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NEW STABLE ELEMENTS IN HOCHSCHILD COHOMOLOGY ALGEBRA

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Abstract. We give a similar result to the embedding of the ordinary cohomology ring of a group into the subalgebra of stable elements in the Hochschild cohomology ring of the group algebra.We take in this case the ordinary cohomology of the centralizer of a representative of a conjugacy class in G, which also embeds into the Hochschild cohomology ring of the group algebra.

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1. PRELIMINARIES

In the next sections we consider G a finite group, $\{x_i \mid i \in \{1, \ldots, r\}\}$ a system of representatives of conjugacy classes of G with the representative x_i, for $i \in \{1, \ldots, r\}$ a fixed indices and R a commutative ring. By [2] we know that there is the additive decomposition of the Hochschild cohomology $\mathrm{HH}^n(RG) \simeq \bigoplus_{i=1}^r \mathrm{H}^n(C_G(x_i), R)$. Thus there are the canonical injections of R-modules $\mathrm{H}^n(C_G(x_i), R) \hookrightarrow \mathrm{HH}^n(RG)$, which extends to the injective homomorphisms of graduates R-algebras

$$\mathrm{H}^*(C_G(x_i), R) \hookrightarrow \mathrm{HH}^*(RG).$$

These homomorphisms are explicitly described in [4] in a more general case. We denote by $K(RG)$ the homotopy category and by $C(RG)$ the category of complexes of finite generated RG-modules.

In Section 2 we define explicitly the injective R-algebras homomorphism $\gamma_{x_i}^G$ from $\mathrm{H}^*(C_G(x_i), R)$ to $\mathrm{HH}^*(RG)$, which if $x_i = 1$,the unity of G, is exactly the "diagonal induction" from the ordinary cohomology $\mathrm{H}^*(G, R)$ to the Hochschild cohomology, from [3]. Next we show that the usual transfer between the cohomology rings $\mathrm{H}^*(C_H(x_i), R)$ and $\mathrm{H}^*(C_G(x_i), R)$ is compatible with $\gamma_{x_i}^G$, where H is a subgroup of G. The compatibility of restriction between this two cohomology rings is not true in general, excepting some particular groups G containing subgroups H.

In Section 3 using the same hypothesis from compatibility of restriction with $\gamma_{x_i}^G$, we prove the embedding of the cohomology algebra $\mathrm{H}^*(C_G(x_i), R)$

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into the algebra of stable elements $\mathrm{HH}^*_M(RG)$, where $M = {}_{RG}RG_{RH}$ is the regular $RG-RH$-bimodule.

Definition 1. The ordinary cohomology of G with coefficients in R is the graduate R-algebra $\mathrm{H}^*(G,R) = \mathrm{Ext}^*_{RG}(R,R)$, where R is the RG-module with trivial action. By standard results in homological algebra we use the isomorphism:

$$\mathrm{H}^n(G,R) \cong \mathrm{Hom}_{K(RG)}(\mathcal{P}_R, \mathcal{P}_R[n]),$$

where $\mathcal{P}_R$ is a projective resolution of R as RG-module with trivial action.

Remark 1. (a) If H is a subgroup of G then $\mathrm{Res}^G_H \mathcal{P}_R$ is a projective resolution of R as RH-module. Thus any element $[\tau] \in \mathrm{H}^n(H,R)$ can be represented by a chain map $\tau : \mathrm{Res}^G_H\mathcal{P}_R \longrightarrow \mathrm{Res}^G_H\mathcal{P}_R[n]$.

(b) For $\Delta G = \{(g,g) \mid g \in G\}$ the diagonal subgroup of $G \times G$, there is the isomorphism $\mathrm{Ind}^{G\times G}_{\Delta G} R \cong RG$. We consider from now $\mathrm{Ind}^{G\times G}_{\Delta G}\mathcal{P}_R$ a projective resolution of RG as $RG-RG$ bimodule (or $R[G \times G]$-module).

