MATHEMATICA, Tome 52 (75), N° 1, 2010, pp. 5–13

NEWTON–LIKE METHOD FOR NONSMOOTH SUBANALYTIC VARIATIONAL INCLUSIONS

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Abstract. We present a new result for the local convergence of Newton–type method to a unique solution of a nonsmooth subanalytic variational inclusions in finite dimensional spaces. Under a center–type conditions [1]–[4] and using the same or less computational cost, we extend the applicability of Newton's method [8], [10].

MSC 2010. 65K10, 65G99, 65H10, 65B05, 47H04, 49M15, 47H17, 14P15.

Key words. Variational inclusions, Aubin–like property, convergence analysis, subanalytic function, Newton's method, Clarke's subdifferential, center conditions, set–valued map.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the generalized equation

$$0 \in F(x) + G(x), \tag{1}$$

where F is a nonsmooth subanalytic function from an open subset $\mathcal{D}$ of $\mathcal{X} = \mathbb{R}^n$ into $\mathcal{X}$, G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{X}$ with closed graph.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of generalized equation (1), introduced by Robinson [15].

In the particular case $G = \{0\}$, (1) is a nonlinear equation in the form

$$F(x) = 0. \tag{2}$$

For example, dynamic systems are mathematically modeled by differential or difference equations, and their solutions usually represent the states of the systems, which are determined by solving equation (2).

Such a study can be of interest in the case $G \neq \{0\}$, for example, to variational inequalities for saddle points (see [17], p. 560). Let A and B be nonempty, closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $L : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be some $\mathcal{C}^1$ convex–concave on $A \times B$. The point $(\bar{x},\bar{y}) \in A \times B$ is a saddle point if the following hold:

$$L(x,\bar{y}) \ \geq \ L(\bar{x},\bar{y}) \ \geq \ L(\bar{x},y), \quad \text{for all}\, x \in A \text{ and}\, y \in B. \tag{3}$$

The saddle point condition (3) is equivalent to

$$0 \in f(x,y) + G(x,y), \tag{4}$$

where f and G are defined on $A \times B$ by $f(x,y) = (\nabla_x L(x,y), -\nabla_y L(x,y))$ and by $G(x,y) = N_A(x) \times N_B(y)$, with N_A (resp. N_B) the normal cone to the set A (resp. B). Hence, the variational problem (3) corresponds to generalized equation in formulation (1) and then the saddle point $(\bar{x}, \bar{y})$ can be approximated by the method investigated in this paper. Another example of application to variational inclusions (see [2]). Let K be a convex set of $\mathbb{R}^n$ and φ is a function from K to $\mathbb{R}^n$. The variational inequality problem consists in seeking k_0 in K such that

$$\text{for each}\, k \in K, \quad (\varphi(k_0), k - k_0) \geq 0, \tag{5}$$

where $(.,.)$ denotes the usual scalar product on $\mathbb{R}^n$. Let $\mathcal{I}_K$ denote the convex indicator function of K and ∂ the subdifferential operator. Then the problem (5) is equivalent to

$$0 \in \varphi(k_0) + \mathcal{H}(k_0), \tag{6}$$

with $\mathcal{H} = \partial \mathcal{I}_K$ (also called the normal cone of K). The variational inequality problem (5) is equivalent to (6) which is a generalized equation in formulation (1). Consequently, we can also approximate the solution k_0 of variational inequality (5) using our algorithm (8).

Most of the numerical approximation methods require the expensive computation of the Fréchet–derivative $F'(x)$ of operator F at each step, for example Newton's method:

$$0 \in F(x_n) + F'(x_n)\,(x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0 \in \mathcal{D},\ n \geq 0). \tag{7}$$

A comprehensive bibliography of these methods is given in [2], [3]. In this study, we are interested in numerical method for solving generalized equation (1) when the involved function F is nonsmooth and subanalytic. We proceed by replacing in method (7) the term $F'(x_n)$ by $\Delta F(x_n)$, where $\Delta F(x) \in \partial F(x)$, $\partial F(x)$ denotes the Clarke Jacobian of F at the point $x \in \mathcal{D}$.

