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THEORETICAL ASPECTS AND SIMULATION OF A GENERALIZED SURPLUS PROCESS WITH A LOGARITHMIC BARRIER

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Abstract. In this article we consider a generalization of the classical Lundberg surplus process. In the presence of the logarithmic dividend barrier we assume that the company also receives interest on its reserve with a constant interest rate. We derive equations for the survival probability and the expected sum of discounted dividend payments. We give important theoretical results concerning the existence and uniqueness of the corresponding solutions. We use Monte Carlo (MC), Quasi-Monte Carlo (QMC) techniques and the direct simulation approach in order to estimate these quantities. We also perform numerical tests, in which we compare the accuracy of these algorithms.

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Key words. Generalized risk process, Monte Carlo method, quasi-Monte Carlo method, logarithmic dividend barrier, survival probability, expected discounted dividend payments.

1. INTRODUCTION

Let us consider at the beginning the classical Lundberg risk model that describes the surplus process of an insurance portfolio, which assumes independent and identically distributed claims $X_j, j = 1, 2, \ldots$, having the same distribution function $F(y)$. The claims occur according to a homogeneous Poisson process $N(t)$, with intensity λ, which counts the claims up to time t. The risk process is described by the following relation:

$$R_t = u + ct - \sum_{j=1}^{N(t)} X_j, \tag{1}$$

where c is a constant premium intensity. For more details on risk theory see Gerber [9], Asmussen [4], Embrechts et al. [8] and Grandell [11]. We assume that the expected value of the individual claim amounts $\mu = E(X_j)$ is finite, and the net profit condition $c > \lambda\mu$ is fulfilled, which "guarantees" survival of the insurance company (see [4] or [11]). In the sequel we consider a generalization of the classical Lundberg surplus process. The company also receives interest on its reserve R_t with a constant interest rate $i > 0$. Hence the risk process R_t, at time t, is described as follows:

$$\mathrm{d}R_t = c\mathrm{d}t + iR_t\mathrm{d}t - \mathrm{d}Y_t, \tag{2}$$

where $Y_t = \sum_{j=1}^{N(t)} X_j$. Accordingly to [19], we obtain

$$R_t = u\mathrm{e}^{it} + c\int_0^t \mathrm{e}^{is}\mathrm{d}s - \int_0^t \mathrm{e}^{i(t-s)}\mathrm{d}Y_s, \tag{3}$$

where $R_0 = u$ is the initial capital.

In the paper [15] a modification of the classical model was introduced: Whenever the surplus process R_t defined in relation (1) reaches a logarithmic time-dependent barrier b_t of the form

$$b_t = \ln(\mathrm{e}^b + at),\ b \geq 0,\ a > 0, \tag{4}$$

dividends are paid out to shareholders, with intensity $c - \mathrm{d}b_t$, and the surplus stays on the barrier, until the occurrence of the next claim. In our generalized model, with interest paid on the company's reserve, the dividends will be paid with intensity $(c+iR_t)-\mathrm{d}b_t$. Thus, the dynamics of the risk process R_t defined in (2) with the logarithmic barrier (4), are given by

$$\begin{cases} \mathrm{d}R_t = (c+iR_t)\mathrm{d}t - \mathrm{d}Y_t, \text{ if } R_t < b_t \text{ (bellow barrier)} \\ \mathrm{d}R_t = \mathrm{d}b_t - \mathrm{d}Y_t, \text{ if } R_t = b_t \text{ (on the barrier).} \end{cases} \tag{5}$$

Together with the initial capital $R_0 = u,\ 0 \leq u < b < \infty$, this determines entirely the risk process $\{R_t,\ t \geq 0\}$ (see Figure 1.1).

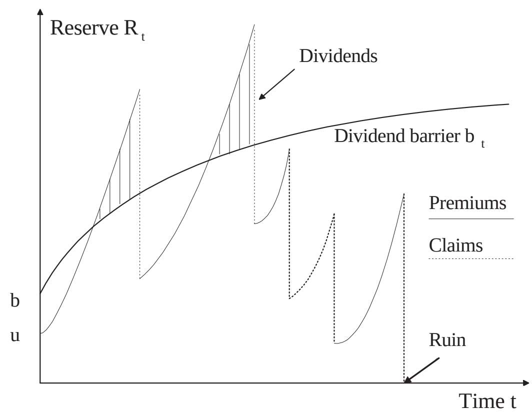

Fig. 1.1 – A sample path of the surplus process R_t

Two crucial quantities in the risk theory are the probability of survival

$$\phi(u,b) = P(R_t \geq 0, \forall t \geq 0 | R_0 = u, b_0 = b), \tag{6}$$

or, alternatively, the probability of ruin $\psi(u,b) = 1-\phi(u,b)$, and the expected sum of discounted dividend payments $V(u,b)$, which will be defined later in this paper.