(c) By [2] we have: $\mathrm{HH}^*(RG) = \mathrm{Ext}^*_{RG\otimes RG^{\mathrm{op}}}(RG) \cong \mathrm{Ext}^*_{R[G\times G]}(RG)$. Similar to Definition 1 we work with:

$$\mathrm{HH}^n(RG) = \mathrm{Hom}_{K(RG)}(\mathcal{P}_{RG}, \mathcal{P}_{RG}[n]),$$

where $\mathcal{P}_{RG}$ is a projective resolution of RG as $R[G \times G]$-modules. By (b) we consider an element $[\tau] \in \mathrm{HH}^n(RG)$ represented by a chain map $\tau : \mathrm{Ind}^{G\times G}_{\Delta G}\mathcal{P}_R \longrightarrow \mathrm{Ind}^{G\times G}_{\Delta G}\mathcal{P}_R[n]$.

From group theory we recall the following result:

Remark 2. If $K \leq H \leq G$ subgroups, $[G/H]$ a system of representatives of left cosets of H in G, $[H/K]$ a system of representatives of left cosets of K in H then there is a system of representatives of left cosets of K in G such that:

$$[G/K] = \{xy \mid x \in [G/H], y \in [H/K]\}.$$

By [3, Definition 2.9] if A, B are R symmetric algebras and X is a bounded complex of $A-B$-bimodules, projective as left and right modules, we may define the transfer associated to X denoted t_X. In the definition of t_X from $\mathrm{HH}^*(B)$ to $\mathrm{HH}^*(A)$ we use the adjunctions maps. From [3, Example 2.6] we explicitly give the definition of the adjunctions maps in the case of $M = (RG)_H$ considered as $RG-RH$- bimodule, obtained by restriction of the regular ${}_{RG}RG_{RG}$-bimodule. The dual M^* is ${}_{RH}RG_{RG}$ as $RH-RG$-bimodule. We consider $\mathrm{Ind}^{H\times H}_{\Delta H} R$ as $RH-RH$-bimodule by

$$h_1 \cdot [(x,y) \otimes_{R\Delta H} 1_R] \cdot h_2 = (h_1 x, h_2^{-1} y) \otimes_{R\Delta H} 1_R,$$

where $h_1, h_2, x, y \in H$.

Remark 3. By [3, Example 2.6] we know that for $a \in RG$:

$$\varepsilon_{M^*} : RG \longrightarrow RG \otimes_{RH} RH \otimes_{RH} RG, \varepsilon_{M^*}(a) = \sum_{g\in[G/H]} ag \otimes_{RH} 1 \otimes_{RH} g^{-1}.$$

Using the isomorphisms from Remark 1 (b) we can define:

$$\varepsilon_{M^*} : \mathrm{Ind}_{\Delta G}^{G\times G} R \longrightarrow RG \otimes_{RH} \mathrm{Ind}_{\Delta H}^{H\times H} R \otimes_{RH} RG,$$

$$\varepsilon_{M^*}((x,y)\otimes_{R\Delta G} 1_R) = \sum_{g\in[G/H]} xg \otimes_{RH} [(1,1)\otimes_{R\Delta H} 1_R] \otimes_{RH} g^{-1}y^{-1},$$

where $(x,y)\otimes_{R\Delta G} 1_R \in \mathrm{Ind}_{\Delta G}^{G\times G} R$. Similarly:

$$\eta_M : RG \otimes_{RH} \mathrm{Ind}_{\Delta H}^{H\times H} R \otimes_{RH} RG \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G} R,$$

$$\eta_M(a \otimes_{RH} [(x,y)\otimes_{R\Delta H} 1_R] \otimes_{RH} b) = (ax, b^{-1}y)\otimes_{R\Delta G} 1_R,$$

where $a,b\in G$ and $x,y\in H$.

Remark 4. ε_{M^*} and η_M lifts to homomorphisms of complexes of $R[G\times G]$-modules:

$$\varepsilon_{M^*} : \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R \longrightarrow RG \otimes_{RH} \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R \otimes_{RH} RG$$

$$\varepsilon_{M^*}((x,y)\otimes_{R\Delta G} z) = \sum_{g\in[G/H]} xg \otimes_{RH} [(1,1)\otimes_{R\Delta H} g^{-1}z] \otimes_{RH} g^{-1}y^{-1},$$

where $x,y\in G$ and $z\in\mathcal{P}_R$.

$$\eta_M : RG \otimes_{RH} \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R \otimes_{RH} RG \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R$$

$$\eta_M(a \otimes_{RH} [(x,y)\otimes_{R\Delta H} z] \otimes_{RH} b) = (ax, b^{-1}y)\otimes_{R\Delta G} z,$$

where $a,b\in G$ and $x,y\in H, z\in\mathcal{P}_R$.