In this paper, for approximating $x^\star$, we consider Newton–like method

$$0 \in F(x_n) + \Delta F(x_n)\,(x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0 \in \mathcal{D},\ n \geq 0). \tag{8}$$

In the nonlinear equations case (i.e., $G = \{0\}$ in (1)), the method (8) becomes

$$0 = F(x_n) + \Delta F(x_n)\,(x_{n+1} - x_n), \quad (x_0 \in \mathcal{D},\ n \geq 0), \tag{9}$$

which considered by Bolte et al. [10] for globally subanalytic mappings.

Here, we are motivated by the works in [10], [8]. Using a center–type conditions [2], [3], we extend the applicability of Newton's method [10], [8]. We prove that Newton's method (8) for globally subanalytic mappings converges superlinearly.

The structure of this paper is the following. In section 2, we collect a number of basic definitions on subanalyticity of sets and functions and recall a fixed points theorem for set–valued maps. In section 3 we show an existence–convergence theorem of sequence given by (8). Some remarks are also presented.

2. BACKGROUND MATERIAL

In order to make the paper as self–contained as possible we reintroduce some definitions and some results on fixed point theorems [5]–[17]. Let us begin with some notations that will used throughout this paper. We let $\mathcal{Z}$ be a Banach space equiped with the norm $\|\cdot\|$. The distance from a point x to a set A in $\mathcal{Z}$ is defined by $\operatorname{dist}(x, A) = \inf_{y\in A} \| x - y \|$, with the convention $\operatorname{dist}(x, \emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). Given a subset C of $\mathcal{Z}$, we denote by $e(C, A)$ the Hausdorff–Pompeiu excess of C into A, defined by

$$e(C, A) = \sup_{x\in C} \operatorname{dist}(x, A),$$

with the conventions $e(\emptyset, A) = 0$ and $e(C, \emptyset) = +\infty$ whenever $C \neq \emptyset$. For a set–mapping $\Lambda : \mathcal{X} \rightrightarrows \mathcal{X}$, we denote by $\operatorname{gph}\Lambda$ the set $\{(x, y) \in \mathcal{X} \times \mathcal{X},\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y)$ the set $\{x \in \mathcal{X},\ y \in \Lambda(x)\}$. The norm in the Banach space $\mathcal{X}$ will be denoted by $\|\cdot\|$ and the closed ball centered at x with radius r by $\mathbb{B}_r(x)$. For each $n \in I\!N$, we define $\tau_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$\tau_n(x_1, x_2, \cdots, x_n) = \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \frac{x_2}{\sqrt{1 + x_2^2}}, \cdots, \frac{x_n}{\sqrt{1 + x_n^2}}\right). \tag{10}$$

We also need to define the pseudo–Lipschitzian concept of set–valued maps, introduced by Aubin [7] and also known as Lipschitz–like property [14]:

Definition 1. A set–valued Γ is pseudo–Lipschitz around $(\overline{x}, \overline{y}) \in \operatorname{gph}\Gamma$ with modulus M if there exist constants a and b such that

$$\sup_{z\in\Gamma(y')\cap\mathbb{B}_a(\overline{y})} \operatorname{dist}(z, \Gamma(y'')) \leq M \parallel y' - y'' \parallel, \text{ for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{11}$$

In the term of excess, we have an equivalent definition of pseudo–Lipschitzian property replacing the inequality (11) by

$$e(\Gamma(y') \cap \mathbb{B}_a(\overline{y}), (y'')) \leq M \parallel y' - y'' \parallel, \text{ for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{12}$$

Pseudo–Lipschitzian property play a crutial role in many aspects of variational analysis and applications [14], [17]. Let us note that the Lipschitz–like of Γ is equivalent to the metric regularity of Γ^{-1}, which is a basic well–posedness property in optimization problems. Other characterization is by Mordukhovich [14] via the concept of coderivative $\mathcal{D}^\star\Gamma(x/y)$, i.e.,

$$v \in \mathcal{D}^\star\Gamma(x/y)(u) \Longleftrightarrow (v, -u) \in N_{\operatorname{gph}\Gamma}(x, y). \tag{13}$$