Dividend barrier models have a long history in risk theory, going back to Finetti [7]. Gerber [10] was the one who introduced the linear dividend barrier, in order to overcome the deficiency of horizontal barrier models that lead to ruin with probability 1. He derived an upper bound for the probability of ruin by martingale methods, and in [10], he obtained exact solutions for the probability of ruin and for the expected sum of discounted dividend payments $V(u, b)$, in terms of infinite series, in the special case of exponentially distributed claim amounts. This result was generalized to arbitrary Erlang claim distributions by Siegl and Tichy [18], who proposed a suitable solution algorithm. The convergence of this algorithm was proved by Albrecher and Tichy in [3]. More general dividend barrier models were recently introduced by Albrecher and Kainhofer in [1]. In paper [2] a generalized version of a non-linear dividend barrier model of an insurance portfolio was introduced and investigated.

As we have already mentioned, we introduced in our paper [15] a general dividend barrier model based on a logarithmic dividend barrier. For this model, we derived the integro-differential equations for the probability of survival $\phi(u, b)$ and for the expected sum of discounted dividend payments $V(u, b)$ and we gave theoretical results concerning the existence and uniqueness of the corresponding solutions. We also developed Monte Carlo (MC) and Quasi-Monte Carlo (QMC) algorithms, in order to obtain these quantities for the logarithmic dividend barrier model.

In this article, we investigate theoretically and numerically the generalized version of the logarithmic dividend barrier model from [15], in which we assume that the company also receives interest on its reserve R_t. In Section 2 we derive new equations for the survival probability $\phi(u, b)$ and for the expected sum of discounted dividend payments $V(u, b)$, which extend the equations from [15]. In Section 3 we present important theoretical results concerning the existence and uniqueness of the corresponding solutions. In the last section, detailed numerical results are presented for estimating these solutions. We develop Monte Carlo (MC) and Quasi-Monte Carlo (QMC) algorithms, and we perform numerical tests using these algorithms. The direct simulation approach for our generalized risk process is also presented.

2. FORMULATION OF THE INTEGRO-DIFFERENTIAL MODEL

In this section we derive the integro-differential equation for the two quantities of interest, $\phi(u, b)$ and $V(u, b)$. We consider the generalized risk model (2) extended by a logarithmic dividend barrier

$$b_t = \ln(\mathrm{e}^b + at), \ b \geq 0, \ a > 0. \tag{7}$$

Following a procedure developed by Gerber in [10], we prove the following theoretical result.

THEOREM 1. *The survival probability $\phi(u, b)$ defined in (1) can be expressed as the solution of the following integro-differential equation*

$$(c + iu)\frac{\partial \phi}{\partial u} + \frac{a}{\mathrm{e}^b}\frac{\partial \phi}{\partial b} - \lambda\phi(u, b) + \lambda \int_0^u \phi(u - y, b)\mathrm{d}F(y) = 0, \tag{8}$$

with the boundary condition

$$\left.\frac{\partial\phi}{\partial u}\right|_{u=b} = 0. \tag{9}$$

Proof. A way to get the above equation for $\phi(u,b)$ is to use a differential argument (see e.g. [6]). We consider the infinitesimal time interval $(0, dt)$. Then, we consider R_t within time interval $(0, dt)$ and split the four possible cases as follows:

(1) no claim in $(0, dt)$,
(2) one claim in $(0, dt)$ but the amount X_1 to be paid does not cause ruin,
(3) one claim in $(0, dt)$ and the amount X_1 to be paid causes ruin,
(4) more than one claim occurs in $(0, dt)$.

From Hipp [12] we know that the classical risk process R_t under a constant interest rate i, modified by us with the logarithmic dividend barrier, is a Markov process with stationary and independent increments. Hence, we get

$$\begin{aligned}\phi(u,b) &= (1-\lambda dt + o(dt))\phi(u+cdt+iudt,\ \ln(\mathrm{e}^b + adt)) \\ &+ (\lambda dt + o(dt))\int_0^{u+cdt+iudt}\phi(u+cdt+iudt-y,\ \ln(\mathrm{e}^b+adt))\mathrm{d}F(y) \\ &+ (\lambda dt + o(dt))\cdot 0 + o(dt),\end{aligned} \tag{10}$$

provided that $u < b$. As usual $\frac{o(dt)}{dt} \to 0$ as $dt \to 0$.
By the Taylor expansion of the function ϕ in a neighborhood of the point (u, b) we get

$$\begin{aligned}\phi(u+cdt+iudt,\ \ln(\mathrm{e}^b+adt)) &= \phi(u,b) + (cdt+iudt)\frac{\partial\phi}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial\phi}{\partial b} + o(dt).\end{aligned} \tag{11}$$

