^{-1}(\Pi Y_\mu \times V_{\lambda_0}) = \Pi X_\mu \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is clopen in ΠX_λ and hence $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is clopen in X_{λ_0}. This implies that f_{λ_0} is perfectly α-irresolute. □

In [3], Caldas et al. defined a function $f : X \to Y$ to be perfectly contra α-irresolute if $f^{-1}(V)$ is an α-open and α-closed set of X for each α-open set of Y and proved that a function $f : X \to Y$ is perfectly contra α -irresolute if and only if $f^{-1}(V)$ is clopen set of X for each α-open set of Y. Thus, perfectly α-irresoluteness is equivalent to perfectly contra α-irresoluteness.

DEFINITION 6. A filter base $\mathcal{F}$ is said to be α-convergent [7] (resp. c-convergent [6]) to a point x in X, if for any α-open (resp. closed) set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset U$.

THEOREM 12. *If a function $f : X \to Y$ is perfectly α-irresolute, then for each point $x \in X$ and each filter base $\mathcal{F}$ in X c-converging to x, the filter base $f(\mathcal{F})$ is α-convergent to $f(x)$.*

Proof. Suppose that $x \in X$ and $\mathcal{F}$ is any filterbase in X which c-converges to x. Let V be any α-open set of Y with $f(x) \in V$. Since f is perfectly α-irresolute, $f^{-1}(V)$ is clopen in X and $x \in f^{-1}(V)$. Since $\mathcal{F}$ is c-convergent to x, there exists $B \in \mathcal{F}$ such that $B \subset f^{-1}(V)$. Therefore, we have $f(B) \subset V$. This shows that $f(\mathcal{F})$ is α-convergent to $f(x)$. □

4. FURTHER PROPERTIES

DEFINITION 7. [11]A space (X, τ) is said to be α-T_0 if (X, τ^{α}) is T_0.

THEOREM 13. *Let $f : X \to Y$ be a perfectly α-irresolute function from a space X into an α-T_0-space Y. Then f is constant on each component of X.*

Proof. Let a and b be two points of X that lie in the same component of X. Assume that $f(a) \neq f(b)$. Since Y is α-T_0- space, there exists an α-open set U containing say $f(a)$ but not $f(b)$. By perfectly α-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing a and b, respectively. This is a contradiction in view of the fact that b belongs to the component of a. Hence the result. □

COROLLARY 1. *Let $f : X \to Y$ be a perfectly α-irresolute function and Y be an α-T_0-space. If A is non-empty connected subset of X, then $f(A)$ is a single point.*

THEOREM 14. *A space X is connected if and only if every perfectly α-irresolute function from space X into any α-T_0-space Y is constant.*

Proof. We only prove the "if" part. Suppose that X is not connected. Then there exists a proper non-empty clopen subset A of X. Let $Y = \{x, y\}$ and σ be discrete topology on Y. Let $f : X \to Y$ be a function such that $f(A) = \{x\}$ and $f(X-A) = \{y\}$. Then f is non-constant, perfectly α-irresolute and Y is α-T_0, which is a contradiction by Theorem 13. Hence X must be connected. □

THEOREM 15. *If $f : (X, \tau) \to (Y, \sigma)$ is a perfectly α-irresolute surjection and if (X, τ) is a connected space, then (Y, σ^{α}) is an indiscrete space.*

Proof. Suppose that (Y, σ^{α}) is not indiscrete. Let A be a proper non-empty α-open subset of Y. Then $f^{-1}(A)$ is a proper non-empty clopen subset of X. This is a contradiction with the fact that (X, τ) is connected. □

By using Lemma 1 and Theorem 15, we have the following corollary.

COROLLARY 2. *If $f : X \to Y$ is perfectly α-irresolute surjection and X is connected, then Y is connected.*

Note that the topological space consisting of two points with the discrete topology is usually denoted by 2.

THEOREM 16. *The following are equivalent for a topological space X:*
(1) *X is connected;*

(2) *Every perfectly α-irresolute function from X into an α-T_0 space Y is constant;*

(3) *Every perfectly α-irresolute function $f : X \to 2$ is constant;*

(4) *There is no perfectly α-irresolute function $f : X \to 2$ is surjective.*

Proof. (1)$\Leftrightarrow$(2): Theorem 14.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are obvious.