Then the Mordukhovich criterion says that Γ is pseudo–Lipschitz around $(\overline{x}, \overline{y})$ if and only if

$$\parallel \mathcal{D}^\star\Gamma(\overline{x}/\overline{y}) \parallel^+ = \sup_{u\in\mathbb{B}_1(0)} \sup_{v\in\mathcal{D}^\star\Gamma(\overline{x}/\overline{y})(u)} \parallel v \parallel < \infty. \tag{14}$$

For some characterizations and applications of the Lipschitz–like property the reader is referred to [7], [12], [14], [16], [17] and the references given there.

We recall the following definition of semianalyticity subsets and subanalyticity functions [9], [10] [11], [18].

DEFINITION 2. (a) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called semianalytic if each point of $\mathbb{R}^n$ admits a neighborhood $\mathcal{V}$ for which $\mathcal{A} \cap \mathcal{V}$ assumes the following form:

$$\bigcup_{i=1}^{i=p} \bigcap_{i=1}^{i=q} \{x \in \mathcal{V} \,:\, f_{ij}(x) = 0,\, g_{ij}(x) > 0\}, \tag{15}$$

where the functions $f_{ij},\ g_{ij} \,:\, \mathcal{V} \longrightarrow \mathbb{R}$ are real–analytic for all $1 \leq i \leq p$, $1 \leq j \leq q$.

(b) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called subanalytic if each point of $\mathbb{R}^n$ admits a neighborhood $\mathcal{V}$ such that:

$$\mathcal{A} \cap \mathcal{V} = \{x \in \mathbb{R}^n \,:\, (x, y) \in \mathcal{B}\}, \tag{16}$$

where $\mathcal{B}$ is a bounded semianalytic subset of $\mathbb{R}^n \times \mathbb{R}^m$ for some $m \geq 1$.

(c) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called globally subanalytic if its image by τ_n defined by (10) is a subanalytic subset of $\mathbb{R}^n$.

(d) $F \,:\, \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called subanalytic, if its graph is a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}^n$.

(e) $F \,:\, \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called globally subanalytic, if its graph is a globally subanalytic subset of $\mathbb{R}^n \times \mathbb{R}^n$.

We need also the following fixed point theorem [12].

LEMMA 1. *Let ϕ be a set–valued map from $\mathcal{X}$ into the closed subsets of $\mathcal{X}$. We suppose that for $\eta_0 \in \mathcal{X}$, $r \geq 0$ and $0 \leq \lambda < 1$ the following properties hold*

(1) $\text{dist}\ (\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$.

(2) $e(\phi(x_1) \cap \mathbb{B}_r(\eta_0), \phi(x_2)) \leq \lambda \parallel x_1 - x_2 \parallel, \quad \forall x_1, x_2 \in \mathbb{B}_r(\eta_0)$.

Then ϕ has a fixed point in $\mathbb{B}_r(\eta_0)$. That is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $\mathbb{B}_r(\eta_0)$.

Finally, we recall a definition concerning directional differentiability and Clarke's Jacobian in finite dimensional spaces.

DEFINITION 3. (a) A mapping $F \,:\, \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be directionally differentiable at $x \in \mathcal{D}$ along direction d if the following limit

$$F'(x; d) := \lim_{t \downarrow 0} \frac{F(x + t\, d) - F(x)}{t} \tag{17}$$

exists.

Note that every definable locally Lipschitz mapping F admits directional derivatives.

(b) For $F \,:\, \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a locally Lipschitz continuous function, the limiting Jacobian of F at $x \in \mathcal{D}$ is defined by

$$\partial F(x) = \{\mathcal{M} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \,:\, \exists\, u^k \in \mathcal{D},\ \lim_{k \to \infty} F'(u^k) = \mathcal{M}\}. \tag{18}$$

(c) Let $F : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a locally Lipschitz continuous function. Clarke's Jacobian of F at $x \in \mathcal{D}$ is defined by

$$\partial^\circ F(x) = \overline{\mathrm{co}}\, \partial F(x), \tag{19}$$

where $\overline{\mathrm{co}}\, \mathcal{A}$ is the closed convex envelope of $\mathcal{A} \subseteq \mathbb{R}^n$.