By the Taylor expansion of the function ϕ in a neighborhood of the point $(u-y, b)$ and by collecting the terms of order dt, we obtain

$$\begin{aligned}\phi(u+cdt+iudt-y,\ \ln(\mathrm{e}^b+adt)) &= \phi(u-y,b) + (cdt+iudt)\frac{\partial\phi}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial\phi}{\partial b} + o(dt).\end{aligned} \tag{12}$$

Substituting (11) and (12) in the equality (10), we obtain

$$\begin{aligned}\phi(u,b) &= \lambda dt\int_0^{u+cdt+iudt}\Big[\phi(u-y,b) + (cdt+iudt)\frac{\partial\phi}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial\phi}{\partial b}\Big]\mathrm{d}F(y) \\ &+ (1-\lambda dt)\Big[\phi(u,b) + (cdt+iudt)\frac{\partial\phi}{\partial u} + (\ln(\mathrm{e}^b+adt)-b)\frac{\partial\phi}{\partial b}\Big] + o(dt).\end{aligned}$$

Dividing by dt and letting $dt \to 0$, we get

$$(c+iu)\frac{\partial\phi}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial\phi}{\partial b}-\lambda\phi(u,b)+\lambda\int_0^u\phi(u-y,b)\mathrm{d}F(y)=0. \tag{13}$$

In the case $u=b$, we obtain the following equation

$$\frac{a}{\mathrm{e}^b}\frac{\partial\phi}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial\phi}{\partial b}-\lambda\phi(u,b)+\lambda\int_0^u\phi(u-y,b)\mathrm{d}F(y)=0. \tag{14}$$

From relations (13) and (14) we get the boundary condition $\frac{\partial\phi}{\partial u}\Big|_{u=b}=0.$ □

Furthermore, we have the natural requirement

$$\lim_{b\to\infty}\phi(u,b)=\phi(u), \tag{15}$$

where $\phi(u)$ is the survival probability in the absence of the barrier.

For $0\le u<b$, we define the time of ruin of the surplus process R_t as follows

$$T(u,b)=\inf\{t:\ R_t<0|R_0=u,b_0=b\}. \tag{16}$$

Let $i>0$ be the risk less interest rate. We define the quantity

$$D(u,b)=\int_0^{T(u,b)}\mathrm{e}^{-it}\mathrm{d}D(t), \tag{17}$$

which represents the present value of all dividends until the time of ruin $T(u,b)$. The term $D(t)$ expresses the aggregate dividends paid to shareholders by time t and is defined as

$$\mathrm{d}D(t)=\Big(\mathrm{e}^{it}(c+iu)-\frac{a}{\mathrm{e}^b+at}\Big)\mathrm{d}t,\quad \text{whenever}\quad R_t=b_t. \tag{18}$$

Finally, the expected sum of discounted dividend payments is

$$V(u,b)=E[D(u,b)]. \tag{19}$$

Theorem 2. *The expected sum of discounted dividend payments $V(u,b)$ can be expressed as the solution of the following integro-differential equation*

$$(c+iu)\frac{\partial V}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial V}{\partial b}-(\lambda+i)V(u,b)+\lambda\int_0^u V(u-y,b)\mathrm{d}F(y)=0, \tag{20}$$

with the boundary condition

$$\frac{\partial V}{\partial u}\Big|_{u=b}=1. \tag{21}$$

Proof. Let us consider the infinitesimal time interval $(0,dt)$. Conditioning on the occurrence of the first claim within this interval, and denoting by T_1

the occurrence time of this claim, we obtain by the strong Markov property of the Markov risk process R_t with constant interest rate i

(22)

$$\begin{aligned} V(u,b) &= E(\mathrm{e}^{-idt}V(R_{dt},b_{dt})) = \mathrm{e}^{-idt}E(V(R_{dt},b_{dt})) \\ &= \mathrm{e}^{-idt}\{P(T_1 > dt)E(V(u+cdt+iudt,\ \ln(\mathrm{e}^b+adt))) \\ &+ P(T_1 \le dt)E(V(u+cdt+iudt-X_1,\ \ln(\mathrm{e}^b+adt)))\} \\ &= \mathrm{e}^{-idt}\{(1-\lambda dt+o(dt))V(u+cdt+iudt,\ \ln(\mathrm{e}^b+adt)) \\ &+ (\lambda dt + o(dt))\int_0^{u+cdt+iudt} V(u+cdt+iudt-y,\ \ln(\mathrm{e}^b+adt))\mathrm{d}F(y)\}. \end{aligned}$$

We have

$$\mathrm{e}^{-idt} = 1 - idt + o(dt). \tag{23}$$

By using the Taylor expansion of the function V in a neighborhood of the point (u,b) and collecting the terms of order dt, we obtain

$$\begin{aligned} V(u+cdt+iudt,\ \ln(\mathrm{e}^b+adt)) &= V(u,b) + (cdt+iudt)\frac{\partial V}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial V}{\partial b} + o(dt). \end{aligned} \tag{24}$$