(4)$\Rightarrow$(1): Suppose that X is not connected. Then there exists a nonvoid proper clopen open subset W of X. We define the function $f : X \to (\{a, b\}, \tau_{discrete})$ as $f(x) = a$ for $x \in W$ and $f(x) = b$ for $x \in X - W$. The function f is perfectly α-irresolute and surjective. This is a contradiction with the hypothesis (4). Hence, X is connected if there is no perfectly α-irresolute function $f : X \to 2$ is surjective. □

Definition 8. A space X is said to be ultra Hausdorff [20] (resp. α-T_2 or α-Hausdorff [9]) if every two distinct points of X can be separated by disjoint clopen (resp. α-open) sets.

Theorem 17. *If $f : X \to Y$ is a perfectly α-irresolute injection and Y is α-T_0, then X is ultra Hausdorff.*

Proof. Let x_1, $x_2 \in X$ and $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$. Since Y is α-T_0, there exists an α-open set U containing say $f(x_1)$ but not $f(x_2)$. By perfectly α-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing x_1 and x_2, respectively. Thus X is ultra Hausdorff. □

The quasi-topology denoted by τ_q on X is the topology having as base the clopen subsets of (X, τ). A subset A of X is called quasi-open if $A \in \tau_q$. The complement of a quasi-open set is called quasi-closed. For a given topological space (X, τ), the space (X, τ_q) is called the ultra regular kernel of X [20].

Theorem 18. *Let Y be an α-T_2 space.*

(1) *If f, $g : X \to Y$ are perfectly α-irresolute functions, then the set $A = \{x \in X : f(x) = g(x)\}$ is quasi-closed in X.*

(2) *If $f : X \to Y$ is perfectly α-irresolute function, then the subset $E = \{(x, y) : f(x) = f(y)\}$ is quasi-closed in $X \times X$.*

Proof. (1) Let $x \notin A$, then $f(x) \neq g(x)$. Since Y is α-T_2, there exist α-open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $g(x) \in V_2$ and $V_1 \cap V_2 = \varnothing$. Since f and g are perfectly α-irresolute, $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are clopen sets. Put $U = f^{-1}(V_1) \cap g^{-1}(V_2)$. Then U is a clopen set containing x and $U \cap A = \varnothing$. Then every point of $X - A$ has a clopen neighbourhood disjoint from A. Hence $X - A$ is a union of clopen sets or equivalently A is quasi-closed.

(2) Let $(x, y) \notin E$. Then $f(x) \neq f(y)$. Since Y is α-T_2, there exist α-open sets V_1 and V_2 containing $f(x)$ and $f(y)$ respectively, such that $V_1 \cap V_2 = \emptyset$ Since f is perfectly α-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets. Put $U = f^{-1}(V_1) \times f^{-1}(V_2)$. Then U is a clopen set containing (x, y) and $U \cap E = \varnothing$. Thus we have that $(x, y) \notin \tau_q$-Cl(E). This completes the proof. □

Definition 9. A space X is called α-regular [5] if for any closed set $F \subseteq X$ and any point $x \in X - F$, there exist disjoint α-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 19. [5]*A space X is α-regular if and only if for every point x of X and every open set V containing x, there exists an α-open set U such that $x \in U \subseteq \tau^{\alpha}$-$\mathrm{Cl}(U) \subseteq V$.*

Definition 10. [13]A function $f : X \to Y$ is called:
(a) α-closed if for each closed subset K of X, $f(K)$ is α-closed in Y.
(b) α-open if for each open subset U of X, $f(U)$ is α-open in Y.

Theorem 20. *A function $f : X \to Y$ is α-closed if and only if for each subset S of Y and for each open subset U of X with $f^{-1}(S) \subseteq U$, there exists an α-open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. ($\Rightarrow$): Suppose that f is α-closed. Let $S \subseteq Y$ be any set and U be an open subset of X with $f^{-1}(S) \subseteq U$. Then $Y - f(X - U)$ is an α-open set in Y. Set $V = Y - f(X - U)$. Then $S \subseteq V$ and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - f^{-1}(f(X - U)) \subseteq U$.
($\Leftarrow$): Let K be any closed subset of X and $S = Y - f(K)$. Then $f^{-1}(S) \subseteq X - K$. By hypothesis, there exists an α-open set V in Y containing S such that $f^{-}(V) \subseteq X - K$. Then, we have $K \subseteq X - f^{-1}(V)$ and $Y - V = f(K)$. Since, $Y - V$ is α-closed, $f(K)$ is α-closed and thus f is an α-closed map. $\square$