3. LOCAL CONVERGENCE OF METHOD (8)

Before presenting our main result of convergence of method (8), we give a variant of the result for subanalytic mappings established by Bolte, Daniilidis and Lewis [10, Lemma 3.3]:

LEMMA 2. *Let $F : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a locally Lipschitz subanalytic function and $x \in \mathcal{D}$. Then, there exists a positive rational number γ and a constant $C_x > 0$, such that:*

$$\| F(y) - F(x) - \Delta(y)\,(y - x) \| \leq C_x \; \| y - x \|^{1+\gamma} \tag{20}$$

where $\Delta(y)$ is any element of $\partial^\circ F(y)$.

In particular, there exists a positive rational number $\gamma^\star$, and a constant $C_{x^\star} > 0$, such that:

$$\| F(y) - F(x^\star) - \Delta(y)\,(y - x^\star) \| \leq C_{x^\star} \; \| y - x^\star \|^{1+\gamma^\star} \tag{21}$$

where $\Delta(y)$ is any element of $\partial^\circ F(y)$.

We will be concerned with the existence and the convergence of the sequence defined by (8) to the solution $x^\star$ of (1). The main result of this study is as follows.

THEOREM 1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ be a locally Lipschitz subanalytic function. Let $x^\star$ be a locally unique solution of (1). Let $C_{x^\star}$ and $\gamma^\star$ be constants given by (21).*

Assume:

($\mathcal{P}1$) *There exists $K^\star > 0$, such that for all x in $\mathcal{D}$*

$$\| \Delta F(x) - \Delta F(x^\star) \| \leq K^\star \; \| x - x^\star \|$$

for all $\Delta F(x) \in \partial^\circ F(x)$ and $\Delta F(x^\star) \in \partial^\circ F(x^\star)$;

($\mathcal{P}2$) *For all $\Delta F(x^\star) \in \partial^\circ F(x^\star)$, the set–valued map $(G + \Delta F(x^\star)\,(. - x^\star))^{-1}$ is pseudo–Lipschitz around $(-F(x^\star), x^\star)$ with constants M, a and b (these constants are given in Definition 1).*

Then, for every constant C satisfying

$$C \geq C_0 = M\, C_{x^\star}, \tag{22}$$

there exists $\delta > 0$ with

$$\delta < \delta_0 = \min \left\{ a;\; \sqrt[\gamma^\star]{\frac{1}{C}};\; \sqrt[1+\gamma^\star]{\frac{b}{2\, C_{x^\star}}};\; \sqrt{\frac{b}{4\, K^\star}} \right\} \tag{23}$$

such that for every starting point x_0 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$), the sequence (x_k) defined by (8) converges to $x^\star$.

Moreover (x_k) *satisfies the following inequality for* $k \geq 0$*:*

$$\| x_{k+1} - x^\star \| \leq C \ \| x_k - x^\star \|^{1+\gamma^\star} . \tag{24}$$

We need to introduce some notations [5], [6]. First, define the set–valued maps $Q : \mathcal{X} \rightrightarrows \mathcal{X}$ and $\psi_k : \mathcal{X} \rightrightarrows \mathcal{X}$ by

$$Q(x) = F(x^\star) + \Delta F(x^\star)\,(x - x^\star) + G(x),\ \psi_k(x) = Q^{-1}(Z_k(x)), \quad k \geq 0 \tag{25}$$

where Z_k is a mapping from $\mathcal{X}$ to $\mathcal{X}$ defined by

$$Z_k(x) = F(x^\star) - F(x_k) + \Delta F(x^\star)\,(x - x^\star) - \Delta F(x_k)\,(x - x_k), \quad k \geq 0. \tag{26}$$

Remark 1. The proof of Theorem 1 is given by induction on k. We first state a result involving the starting point x_0. Let us note that the point x_1 is a fixed point of ψ_0 if and only if $0 \in F(x_1) + \Delta F(x_0)\,(x_1 - x_0) + G(x_1)$. Once x_k is computed, we show that the function ψ_k has a fixed point x_{k+1} in $\mathcal{X}$. This process is useful to prove the existence of a sequence (x_k) satisfying (8).