By [3, Definition 2.9] using Remarks 3, 4 we can define the transfer associated to M.

Definition 2. The transfer associated to M is the unique graded linear map

$$t_M : \mathrm{HH}^n(RH) \longrightarrow \mathrm{HH}^n(RG),$$

sending for any $n\geq 0$, the homotopy class $[\tau]$ of a chain map

$$\tau : \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R[n]$$

to the homotopy class $[\eta_M[n]\circ(Id_M \otimes_{RH} \tau \otimes_{RH} Id_{M^*})\circ\varepsilon_{M^*}]$.

Explicitly we have for any element $(x,y)\otimes_{R\Delta G} z \in \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R$:

$$t_M(\tau)((x,y)\otimes_{R\Delta G} z) = \sum_{g\in[G/H]} \eta_M[n](xg\otimes_{RH}\tau((1,1)\otimes_{R\Delta H} g^{-1}z)\otimes_{RH} g^{-1}y^{-1}).$$

Similarly by [3]:

Remark 5. The adjunction maps associated to M and M^*:

$$\varepsilon_M : \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R, \quad \varepsilon_M((x,y)\otimes_{R\Delta H} z) = (x,y)\otimes_{R\Delta G} z,$$

where $x,y\in H$ and $z\in\mathcal{P}_R$. The map

$$\eta_{M^*} : \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta H}^{H\times H} \mathcal{P}_R,$$

is the natural projection mapping $(x,y)\otimes_{R\Delta G} z$ to $(x,y)\otimes_{R\Delta H} z$ if $x\in H$ and $y\in H$ and to 0 if $x\notin H$ or $y\notin H$, where $(x,y)\otimes_{R\Delta G} z\in \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R$.

By [3, Definition 2.9] using Remark 5 we define t_{M^*}:

Definition 3. The transfer associated to M^* is the unique graded linear map:

$$t_{M^*}:\mathrm{HH}^*(RG)\longrightarrow \mathrm{HH}^*(RH),$$

sending for any $n\geq 0$, the homotopy class $[\tau]$ of a chain map

$$\tau:\mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R\longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R[n]$$

to the homotopy class $[\eta_{M^*}[n]\circ\tau\circ\varepsilon_M]$. Explicitly:

$$t_{M^*}(\tau):\mathrm{Ind}_{\Delta H}^{H\times H}\mathcal{P}_R\longrightarrow \mathrm{Ind}_{\Delta H}^{H\times H}\mathcal{P}_R[n],$$

$$t_{M^*}(\tau)((x,y)\otimes_{R\Delta H} z)=(\eta_{M^*}[n]\circ\tau)((x,y)\otimes_{R\Delta G} z).$$

Remark 6. As in [3, Proposition 4.5] there is an injective homomorphisms of R-algebras

$$\delta_G:\mathrm{H}^*(G,R)\longrightarrow \mathrm{HH}^*(RG), \delta_G([\tau])=[\mathrm{Ind}_{\Delta G}^{G\times G}(\tau)],$$

where $[\tau]\in\mathrm{H}^n(G,R)$ corresponding to $\tau:\mathcal{P}_R\longrightarrow\mathcal{P}_R[n]$. Explicitly we have:

$$\mathrm{Ind}_{\Delta G}^{G\times G}(\tau):\mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R\longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R[n],$$

$$\mathrm{Ind}_{\Delta G}^{G\times G}(\tau)((x,y)\otimes_{R\Delta G} z)=(x,y)\otimes_{R\Delta G}\tau(z)$$

where $x,y\in G$ and $z\in\mathcal{P}_R$.