By the Taylor expansion of the function V in a neighborhood of the point $(u-y,b)$ and collecting the terms of order dt, we get

$$\begin{aligned} V(u+cdt+iudt-y,\ \ln(\mathrm{e}^b+adt)) &= V(u-y,b) + (cdt+iudt)\frac{\partial V}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial V}{\partial b} + o(dt). \end{aligned} \tag{25}$$

Replacing (24) and (25) into (22), we obtain

$$\begin{aligned} V(u,b) &= (1-idt)\lambda dt\int_0^{u+cdt+iudt}\Big[V(u-y,b) + (cdt+iudt)\frac{\partial V}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial V}{\partial b}\Big]\mathrm{d}F(y) + o(dt) \\ &+ (1-idt)(1-\lambda dt)\Big[V(u,b) + (cdt+iudt)\frac{\partial V}{\partial u} \\ &+ (\ln(\mathrm{e}^b+adt)-b)\frac{\partial V}{\partial b}\Big] + o(dt). \end{aligned}$$

Dividing by dt and letting $dt \to 0$, we finally get

$$(c+iu)\frac{\partial V}{\partial u} + \frac{a}{\mathrm{e}^b}\frac{\partial V}{\partial b} - (\lambda+i)V(u,b) + \lambda\int_0^u V(u-y,b)\mathrm{d}F(y) = 0.$$

In the case $u = b$ the boundary condition (21) can be deduced by similar arguments. □

There has been some interest in the actuarial field for models where dividends are paid also after a ruin event. In our model with a logarithmic barrier, if we allow that the dividends are paid after a ruin event, then we obtain a similar equation with (20) for the expected value $W(u,b)$ of the discounted dividend payments

$$(c+iu)\frac{\partial W}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial W}{\partial b}-(\lambda+i)W(u,b)+\lambda\int_0^u W(u-y,b)\mathrm{d}F(y)=0, \tag{26}$$

with the boundary condition

$$\frac{\partial W}{\partial u}\Big|_{u=b}=1. \tag{27}$$

3. THEORETICAL RESULTS FOR THE BOUNDARY VALUE PROBLEM

It is known that, even for the particular situation of the exponentially distributed claim amounts, the problem of obtaining analytical results is not so easy. Next we present some theoretical results concerning the solution of the boundary value problems developed in the previous section.

We first give two important theoretical results in which we prove that the boundary value problem (20) and (21) has a unique bounded solution.

THEOREM 3. *A function $(u,b)\mapsto V(u,b)$ is the solution of the integro-differential equation*

$$(c+iu)\frac{\partial V}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial V}{\partial b}-(\lambda+i)V(u,b)+\lambda\int_0^u V(u-y,b)\mathrm{d}F(y)=0, \tag{28}$$

with the boundary condition

$$\frac{\partial V}{\partial u}\Big|_{u=b}=1, \tag{29}$$

if and only if it is a fixed point of the following operator

(30)

$$Ag(u,b)=$$
$$\int_0^{t^*}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{(u+\frac{c}{i})\mathrm{e}^{it}-\frac{c}{i}} g\Big((u+\frac{c}{i})\mathrm{e}^{it}-\frac{c}{i}-y,\ \ln(\mathrm{e}^b+at)\Big)\mathrm{d}F(y)\mathrm{d}t$$
$$+\int_{t^*}^{\infty}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\ln(\mathrm{e}^b+at)} g(\ln(\mathrm{e}^b+at)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t$$
$$+\int_{t^*}^{\infty}\lambda\mathrm{e}^{-\lambda t}\int_{t^*}^{t}\mathrm{e}^{-is}\Big(\mathrm{e}^{is}(c+iu)-\frac{a}{\mathrm{e}^b+as}\Big)\mathrm{d}s\mathrm{d}t,$$

where t^* is the unique positive solution of the equation

$$(u+\frac{c}{i})\mathrm{e}^{it}-\frac{c}{i}=\ln(\mathrm{e}^b+at),$$

and $(u,b)\mapsto g(u,b)$ is a bounded function with $0\le u<b<\infty$.

Proof. In our demonstration we follow an idea from [19].

Let $V(u,b)$ be the expected sum of the discounted dividend payments. Then the function $(u,b) \mapsto V(u,b)$ satisfies the differential equation (20) according to Theorem 2. Along this proof, we will use the representation (3) of the risk process R_t.

There can happen two situations of interest:

(1) the first claim is before the process hits the dividend barrier,
(2) the first claim is after the process hits the barrier (in which case we have an aditional term representing the discounted sum of dividend payments until the first claim occurs).