Proposition 1. *Under the assumptions of Theorem* 1*, and for every distinct starting points* x_0 *in* $\mathbb{B}_\delta(x^\star)$ *(with* $x_0 \neq x^\star$*), the set–valued map* ψ_0 *has a fixed point* x_1 *in* $\mathbb{B}_\delta(x^\star)$ *satisfying*

$$\| x_1 - x^\star \| \leq C \ \| x_0 - x^\star \|^{1+\gamma^\star} \tag{27}$$

where C *and* δ *are given by Theorem* 1.

Proof. By hypothesis $(\mathcal{P}2)$ we have

$$e(Q^{-1}(y') \cap \mathbb{B}_a(x^\star), Q^{-1}(y'')) \leq M \ \| y' - y'' \|, \forall y', y'' \in \mathbb{B}_b(0). \tag{28}$$

Moreover, by Lemma 2 (see (21)) we obtain the following

$$\begin{aligned} \| Z_0(x^*) \| &= \| F(x_0) - F(x^\star) - \Delta f(x_0)\ (x_0 - x^\star) \| \\ &\leq\ C_{x^\star} \ \| x_0 - x^\star \|^{1+\gamma^\star} . \end{aligned} \tag{29}$$

By (23) we have $Z_0(x^\star) \in \mathbb{B}_b(0)$.
Hence from (28) one gets

$$\begin{aligned} e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \psi_0(x^\star)\Big) &=\ e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), Q^{-1}[Z_0(x^\star)]\Big) \\ &\leq M\, C_{x^\star} \ \| x_0 - x^\star \|^{1+\gamma^\star} \\ &= C_0 \ \| x_0 - x^\star \|^{1+\gamma^\star} . \end{aligned} \tag{30}$$

According to the definition of excess e and using (30), we have

$$\operatorname{dist}(x^\star, \psi_0(x^\star)) \leq e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \psi_0(x^\star)\Big), \tag{31}$$

and

$$\operatorname{dist}(x^\star, \psi_0(x^\star)) \leq C_0 \ \| x_0 - x^\star \|^{1+\gamma^\star} . \tag{32}$$

Since $C > C_0$, there exists $\lambda \in [0,1[$ such that $C\,(1-\lambda) \geq C_0$, and

$$\text{dist}\,(x^\star, \psi_0(x^\star)) \leq C\,(1-\lambda)\ \| x_0 - x^\star \|^{1+\gamma^\star}\ . \tag{33}$$

Identifying η_0, ϕ and r in Lemma 1 by $x^\star$, ψ_0 and $r_0 = C\ \| x_0 - x^\star \|^{1+\gamma^\star}$ respectively, we can deduce from the inequality (33) that the first assumption in Lemma 1 is satisfied.

We prove now that the second assumption of Lemma 1 is verified.

Using (23), we have $r_0 \leq \delta \leq a$, and moreover for $x \in \mathbb{B}_\delta(x^\star)$ we get in turn

$$\begin{aligned} \| Z_0(x) \| &= \| F(x^\star) - F(x_0) + \Delta F(x^\star)(x - x^\star) - \Delta F(x_0)(x - x_0) \| \\ &\leq \| F(x_0) - F(x^\star) - \Delta F(x_0)(x_0 - x^\star) \| \\ &\quad + \| (\Delta F(x_0) - \Delta F(x^\star))(x - x^\star) \| \\ &\leq \| F(x_0) - F(x^\star) - \Delta F(x_0)(x_0 - x^\star) \| \\ &\quad + \| \Delta F(x_0) - \Delta F(x^\star) \| \| x - x^\star \|\ . \end{aligned}$$