We recall the definition of the transfer map in group cohomology. By [3] we know that:

$$\mathrm{tr}_H^G:\mathrm{H}^*(H,R)\longrightarrow \mathrm{H}^*(G,R),\quad \mathrm{tr}_H^G([\tau]=[\mathrm{Tr}_H^G(\tau)],$$

where $[\tau]\in\mathrm{H}^n(H,R)$ represented by $\tau:\mathrm{Res}_H^G\mathcal{P}_R\longrightarrow\mathrm{Res}_H^G\mathcal{P}_R[n]$. Explicitly, the chain map $\mathrm{Tr}_H^G(\tau)$ is

$$\mathrm{Tr}_H^G(\tau):\mathcal{P}_R\longrightarrow\mathcal{P}_R[n],\quad \mathrm{Tr}_H^G(\tau)(a)=\sum_{g\in[G/H]} g\tau(g^{-1}a),\quad a\in\mathcal{P}_R.$$

2. THE GENERALIZATION OF THE DIAGONAL INDUCTION

From [4] we know that there is an injective ring homomorphism

$$\gamma_i:\mathrm{H}^*(C_G(x_i),R)\longrightarrow \mathrm{H}^*(G,RG),$$

where RG is a G-module by conjugation ($ga=gag^{-1}, g\in G, a\in RG$). γ_i is defined in [4] using cocycles, by the diagram:

$$\mathrm{H}^*(C_G(x_i),R)\xrightarrow{\ \theta_{x_i}\ }\mathrm{H}^*(C_G(x_i),RG)\xrightarrow{\ \mathrm{tr}_{C_G(x_i)}^G\ }\mathrm{H}^*(G,RG)$$

We will restate the definition of this homomorphism, which we denote $\gamma_{x_i}^G$ using the description of ordinary cohomology and Hochschild cohomology by chain map. We also keep in mind the isomorphism from [4]: $\mathrm{H}^*(G, RG) \cong \mathrm{HH}^*(RG)$.

If $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ is represented by the chain map

$$\tau : \mathrm{Res}^G_{C_G(x_i)} \mathcal{P}_R \longrightarrow \mathrm{Res}^G_{C_G(x_i)} \mathcal{P}_R[n]$$

we define the map:

$$\gamma_{x_i}^G(\tau) : \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G} \mathcal{P}_R[n],$$

$$\gamma_{x_i}^G(\tau)((x,y) \otimes_{R\Delta G} z) = (x,y) \sum_{g \in [G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \tau(g^{-1}z),$$

where $x, y \in G, z \in \mathcal{P}_R$.

Proposition 1. *For every chain map τ, the map $\gamma_{x_i}^G(\tau)$ is well defined and is a chain map.*

Proof. First we prove that the definition of $\gamma_{x_i}^G$ is independent of the choice of representatives $[G/C_G(x_i)]$. If A is a set of different representatives then for any $a \in A$ there is $g \in [G/C_G(x_i)]$ such that $a = gb_a$ where $b_a \in C_G(x_i)$. Then:

$$\begin{aligned}\gamma_{x_i}^G(\tau)((x,y) \otimes_{R\Delta G} z) &= (x,y) \sum_{a \in A} (ax_i, a) \otimes_{R\Delta G} \tau(a^{-1}z) \\ &= (x,y) \sum_{g \in [G/C_G(x_i)]} (gb_a x_i, gb_a) \otimes_{R\Delta G} \tau(b_a^{-1} g^{-1} z) \\ &= (x,y) \sum_{g \in [G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \tau(g^{-1}z)\end{aligned}$$

where $x, y \in G, z \in \mathcal{P}_R$.The last equality is clear since $b_a \in C_G(x_i)$ for all $a \in A$.The second part is obvious since τ is a chain map. □

By Proposition 1 we have the following definition:

Definition 4. Let G be a finite group and x_i a representative of a conjugacy class of G. The injective R-algebra homomorphism

$$\gamma_{x_i}^G : \mathrm{H}^*(C_G(x_i), R) \longrightarrow \mathrm{HH}^*(RG)$$

is the unique graduated linear map given by $\gamma_{x_i}^G([\tau]) = [\gamma_{x_i}^G(\tau)]$, where $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ is represented by the chain map

$$\tau : \mathrm{Res}^G_{C_G(x_i)} \mathcal{P}_R \longrightarrow \mathrm{Res}^G_{C_G(x_i)} \mathcal{P}_R[n].$$

If $x_i = 1$ then $C_G(1) = G$ and by Remark 6 we have that $\gamma_1^G = \delta_G$. The next proposition is a generalization of [3, Proposition 4.6], since we obtain it for $x_i = 1$.