If we condition on the time of occurrence T_1 and the amount X_1 of the first claim, we can express $V(u,b)$, by using the total probability formula, as follows:

$$
\begin{aligned}
&V(u,b) = E(\mathrm{e}^{-iT_1} V(R_{T_1}, b_{T_1}))\\
&= E(\mathrm{e}^{-iT_1} V(u\mathrm{e}^{iT_1} + c\int_0^{T_1} \mathrm{e}^{is}\mathrm{d}s - X_1, \ln(\mathrm{e}^b + aT_1)))\\
&= \int_0^{t^*} \mathrm{e}^{-it} E(V(u\mathrm{e}^{it} + c\int_0^t \mathrm{e}^{is}\mathrm{d}s - X_1, \ln(\mathrm{e}^b + at)))\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \mathrm{e}^{-it} E(V(\ln(\mathrm{e}^b + at) - X_1, \ln(\mathrm{e}^b + at)))\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \lambda\mathrm{e}^{-\lambda t} \int_{t^*}^{t} \mathrm{e}^{-is}\Big(c + i\big(u\mathrm{e}^{is} + c\frac{\mathrm{e}^{is}}{i} - \frac{c}{i}\big) - \frac{a}{\mathrm{e}^b + as}\Big)\mathrm{d}s\mathrm{d}t\\
&= \int_0^{t^*} \lambda\mathrm{e}^{-(\lambda+i)t} \int_0^{u\mathrm{e}^{it} + c\int_0^t \mathrm{e}^{is}ds} V(u\mathrm{e}^{it} + c\int_0^t \mathrm{e}^{is}\mathrm{d}s - y, \ln(\mathrm{e}^b + at))\mathrm{d}F(y)\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \lambda\mathrm{e}^{-(\lambda+i)t} \int_0^{\ln(\mathrm{e}^b + at)} V(\ln(\mathrm{e}^b + at) - y, \ln(\mathrm{e}^b + at))\mathrm{d}F(y)\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \lambda\mathrm{e}^{-\lambda t} \int_{t^*}^{t} \mathrm{e}^{-is}\Big((c + iu)\mathrm{e}^{is} - \frac{a}{\mathrm{e}^b + as}\Big)\mathrm{d}s\mathrm{d}t\\
&= \int_0^{t^*} \lambda\mathrm{e}^{-(\lambda+i)t} \int_0^{(u+\frac{c}{i})\mathrm{e}^{it} - \frac{c}{i}} V\Big(\big(u + \frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i} - y, \ln(\mathrm{e}^b + at)\Big)\mathrm{d}F(y)\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \lambda\mathrm{e}^{-(\lambda+i)t} \int_0^{\ln(\mathrm{e}^b + at)} V(\ln(\mathrm{e}^b + at) - y, \ln(\mathrm{e}^b + at))\mathrm{d}F(y)\mathrm{d}t\\
&+ \int_{t^*}^{\infty} \lambda\mathrm{e}^{-\lambda t} \int_{t^*}^{t} \mathrm{e}^{-is}\Big((c + iu)\mathrm{e}^{is} - \frac{a}{\mathrm{e}^b + as}\Big)\mathrm{d}s\mathrm{d}t\\
&= AV(u,b).
\end{aligned}
$$

Hence the expected sum of the discounted dividend payments $(u,b) \mapsto V(u,b)$, that can be expressed as the solution of the integro-differential equation (28), is a fixed point of the operator (30). □

THEOREM 4. *The integro-differential equation*

$$(c+iu)\frac{\partial V}{\partial u}+\frac{a}{\mathrm{e}^b}\frac{\partial V}{\partial b}-(\lambda+i)V(u,b)+\lambda\int_0^u V(u-y,b)\mathrm{d}F(y)=0,$$

with the boundary condition $\frac{\partial V}{\partial u}\Big|_{u=b}=1$ *has a unique bounded solution.*

Proof. First we show that the operator A, defined in Theorem 3, is a contraction. For any two bounded functions g_1 and g_2 we have

$$\begin{aligned}
&|Ag_1(u,b)-Ag_2(u,b)|\\
&=\Big|\int_0^{t^*}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\left(u+\frac{c}{i}\right)\mathrm{e}^{it}-\frac{c}{i}} g_1\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}-y,\ \ln(\mathrm{e}^b+at)\Big)\mathrm{d}F(y)\mathrm{d}t\\
&+\int_{t^*}^{\infty}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\ln(\mathrm{e}^b+at)} g_1(\ln(\mathrm{e}^b+at)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t\\
&+\int_{t^*}^{\infty}\lambda\mathrm{e}^{-\lambda t}\int_{t^*}^{t}\mathrm{e}^{-is}\Big(\mathrm{e}^{is}(c+iu)-\frac{a}{\mathrm{e}^b+as}\Big)\mathrm{d}s\mathrm{d}t\\
&-\int_0^{t^*}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\left(u+\frac{c}{i}\right)\mathrm{e}^{it}-\frac{c}{i}} g_2\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}-y,\ \ln(\mathrm{e}^b-at)\Big)\mathrm{d}F(y)\mathrm{d}t\\
&-\int_{t^*}^{\infty}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\ln(\mathrm{e}^b+at)} g_2(\ln(\mathrm{e}^b+at)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t\\
&-\int_{t^*}^{\infty}\lambda\mathrm{e}^{-\lambda t}\int_{t^*}^{t}\mathrm{e}^{-is}\Big(\mathrm{e}^{is}(c+iu)-\frac{a}{\mathrm{e}^b+as}\Big)\mathrm{d}s\mathrm{d}t\Big|.
\end{aligned}$$