Using Lemma 2 and $(\mathcal{P}2)$ we obtain

$$\begin{aligned} \| Z_0(x) \| &\leq C_{x^\star}\ \| x_0 - x^\star \|^{1+\gamma^\star} + K^\star\ \| x_0 - x^\star \| \| x - x^\star \| \\ &\leq C_{x^\star}\, \delta^{1+\gamma^\star} + K^\star\, \delta^2. \end{aligned} \tag{34}$$

Then by (23), we deduce that for all $x \in \mathbb{B}_\delta(x^\star)$ we have $Z_0(x) \in \mathbb{B}_b(0)$. Then it follows that for all $x', x'' \in \mathbb{B}_{r_0}(x^\star)$ we have

$$e(\psi_0(x') \cap \mathbb{B}_{r_0}(x^\star), \psi_0(x'')) \leq e(\psi_0(x') \cap \mathbb{B}_\delta(x^\star), \psi_0(x'')),$$

which yields by (28) and $(\mathcal{P}2)$:

$$\begin{aligned} e(\psi_0(x') \cap \mathbb{B}_{r_0}(x^\star), \psi_0(x'')) &\leq M\ \| Z_0(x') - Z_0(x'') \| \\ &= M\ \| (\Delta F(x_0) - \Delta F(x^\star))\,(x'' - x') \| \\ &\leq M\,K^\star\ \| x_0 - x^\star \| \| x'' - x' \| \\ &\leq M\,K^\star\,\delta\ \| x'' - x' \|\ . \end{aligned} \tag{35}$$

Without loss of generality, we may suppose that $\delta \leq \dfrac{\lambda}{MK^\star}$. The second condition of Lemma 1 is satisfied. By Lemma 1 we can deduce the existence of a fixed point $x_1 \in \mathbb{B}_{r_0}(x^\star)$ for the map ψ_0. The proof of Proposition 1 is complete. □

Proof. (Proof of Theorem 1) Keep $\eta_0 = x^\star$, and for $k \geq 1$, set:

$$r := r_k = C\ \| x^\star - x_k \|^{1+\gamma^\star}\ .$$

By Remark 1, the application of Proposition 1 to the map ψ_k gives the desired result. □

Remark 2. In order for us to compare our results with the corresponding ones in [8], let us introduce conditions used in [8] to prove a result similar to Theorem 1:

$(\mathcal{P}1)^\star$ There exists $K > 0$, such that $\forall x \in \mathcal{D}$, $\forall \Delta F(x) \in \partial^\circ F(x)$, $\| \Delta F(x) \| \leq K$;

$(\mathcal{P}2)^\star$ For all $\Delta F(x^\star) \in \partial^\circ F(x^\star)$, the set–valued map $(G+\Delta F(x^\star)\,(.-x^\star))^{-1}$ is pseudo–Lipschitz around $(-F(x^\star), x^\star)$ with constants M, a and b (These constants are given in Definition 1), and $2\,M\,K < 1$.

The condition $(\mathcal{P}2)^\star$ used in [8] is stronger than our condition $(\mathcal{P}2)$ (we do not use in our hypothesis $(\mathcal{P}2)$ the additional condition $2\,M\,K < 1$). Moreover, the constant K given in $(\mathcal{P}1)^\star$ is not easy to compute. This is another motivation for introducing our center–condition $(\mathcal{P}1)$ [2], [3].

Another advantage of our method, we give in our Theorem 1 a finer error estimate on the distances $\| x_{k+1} - x^\star \|$ $(k \geq 0)$ than that given in [8, Theorem 3.1]. The convergence in [8] is given by:

$$\| x_{k+1} - x^\star \| \leq \rho \, \| x_k - x^\star \|^{1+\gamma} \qquad (k \geq 0) \tag{36}$$

where γ is given by strong estimate (20) in Lemma 2 and ρ is a positive constant. In our Theorem 1, we obtain a finer estimate than (36):

$$\| x_{k+1} - x^\star \| \leq C \, \| x_k - x^\star \|^{1+\gamma^\star} \qquad (k \geq 0) \tag{37}$$

by using only $\gamma^\star$ given by center–estimate (21).