PROPOSITION 2. *Let G be a finite group, x_i a representative of a conjugacy class of G, H a subgroup of G such that $x_i \in H$. Then x_i is a representative of a conjugacy class of H, $C_H(x_i) \le C_G(x_i)$ and the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{H}^*(C_H(x_i), R) & \xrightarrow{\mathrm{tr}_{C_H(x_i)}^{C_G(x_i)}} & \mathrm{H}^*(C_G(x_i), R) \\ {\scriptstyle \gamma_{x_i}^H}\downarrow & & \downarrow{\scriptstyle \gamma_{x_i}^G} \\ \mathrm{HH}^*(RH) & \xrightarrow{\quad t_M \quad} & \mathrm{HH}^*(RG) \end{array}$$

Proof. Let $\mathcal{P}_R$ be a projective resolution of R as trivial RG-module, $[\tau] \in \mathrm{H}^n(C_H(x_i), R)$ represented by the chain map

$$\tau : \mathrm{Res}_{C_H(x_i)}^G \mathcal{P}_R \longrightarrow \mathrm{Res}_{C_H(x_i)}^G \mathcal{P}_R[n]$$

and the elements $x, y \in G$, $z \in \mathcal{P}_R$. By Definition 4 we have that $(\gamma_{x_i}^G \circ \mathrm{tr}_{C_H(x_i)}^{C_G(x_i)})(\tau)$ is a chain map representing an element in $\mathrm{HH}^n(RG)$ defined by:

$$\gamma_{x_i}^G(\mathrm{tr}_{C_H(x_i)}^{C_G(x_i)})(\tau)((x,y) \otimes_{R\Delta G} z)$$

$$\begin{aligned} &= (x,y) \sum_{g \in [G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \mathrm{tr}_{C_H(x_i)}^{C_G(x_i)}(\tau)(g^{-1}z) \\ &= (x,y) \sum_{g \in [G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \sum_{h \in [C_G(x_i)/C_H(x_i)]} h\tau(h^{-1}g^{-1}z) \\ &= (x,y) \sum_{g \in [G/C_G(x_i)]} \sum_{h \in [C_G(x_i)/C_H(x_i)]} (gx_ih, gh) \otimes_{R\Delta G} \tau((gh)^{-1}z). \end{aligned}$$

Using Remark 2 and the fact that $hx_i = x_ih$ we obtain that:

$$(\gamma_{x_i}^G \circ \mathrm{tr}_{C_H(x_i)}^{C_G(x_i)})(\tau)((x,y) \otimes_{R\Delta G} z) = (x,y) \sum_{m \in [G/C_H(x_i)]} (mx_i, m) \otimes_{R\Delta G} \tau(m^{-1}z).$$

By Definition 2 and 4, using Remark 4 we have that:

$$t_M(\gamma_{x_i}^H(\tau))((x,y) \otimes_{R\Delta G} z)$$

$$\begin{aligned} &= \sum_{g \in [G/H]} \eta_M[n](xg \otimes_{RH} \gamma_{x_i}^H(\tau)((1,1) \otimes_{R\Delta H} g^{-1}z) \otimes_{RH} g^{-1}y^{-1}) \\ &= \sum_{g \in [G/H]} \eta_M[n](xg \otimes_{RH} \sum_{h \in [H/C_H(x_i)]} (hx_i, h) \otimes_{R\Delta H} \tau(h^{-1}g^{-1}z) \otimes_{RH} g^{-1}y^{-1}) \\ &= \sum_{g \in [G/H]} \sum_{h \in [H/C_H(x_i)]} (xghx_i, ygh) \otimes_{R\Delta G} \tau(h^{-1}g^{-1}z) \\ &= (x,y) \sum_{m \in [G/C_H(x_i)]} (mx_i, m) \otimes_{R\Delta G} \tau(m^{-1}z), \end{aligned}$$

which concludes the proof. □

If $x_i = 1$ we are under the hypothesis of Proposition 2 and we obtain as corollary [3, Proposition 4.6].