Theorem 21. *If f is a perfectly α-irresolute, α-open injective function from a regular space X onto a space Y, then Y is α-regular.*

Proof. Let F be a closed set in Y with $y \notin F$. Take $y = f(x)$. Since F is also α-closed and f is perfectly α-irresolute, $f^{-1}(F)$ is clopen and so closed set in X and $x \notin f^{-1}(F)$. By the regularity of X, there exist disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$. We obtain that $y = f(x) \in f(U)$ and $F \subseteq f(V)$ such that $f(U)$ and $f(V)$ are disjoint α-open sets. Thus Y is α-regular. $\square$

Theorem 22. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous, α-open, α-closed surjection. If (X, τ) is regular, then (Y, σ) is α-regular.*

Proof. Let $y \in Y$ and V be an open set in Y with $y \in V$. Take $y = f(x)$. Since f is continuous and (X, τ) is regular, there exists an open set U such that $x \in U \subseteq \mathrm{Cl}(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(\mathrm{Cl}(U)) \subseteq V$. By assumptions, $f(U)$ is α-open and $f(\mathrm{Cl}(U))$ is α-closed set in (Y, σ). Therefore, we have $y \in f(U) \subseteq \sigma^{\alpha}$-$f(U) \subseteq V$. This shows that (Y, σ) is α-regular. $\square$

Definition 11. A space X is called α-normal [15] if for every pair of disjoint closed subsets F_1 and F_2 of X, there exist disjoint α-open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Theorem 23. *If f is a perfectly α-irresolute, α-open injective function from a normal space X onto a space Y, then Y is α-normal.*

Proof. Let F_1 and F_2 be disjoint closed sets in Y. Since F_1 and F_2 are also α-closed and f is perfectly α-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint clopen and so closed sets in X. By normality of X, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. We obtain that $F_1 \subseteq f(U)$ and $F_2 \subseteq f(V)$ such that $f(U)$ and $f(V)$ are disjoint α-open sets. Thus Y is α-normal. □

THEOREM 24. *Let $f : X \to Y$ be a continuous α-closed surjection. If X is a normal space, then Y is normal.*

Proof. Let F_1 and F_2 be disjoint closed sets of Y. Since f is continuous and X is normal, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. By Theorem 20, there exist α-open sets G and H such that $F_1 \subseteq G$, $F_2 \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \varnothing$ and hence $G \cap H = \varnothing$. Since G and H are disjoint α-open sets, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H))) = \varnothing$ and $F_1 \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G)))$ and $F_2 \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H)))$. Thus Y is normal. □

COROLLARY 3. *Let $f : X \to Y$ be a continuous α-closed surjection. If X is a normal space, then Y is α-normal.*

DEFINITION 12. A space (X, τ) is called mildly compact [20] (resp. α-compact [10]) if every clopen (resp. α-open) cover of X has a finite subcover.

THEOREM 25. *Let $f : X \to Y$ be a perfectly α-irresolute surjection. If X is mildly compact, then Y is α-compact.*

Proof. Let $f : X \to Y$ be perfectly α-irresolute and let X be mildly compact. Let $\{V_i\}_{i\in I}$ be an α-open cover of Y. Since f is perfectly α-irresolute, $\{f^{-1}(V_i)\}_{i\in I}$ is a clopen cover of X so there is a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(V_i)$. Therefore, $Y = \cup_{i\in I_0} V_i$ since f is surjective. Thus Y is α-compact. □

5. QUASI-α-CLOSEDNESS OF GRAPHS

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

DEFINITION 13. The graph $G(f)$ of a function $f : X \to Y$ is said to be quasi-α-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an α-open set V containing y such that $(U \times V) \cap G(f) = \varnothing$.