Using some known integral properties and the distribution function properties, we obtain

$$\begin{aligned}
&|Ag_1(u,b)-Ag_2(u,b)|\\
&\le\Big|\int_0^{t^*}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\left(u+\frac{c}{i}\right)\mathrm{e}^{it}-\frac{c}{i}} g_1\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}-y,\ \ln(\mathrm{e}^b+at)\Big)\mathrm{d}F(y)\mathrm{d}t\\
&-\int_0^{t^*}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\left(u+\frac{c}{i}\right)\mathrm{e}^{it}-\frac{c}{i}} g_2\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}-y,\ \ln(\mathrm{e}^b+at)\Big)\mathrm{d}F(y)\mathrm{d}t\Big|\\
&+\Big|\int_{t^*}^{\infty}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\ln(\mathrm{e}^b+at)} g_1(\ln(\mathrm{e}^b+at)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t\\
&-\int_{t^*}^{\infty}\lambda\mathrm{e}^{-(\lambda+i)t}\int_0^{\ln(\mathrm{e}^b+at)} g_2(\ln(\mathrm{e}^b+at)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t\Big|\\
&\le\int_0^{t^*}\big|\lambda\mathrm{e}^{-(\lambda+i)t}\big|\int_0^{\infty}\|g_1-g_2\|_\infty\mathrm{d}F(y)\mathrm{d}t\\
&+\int_{t^*}^{\infty}\big|\lambda\mathrm{e}^{-(\lambda+i)t}\big|\int_0^{\infty}\|g_1-g_2\|_\infty\mathrm{d}F(y)\mathrm{d}t
\end{aligned}$$

$$= \int_0^{t^*} \left|\lambda e^{-(\lambda+i)t}\right| \|g_1 - g_2\|_\infty dt + \int_{t^*}^{\infty} \left|\lambda e^{-(\lambda+i)t}\right| \|g_1 - g_2\|_\infty dt$$
$$= \|g_1 - g_2\|_\infty \int_0^\infty \lambda e^{-(\lambda+i)t} dt = \frac{\lambda}{\lambda+i}\|g_1 - g_2\|_\infty,$$

where $\|g(u,b)\|_\infty = \sup_{0\le u<b<\infty} |g(u,b)|$.

As $\frac{\lambda}{\lambda+i} < 1$, it follows that the operator A is a contraction. Hence, according to Banach's Theorem, the fixed point of the operator is unique.

Applying Theorem (3), we finally obtain that equation (28) has a unique bounded solution. □

Proceeding in a similar way, we can show that the equation (26), with the boundary condition (27), has also a unique bounded solution W.

Theorem 5. *The solution of the integro-differential equation*

$$(c+iu)\frac{\partial\phi}{\partial u} + \frac{a}{e^b}\frac{\partial\phi}{\partial b} - \lambda\phi(u,b) + \lambda\int_0^u \phi(u-y,b)dF(y) = 0, \tag{31}$$

with the boundary condition

$$\frac{\partial\phi}{\partial u}\Big|_{u=b} = 0, \tag{32}$$

is a fixed point of the following operator

(33)

$$A\phi(u,b) = \int_0^{t^*} \lambda e^{-\lambda t} \int_0^{\left(u+\frac{c}{i}\right)e^{it}-\frac{c}{i}} \phi\Big(\big(u+\frac{c}{i}\big)e^{it} - \frac{c}{i} - y,\ \ln(e^b+at)\Big)dF(y)dt$$
$$+ \int_{t^*}^{\infty} \lambda e^{-\lambda t}\int_0^{\ln(e^b+at)} \phi(\ln(e^b+at)-y,\ \ln(e^b+at))dF(y)dt.$$

Proof. Let $\phi(u,b)$ be the survival probability. Then, $(u,b)\mapsto\phi(u,b)$ satisfies the differential equation (31) according to Theorem 1.