Corollary 1. *For H a subgroup of G the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{H}^*(H,R) & \xrightarrow{\mathrm{tr}_H^G} & \mathrm{H}^*(G,R) \\ \downarrow \delta_H & & \downarrow \delta_G \\ \mathrm{HH}^*(RH) & \xrightarrow{t_M} & \mathrm{HH}^*(RG) \end{array}$$

By [3, Proposition 4.7] we know that δ_G is compatible with res_H^G for all H subgroups of G. In our case we have that $\gamma_{x_i}^G$ is compatible with res_H^G for particular groups.

In the next paragraphs we will work under the following situation :

Situation $(*)$. *Let G be a finite group, H a subgroup of G and x_i an element of H, a representative of a G conjugacy class . Suppose that there is a system of representatives of left cosets of $C_H(x_i)$ in H which is also a system of representatives of left cosets of $C_G(x_i)$ in G.*

The question which arise now is, if there are groups G in situation $(*)$? We give next such an example.

Example 1. Let G be the dihedral group of order $4n$ denoted D_{2n}, where n is an odd nonnegative integer:

$$D_{2n} = \langle x, y \mid x^{2n} = 1, y^2 = 1, yxy = x^{-1}\rangle.$$

We give the description of D_{2n}:

$$D_{2n} = \{1, x, x^2, \ldots, x^{2n-1}, y, xy, x^2y, \ldots, x^{2n-1}y\}.$$

We take $x_i = y$ and $H = \{1, y, x^2, x^4, \ldots, x^{2n-2}, x^2y, x^4y, \ldots, x^{2n-2}y\}$ to be a subgroup of G. By regarding the composition table, we have that $C_G(y) = \{1, y, x^n, x^ny\}$ thus $C_H(y) = \{1, y\}$, since n is odd. There are n left cosets of $C_H(y)$ in H: $\{1, y\}$, $\{x^2, x^2y\}, \ldots, \{x^{2n-2}, x^{2n-2}y\}$ which are included, respectively in the n left cosets of $C_G(y)$ in G: $\{1, y, x^n, x^ny\}$, $\{x, xy, x^{n+1}, x^{n+1}y\}$, $\{x^2, x^2y, x^{n+2}, x^{n+2}y\}, \ldots, \{x^{n-1}, x^{n-1}y, x^{2n-1}, x^{2n-1}y\}$. Obviously if we choose the system $[H/C_H(y)] = \{1, x^2, x^4, \ldots, x^{2n-2}\}$ we see that this is also a system in G, since n is odd. This system will be denoted by $[G/C_G(y)]$.

Moreover, as we can remark by Example 1 we have the following lemma.

Lemma 1. *Under the situation $(*)$ it is true that any system of representatives of left cosets of $C_H(x_i)$ in H is a system of representatives of left cosets of $C_G(x_i)$ in G.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be a system of representatives of left cosets of $C_H(x_i)$ in H which is a system of representatives of left cosets of $C_G(x_i)$ in G. Let $\{b_1, b_2, \ldots, b_n\}$ be a different system of representatives of left cosets of $C_H(x_i)$ in H. Since for any j, k from 1 to n with i different form j we have $b_k^{-1}b_j \notin C_H(x_i)$ it follows that $b_k^{-1}b_j \notin C_G(x_i)$ thus they represent different classes in G. If b_j corresponds to a_j then $b_j^{-1}a_j \in C_H(x_i)$, thus $b_j^{-1}a_j \in C_G(x_i)$, which concludes the proof. □

PROPOSITION 3. *Under the situation* $(*)$ *the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{H}^*(C_G(x_i), R) & \xrightarrow{\mathrm{res}^{C_G(x_i)}_{C_H(x_i)}} & \mathrm{H}^*(C_H(x_i), R) \\ \downarrow{\gamma^G_{x_i}} & & \downarrow{\gamma^H_{x_i}} \\ \mathrm{HH}^*(RG) & \xrightarrow{t_{M^*}} & \mathrm{HH}^*(RH) \end{array}$$

Proof. Let $\mathcal{P}_R$ be a projective resolution of R, $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ represented by the chain map $\tau : \mathrm{Res}^G_{C_G(x_i)}\mathcal{P}_R \longrightarrow \mathrm{Res}^G_{C_G(x_i)}\mathcal{P}_R[n]$ and the elements $x, y \in H$ $z \in \mathcal{P}_R$.