LEMMA 2. *The graph $G(f)$ of a function $f : X \to Y$ is quasi-α-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an α-open set V containing y such that $f(U) \cap V = \varnothing$.*

Proof. It is obvious. □

THEOREM 26. *If $f : X \to Y$ is perfectly α-irresolute and Y is α-T_2, then the graph $G(f)$ of f is quasi-α-closed in $X \times Y$.*

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is α-T_2, there exist α-open sets V_1 and V_2 containing $f(x)$ and y , respectively, such that $V_1 \cap V_2 = \varnothing$. Since f is perfectly α-irresolute, $f^{-1}(V_1)$ is clopen set containing x . Set $U = f^{-1}(V_1)$. Therefore, $f(U) \cap V_2 = \varnothing$ and $G(f)$ is quasi-α-closed in $X \times Y$. □

Definition 14. A subset A of a space X is said to be mildly compact (resp. α-compact [19]) relative to X if for every cover $\{V_\alpha : \alpha \in I\}$ of A by clopen (resp. α-open) sets of X, there exists a finite subset I_0 of I such that $A \subseteq \cup\{V_\alpha : \alpha \in I_0\}$.

Theorem 27. *If a function $f : (X, \tau) \to (Y, \sigma)$ has a quasi-α-closed graph, then $f(K)$ is α-closed in Y for each subset K which is mildly compact relative to X.*

Proof. Suppose $y \notin f(K)$. Then for each $x \in K$, we have $(x, y) \notin G(f)$ and by Lemma 2, there exist a clopen set U_x containing x and an α-open set V_x of Y containing y such that $f(U_x) \cap V_x = \varnothing$. The family of $\{U_x : x \in K\}$ is a cover of K by clopen sets of X and there exists a finite subset K_* of K such that $K \subseteq \cup_{x \in K_*} U_x$. Set $V = \cap_{x \in K_*} V_x$, then V is an α-open set containing y and $f(K) \cap V \subseteq \cup_{x \in K_*} f(U_x) \cap V = \varnothing$. Therefore, we have, $V \cap f(K) = \varnothing$ and hence $y \notin \sigma^\alpha\text{-Cl}(f(K))$. This shows that $f(K)$ is α-closed in (Y, σ). □

Corollary 4. *If $f : X \to Y$ is a perfectly α-irresolute function and Y is α-T_2, then $f(K)$ is α-closed in Y for each subset K which is mildly compact relative to X.*

Theorem 28. *If a function $f : X \to Y$ has a quasi-α-closed graph, $f^{-1}(K)$ is quasi-closed in X for every subset K which is α-compact relative to Y.*

Proof. Let K be α-compact relative to Y and $x \notin f^{-1}(K)$. For each $y \in K$, we have $(x, y) \in (X \times Y) - G(f)$ and there exist a clopen set U_y containing x in X and an α-open set V_y containing y in Y such that $f(U_y) \cap V_y = \varnothing$. The family $\{V_y : y \in K\}$ is an cover of K by α-open sets of Y and there exists a finite number of points, say $y_1, y_2, ..., y_n$ of K such that $K \subseteq \cup\{V_{y_i} : i = 1, 2, ..., n\}$. Set $U = \cap\{U_{y_i} : i = 1, 2, ..., n\}$. Then U is a clopen subset of X containing x and $f(U) \cap K = \varnothing$; hence $U \cap f^{-1}(K) = \varnothing$. This shows that $f^{-1}(K)$ is quasi-closed in X. □

Theorem 29. *Let $f : (X, \tau) \to (Y, \sigma)$ have a quasi-α-closed graph. If f is injective, then (X, τ_q) is T_1.*

Proof. Let x and y be any two distinct points of X. Then, we have $f(x) \neq f(y)$ and so $(x, f(y)) \in (X \times Y) - G(f)$. By the quasi-$\alpha$-closedness of the graph $G(f)$, there exist a clopen set U of X and an α-open set of Y such that $(x, f(y)) \in U \times V$ and $(U \times V) \cap G(f) = \varnothing$. Then, we have $f(U) \cap V = \varnothing$, hence $U \cap f^{-1}(V) = \varnothing$. Therefore, we have $y \notin U$. This implies that (X, τ_q) is T_1. □

Theorem 30. *If $f : X \to Y$ is a perfectly α-irresolute injection with a quasi-α-closed graph, then X is ultra Hausdorff.*

Proof. Let x and y be distinct points in X. Then $f(x) \neq f(y)$ and so $(x, f(y)) \notin G(f)$. Therefore, there exist a clopen set U of X and an α-open set V of Y such that $(x, f(y)) \in U \times V$ and $(U \times V) \cap G(f) = \varnothing$. On the other hand, $y \in f^{-1}(V)$ and $f^{-1}(V)$ is clopen set in X because f is perfectly α -irresolute. Since $(U \times V) \cap G(f) = \emptyset$, $U \cap f^{-1}(V) = \emptyset$. Thus X is ultra Hausdorff. □