But, if we condition on the time of occurrence T_1 and the amount X_1 of the first claim, we can express $\phi(u,b)$, by using the total probability law (see [5]), as follows

$$\phi(u,b) = E(\phi(R_{T_1}, b_{T_1}))$$
$$= E(\phi(ue^{iT_1} + c\int_0^{T_1} e^{is}ds - X_1, \ln(e^b + aT_1)))$$
$$= \int_0^{t^*} E\bigg(\phi\Big(\big(u+\frac{c}{i}\big)e^{it} - \frac{c}{i} - X_1, \ln(e^b+at)\Big)\bigg)\lambda e^{-\lambda t}dt$$
$$+ \int_{t^*}^{\infty} E(\phi(\ln(e^b+at) - X_1, \ln(e^b+at)))\lambda e^{-\lambda t}dt$$

$$
\begin{aligned}
&= \int_0^{t^*} \lambda \mathrm{e}^{-\lambda t} \int_0^{\left(u+\frac{c}{i}\right)\mathrm{e}^{it}-\frac{c}{i}} \phi\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}-y, \ln(\mathrm{e}^b+at)\Big)\mathrm{d}F(y)\mathrm{d}t \\
&+ \int_{t^*}^{\infty} \lambda \mathrm{e}^{-\lambda t} \int_0^{\ln(\mathrm{e}^b+at)} \phi(\ln(\mathrm{e}^b+at)-y, \ln(\mathrm{e}^b+at))\mathrm{d}F(y)\mathrm{d}t \\
&= A\phi(u,b).
\end{aligned}
$$

Hence, the survival probability $(u, b) \mapsto \phi(u, b)$, that can be expressed as the solution of the integro-differential equation (31), is a fixed point of the operator (33). □

Remark 6. Unfortunately, we cannot apply the same approach using the contraction property to prove the uniqueness of the solution. However, we can still obtain a numerical solution for $\phi(u, b)$, by stochastic simulation of the risk process R_t with the logarithmic barrier.

4. NUMERICAL METHODS FOR ESTIMATING THE SOLUTION

4.1. The contracting operator approach. In this paragraph, we present a procedure for approximating the solution of the integro-differential equation (28), based on operator (30). Following a technique used in Tichy [20], the fixed point of the operator (30) can be obtained using the successive approximations method. We start with an arbitrary bounded function $(u, b) \mapsto f(u, b)$ and apply the contracting integral operator A to $f(u, b)$, k times. The number of iterations k in the method should be chosen according to the desired accuracy of the solution. Thus we have

$$
\begin{aligned}
g^{(k)}(u,b) &= A^k g^{(0)}(u,b), \\
g^{(0)}(u,b) &= f(u,b).
\end{aligned}
$$

Hence, we obtain a $2k$-dimensional integral for $g^{(k)}(u, b)$, which we calculate using the MC and QMC methods. For this we need to transform the integration domain of the operator (30) into the unit square $[0, 1]^2$. We consider the following change of variables:

(1) For the first double integral

$$t = -\frac{\ln(1-x_1(1-\mathrm{e}^{-(\lambda+i)t^*}))}{\lambda+i}, \tag{34}$$

$$y = F^{-1}\Big(x_2 \cdot F\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it}-\frac{c}{i}\Big)\Big). \tag{35}$$

(2) For the second double integral

$$t = t^* - \frac{\ln(1-z_1)}{\lambda+i}, \tag{36}$$

$$y = F^{-1}\Big(z_2 \cdot F(\ln(\mathrm{e}^b+at))\Big). \tag{37}$$

We consider the following notation for the inhomogeneous term of operator A defined in (30)

$$p(u,b) := \int_{t^*}^{\infty} \lambda \mathrm{e}^{-\lambda t} \int_{t^*}^{t} \mathrm{e}^{-is}\Big(\mathrm{e}^{is}(c+iu) - \frac{a}{\mathrm{e}^b + as}\Big)\mathrm{d}s\mathrm{d}t. \tag{38}$$

Thus the operator A becomes

$$\begin{aligned} Ag(u,b) = p(u,b) + \frac{\lambda}{\lambda+i} \cdot \Bigg[& (1-\mathrm{e}^{-(\lambda+i)t^*}) \\ & \cdot \int_{[0,1]^2} g\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i} - y,\ \ln(\mathrm{e}^b + at)\Big) F\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i}\Big)\mathrm{d}x_1\mathrm{d}x_2 \\ & + \mathrm{e}^{-(\lambda+i)t^*} \int_{[0,1]^2} g(\ln(\mathrm{e}^b+at) - y,\ \ln(\mathrm{e}^b+at)) F(\ln(\mathrm{e}^b+at))\mathrm{d}z_1\mathrm{d}z_2 \Bigg], \end{aligned} \tag{39}$$

where t and y are determined according to formulas (34) and (35), for the first integral, and according to formulas (36) and (37), for the second integral.