$$\begin{aligned}(\gamma^H_{x_i} \circ \mathrm{res}^{C_G(x_i)}_{C_H(x_i)})(\tau)((x,y) \otimes_{R\Delta H} z) &= (\gamma^H_{x_i} res^{C_G(x_i)}_{C_H(x_i)})(\tau))((x,y) \otimes_{R\Delta H} z) \\ &= (x,y) \sum_{h \in [H/C_H(x_i)]} (hx_i, h) \otimes_{R\Delta H} \tau(h^{-1}z).\end{aligned}$$

The left composition from the diagram is:

$$\begin{aligned} t_{M^*}(\gamma^G_{x_i}(\tau))((x,y) \otimes_{R\Delta H} z) &= \eta_{M^*}[n](\gamma^G_{x_i}(\tau)((x,y) \otimes_{R\Delta G} z) \\ &= \eta_{M^*}[n](x,y) \sum_{g \in [G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \tau(g^{-1}z),\end{aligned}$$

where $x, y \in H$ and $z \in \mathcal{P}_R$.

From Lemma 1, since the last term is independent of the choice of $[G/C_G(x_i)]$ we choose, using situation $(*)$, the system which is in H. Thus the last term is:

$$\sum_{h \in [H/C_H(x_i)]} \eta_{M^*}[n]((xhx_i, yh) \otimes_{R\Delta G} \tau(h^{-1}z))$$

$$= (x,y) \sum_{h \in [H/C_H(x_i)]} (hx_i, h) \otimes_{R\Delta H} \tau(h^{-1}z)$$

where the last equality follows by Remark 5. □

If $x_i = 1$ we are in situation $(*)$ then from Proposition 3 we obtain as corollary [3, Proposition 4.7].

Corollary 2. *For H a subgroup of G and R a commutative ring the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{H}^*(G,R) & \xrightarrow{\mathrm{res}^G_H} & \mathrm{H}^*(H,R) \\ \downarrow{\scriptstyle \delta_G} & & \downarrow{\scriptstyle \delta_H} \\ \mathrm{HH}^*(RG) & \xrightarrow{t_{M^*}} & \mathrm{HH}^*(RH). \end{array}$$

3. STABLE ELEMENTS IN HOCHSCHILD COHOMOLOGY OF GROUP ALGEBRAS

By [3, Proposition 4.8] we know that $\mathrm{Im}\,\delta_G \subset \mathrm{HH}^*_M(RG)$. Thus $\mathrm{Im}\,\delta_G$ is a subalgebra of the graded algebra of stable elements in Hochschild cohomology algebra of RG. We prove in the next theorem, which is the main result of this paper, that under the situation $(*)$ we have a similar embedding $Im\gamma^G_{x_i} \subset \mathrm{HH}^*_M(RG)$ where M is the regular $RG - RH$ bimodule.

First we remind, from [3] the description of the subalgebra of M stable elements in the Hocshschild cohomology algebra of the group algebra. We denote by $\mathrm{HH}^*_M(RG)$ the subalgebra of M-stable elements in $\mathrm{HH}^*(RG)$. An element $\zeta \in \mathrm{HH}^*(RG)$ is called M-stable if there is $\tau \in \mathrm{HH}^*(RH)$ such that for all nonnegative integer n the following diagram is a commutative homotopy:

$$\begin{array}{ccc} \mathcal{P}_{RG} \otimes_{RG} M & \xrightarrow{\cong} & M \otimes_{RH} \mathcal{P}_{RH} \\ \downarrow{\scriptstyle \zeta_n \otimes Id_M} & & \downarrow{\scriptstyle Id_M \otimes \tau_n} \\ \mathcal{P}_{RG}[n] \otimes_{RG} M & \xrightarrow{\cong} & M \otimes_{RH} \mathcal{P}_{RH}[n] \end{array}$$

where ζ_n, τ_n are degree n components of ζ, τ and the horizontal arrows are the natural homotopy equivalences $\mathcal{P}_{RG} \otimes_{RG} M \cong M \otimes_{RH} \mathcal{P}_{RH}$ which lifts the natural isomorphisms $RG \otimes_{RG} M \cong M \otimes_{RH} RH$.