Hence, the claims us ade in the introduction have been justified.

Remark 3. We can enlarge the radius of convergence in Theorem 1 even further as follows: using inequalities (34), (29), we can improve δ given by (23) by considering the constant δ':

$$\delta' < \delta'_0 = \min \left\{ a;\ \sqrt[\gamma^\star]{\frac{1}{C}};\ \sqrt[1+\gamma^\star]{\frac{b}{C_{x^\star}}};\ \delta_1 \right\}$$

where δ_1 is given by $\delta_1 = \max\{\eta > 0 \,:\, C_{x^\star}\,\eta^{1+\gamma^*} + 2\,K^\star\,\eta^2 - b < 0\}$.

CONCLUSION

We provided a Newton–like method to approximate an unique solution for nonsmooth subanalytic variational inclusions in finite dimensional spaces. Moreover, we obtain a local convergence result (see Theorem 1) using center–type conditions and Lipschitz–like concept for set–valued maps.

Under some ideas given in [2, 3] for nonlinear equations, and using some observations (see Remarks 2 and 3), we provided a finer analysis than [8] with finer error estimate on the distances $\| x_n - x^\star \|$ $(n \geq 1)$.

These observations are very important in computational mathematics [2, 3].

REFERENCES

[1] Argyros, I.K., *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397.

[2] Argyros, I.K., *Computational theory of iterative methods*, Studies in Computational Mathematics, **15**, Elsevier, New York, 2007.

[3] Argyros, I.K., *Convergence and applications of Newton–type iterations*, Springer Verlag Publ., New York, 2008.
[4] Argyros, I.K., *A Kantorovich–type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. Appl., **332** (2007), 97–108.
[5] Argyros, I.K. and Hilout, S., *Newton's methods for variational inclusions under conditioned Fréchet derivative*, Appl. Math., **34** (2007), 349–357.
[6] Argyros, I.K. and Hilout, S., *Local convergence of Newton–like methods for generalized equations*, Appl. Math. Comput., **197** (2008), 507–514.
[7] Aubin, J.P. and Frankowska, H., *Set–valued analysis*, Birkhäuser, Boston, 1990.
[8] Cabuzel, C. and Piétrus, A., *Local convergence of Newton's method for subanalytic variational inclusions*, Positivity, **12** (2008), 525–533.
[9] Bolte, J., Daniilidis, A. and Lewis, A., *The Lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems*, SIAM J. Optim., **17** (2006), 1205–1223.
[10] Bolte, J., Daniilidis, A. and Lewis, A., *Tame functions are semismooth*, Math. Program., Ser. B, **117** (2009), 5–19.
[11] Dedieu, J.P., *Penality functions in subanalytic optimization*, Optimization, **26** (1992), 27–32.
[12] Dontchev, A.L. and Hager, W.W., *An inverse function theorem for set–valued maps*, Proc. Amer. Math. Soc., **121** (1994), 481–489.
[13] Hilout, S., *Convergence analysis of a family of Steffensen–type methods for generalized equations*, J. Math. Anal. Appl., **339** (2008), 753–761.
[14] Mordukhovich, B.S., *Variational analysis and generalized differentiation, I. Basic theory, II. Applications*, Vol. **330**, **331**, Springer, Grundlehren Series, 2006.
[15] Robinson, S.M., *Generalized equations and their solutions, part I: basic theory*, Mathematical Programming Study, **10** (1979), 128–141.
[16] Rockafellar, R.T., *Lipschitzian properties of multifunctions*, Nonlinear Anal., **9** (1984), 867–885.
[17] Rockafellar, R.T. and Wets, R.J–B., *Variational analysis*, A Series of Comprehensives Studies in Mathematics, **317**, Springer, 1998.
[18] Van Den Dries, L. and Miller, C., *Geometric categories and O-minimal structures*, Duke Math. J., **84** (1996), 497–540.

Received March 29, 2008
Accepted June 27, 2009

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