Theorem 31. *Let $f : (X, \tau) \to (Y, \sigma)$ has a quasi-α-closed graph. If f is a surjective α-open function, then (Y, σ) is α-T_2.*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. By the quasi-$\alpha$-closedness of the graph $G(f)$, there exist a clopen set U of X and an α-open set V of Y such that $(x, y_2) \in (U \times V)$ and $(U \times V) \cap G(f) = \varnothing$. Then we have $f(U) \cap V = \varnothing$. Since f is α-open, then $f(U)$ is α-open such that $f(x) = y_1 \in f(U)$. This implies that (Y, σ) is α-T_2. □

Definition 15. A topological space X is said to be hyperconnected [21] if every pair nonempty open sets of X has nonempty intersection.

Theorem 32. *Let X be hyperconnected. If $f : X \to Y$ is a perfectly α-irresolute function with a quasi-α-closed graph, then f is constant.*

Proof. Suppose that f is not constant. Then there exist two points x and y of X such that $f(x) \neq f(y)$. Then we have $(x, f(y)) \notin G(f)$. Since $G(f)$ is quasi-α-closed, there exist a clopen set U of X and an α-open set V of Y such that $(x, f(y)) \in U \times V$ and $(U \times V) \cap G(f) = \varnothing$. Then we have $f(U) \cap V = \varnothing$. Therefore, we have $U \cap f^{-1}(V) = \emptyset$. This is a contradiction with the hyperconnectedness of X, since $f^{-1}(V)$ is non-empty open set in X. □

6. RELATIONSHIPS

In this section we investigate relationships among perfectly α-irresolute and other related functions.

Definition 16. A function $f : X \to Y$ is called:
(1) completely continuous [1] if $f^{-1}(V)$ is regular open in X for every open set V of Y.
(2) completely α-irresolute [14] if $f^{-1}(V)$ is regular open in X for every α-open set V of Y.
(3) strongly α-irresolute [8] if $f^{-1}(V)$ is open in X for every α-open set V of Y.

From the definitions, we have the following relationships:

$$\begin{array}{ccccccc}
\text{perf. cont.} & \Rightarrow & \text{comp. cont.} & \Rightarrow & \text{cont.} & \Rightarrow & \alpha\text{-cont.} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{perf. } \alpha\text{-irr.} & \Rightarrow & \text{comp. } \alpha\text{ -irr.} & \Rightarrow & \text{str. } \alpha\text{-irr.} & \Rightarrow & \alpha\text{-irr.} \\
\Downarrow & & & & & & \Downarrow \\
\text{contra } \alpha\text{-irr.} & & \Rightarrow & & \Rightarrow & & \text{slightly } \alpha\text{-cont.}
\end{array}$$

None of these implications is reversible as the following examples show.

Example 1. Let $X = \{a, b, c, d\}$, $\sigma = \{\emptyset, X, \{b\}, \{b, d\}, \{a, b, c\}\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$. Define the function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(d) = a$, $f(b) = b$, $f(c) = c$. Then f is perfectly continuous, but it is not perfectly α-irresolute since $f^{-1}(\{b, c\}) = \{b, c\}$ is not clopen in (X, τ) for $\{b, c\} \in \sigma^\alpha$.

Example 2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f(a) = a$, $f(b) = f(c) = c$. Then f is completely α-irresolute, but it is not perfectly α -irresolute since $\tau = \tau^\alpha$ and $f^{-1}(\{a\}) = \{a\}$ is not clopen in (X, τ) for $\{a\} \in \tau^\alpha$.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(b) = c$, $f(c) = b$. Then f is slightly α-continuous, but it is not α-irresolute since $\tau = \tau^\alpha$ and $f^{-1}(\{b\}) = \{c\} \notin \tau^\alpha$ for $\{a\} \in \sigma^\alpha$.

The other implications are not reversible as shown in related papers [3, 9, 13]. The next theorem is an immediate consequence of Remark 1.

Theorem 33. *Let (Y, σ) be a locally indiscrete space. For a function $f : (X, \tau) \to (Y, \sigma)$, all notions indicated in the upper diagram are equivalent.*

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