Theorem 1. *Under the situation $(*)$ the following statements are true:*

(i) *For any nonnegative integer n and chain map $\tau \in \mathrm{Hom}_{C(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$ we have that the following diagram is a commutative homotopy:*

$$\begin{array}{ccc} RG \otimes_{RH} \mathrm{Ind}^{H\times H}_{\Delta_H}(\mathcal{P}_R) \otimes_{RH} RG & \xrightarrow{\eta_M} & \mathrm{Ind}^{G\times G}_{\Delta G}(\mathcal{P}_R) \\ \downarrow{\scriptstyle Id_M \otimes_{RH} \gamma^H_{x_i}(\tau) \otimes_{RH} Id_{M^*}} & & \downarrow{\scriptstyle \gamma^G_{x_i}(\tau)} \\ RG \otimes_{RH} \mathrm{Ind}^{H\times H}_{\Delta_H}(\mathcal{P}_R[n]) \otimes_{RH} RG & \xrightarrow{\eta_M[n]} & \mathrm{Ind}^{G\times G}_{\Delta G}(\mathcal{P}_R[n]) \end{array}$$

(ii) $\mathrm{Im}\,\gamma^G_{x_i} \subset \mathrm{HH}^*_M(RG)$.

Proof. By definition of η_M from Remark 4 and of $\gamma^G_{x_i}$ from Definition 4 we have for $g_1, g_2 \in G, x, y \in H$ and $z \in \mathcal{P}_R$:

$$\gamma^G_{x_i}(\tau)[\eta_M(g_1 \otimes_{RH} ((x,y) \otimes_{R\Delta H} z) \otimes_{RH} g_2)] = \gamma^G_{x_i}(\tau)((g_1x, g_2^{-1}y) \otimes_{R\Delta G} z)$$

$$= (g_1x, g_2^{-1}y) \sum_{g\in[G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \tau(g^{-1}z),$$

$$\eta_M[n](Id_M \otimes_{RH} \gamma^H_{x_i}(\tau) \otimes_{RH} Id_{M^*}(g_1 \otimes_{RH} ((x,y) \otimes_{R\Delta H} z) \otimes_{RH} g_2))$$
$$= \eta_M[n](g_1 \otimes_{RH} \gamma^H_{x_i}(\tau)((x,y) \otimes_{R\Delta H} z) \otimes_{RH} g_2)$$
$$= \eta_M[n](g_1 \otimes_{RH} (\sum_{h\in[H/C_H(x_i)]} (xhx_i, yh) \otimes_{R\Delta H} \tau(h^{-1}z)) \otimes_{RH} g_2)$$
$$= \sum_{h\in[H/C_H(x_i)]} \eta_M[n](g_1 \otimes_{RH} [(xhx_i, yh) \otimes_{R\Delta H} \tau(h^{-1}z)] \otimes_{RH} g_2)$$
$$= \sum_{h\in[H/C_H(x_i)]} (g_1xhx_i, g_2^{-1}yh) \otimes_{R\Delta G} \tau(h^{-1}z).$$

Since we are under situation $(*)$ the last sum is equal to

$$(g_1x, g_2^{-1}y) \sum_{g\in[G/C_G(x_i)]} (gx_i, g) \otimes_{R\Delta G} \tau(g^{-1}z),$$

which completes the proof of (i).

The statement (ii) is a consequence of (i) and [3, lemma 3.3, 3.3.6]. □

If $x_i = 1$ then we are under the situation $(*)$, and since $\gamma^G_1 = \delta_G$ we obtain in theorem 1 the statement [3, Proposition 4.8, ii)].

REFERENCES

[1] Benson, D.J., *Representations and Cohomology I*, Cambridge University Press, Cambridge, 1991.

[2] Benson, D.J., *Representations and Cohomology II*, Cambridge University Press, Cambridge, 1991.

[3] Linckelmann, M., *Transfer in Hochschild cohomology of blocks of finite groups*, Algebr. Represent. Theory, **2** (1999), 107–135.

[4] Siegel, S.F. and Witherspoon, S.W., *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc., **79** (1999), 131–157.

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