For a given choice of the pair (u,b), the MC estimator is

$$\widetilde{V}_{MC}(u,b) = \frac{1}{N}\sum_{n=1}^{N} g_n^{(k)}(u,b), \tag{40}$$

where each $g_n^{(k)}(u,b)$ $(n=1,\dots N)$ is calculated recursively by

$$g_n^{(0)}(u,b) = f(u,b), \tag{41}$$

$$\begin{aligned} g_n^{(j)}(u,b) = p(u,b) + \frac{\lambda}{\lambda+i} \cdot \Bigg[& (1-\mathrm{e}^{-(\lambda+i)t^*}) F\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it_{1,n}^{(j)}} - \frac{c}{i}\Big) \\ & \cdot g_n^{(j-1)}\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it_{1,n}^{(j)}} - \frac{c}{i} - y_{1,n}^{(j)},\ \ln(\mathrm{e}^b + at_{1,n}^{(j)})\Big) \\ & + \mathrm{e}^{-(\lambda+i)t^*} F(\ln(\mathrm{e}^b + at_{2,n}^{(j)})) \\ & \cdot g_n^{(j-1)}(\ln(\mathrm{e}^b + at_{2,n}^{(j)}) - y_{2,n}^{(j)},\ \ln(\mathrm{e}^b + at_{2,n}^{(j)})) \Bigg], \end{aligned} \tag{42}$$

for $j=1,\dots,k$. The values $t_{1,n}^{(j)}$, $y_{1,n}^{(j)}$, $t_{2,n}^{(j)}$ and $y_{2,n}^{(j)}$, $j=1,\dots,k$, are determined as described below. We first consider the random points $(x_n^{(1)},\dots,x_n^{(2k)})$ and $(z_n^{(1)},\dots,z_n^{(2k)})$ uniformly distributed on $[0,1]^{2k}$. Then $t_{1,n}^{(j)}$ and $y_{1,n}^{(j)}$, $j=1,\dots,k$, are determined according to relations (34) and (35), with x_1 and x_2 being the values $x_n^{(j)}$ and $x_n^{(j+k)}$, respectively, whereas $t_{2,n}^{(j)}$ and $y_{2,n}^{(j)}$, $j=1,\dots,k$, are determined using relations (36) and (37), with z_1 and z_2 being the values $z_n^{(j)}$ and $z_n^{(j+k)}$, respectively.

The procedure is described in the following algorithm:

ALGORITHM 7. *Monte Carlo Algorithm for estimating the solution $V(u,b)$ of equation (28)*

Input data:

- the initial reserve u of the insurance company, and the initial value b of the logarithmic barrier;
- the parameters λ, δ, μ, a, c;
- the integers N, k;
- the bounded function f;

FOR $n = 1, \dots, N$ DO

Generate the random points $(x_n^{(1)}, \dots, x_n^{(2k)})$ and $(z_n^{(1)}, \dots, z_n^{(2k)})$ uniformly distributed on $[0,1]^{2k}$.

Calculate the values $t_{1,n}^{(j)}$ and $y_{1,n}^{(j)}$, $j = 1, \dots, k$, according to relations (34) and (35), with x_1 and x_2 being the values $x_n^{(j)}$ and $x_n^{(j+k)}$, respectively.

Compute the values $t_{2,n}^{(j)}$ and $y_{2,n}^{(j)}$, $j = 1, \dots, k$, using relations (36) and (37), with z_1 and z_2 being the values $z_n^{(j)}$ and $z_n^{(j+k)}$, respectively.

Calculate recursively $g_n^{(k)}(u,b)$, using formulas (41) and (42).

END FOR

Compute the MC estimate

$$\widetilde{V}_{MC}(u,b) = \frac{1}{N} \sum_{n=1}^{N} g_n^{(k)}(u,b).$$

Output data: The value $\widetilde{V}_{MC}(u,b)$, which approximates the expected sum of discounted dividend payments $V(u,b)$.

For a given choice of the pair (u,b), the QMC estimator is

$$\widetilde{V}_{QMC}(u,b) = \frac{1}{N} \sum_{n=1}^{N} g_n^{(k)}(u,b), \tag{43}$$

where each $g_n^{(k)}(u,b)$ $(n = 1, \dots N)$ is calculated recursively using formulas (41) and (42).

In the case of QMC method, the values $t_{1,n}^{(j)}$, $y_{1,n}^{(j)}$, $t_{2,n}^{(j)}$ and $y_{2,n}^{(j)}$, $j = 1, \dots, k$, in formulas (41) and (42), are determined as follows. We first consider the low-discrepancy sequences of points on $[0,1]^{2k}$, $x = (x_1, \dots, x_N)$ for the first integral and $z = (z_1, \dots, z_N)$ for the second integral. Each point of the sequence x is of the form $x_n = (x_n^{(1)}, \dots, x_n^{(2k)})$ $(n = 1, \dots, N)$, and each point of the sequence z is of the form $z_n = (z_n^{(1)}, \dots, z_n^{(2k)})$ $(n = 1, \dots, N)$.

The values $t_{1,n}^{(j)}$ and $y_{1,n}^{(j)}$, $j = 1,\dots,k$, are determined next according to relations (34) and (35), with x_1 and x_2 being the values $x_n^{(j)}$ and $x_n^{(j+k)}$, respectively, whereas $t_{2,n}^{(j)}$ and $y_{2,n}^{(j)}$, $j = 1,\dots,k$, are determined using relations (36) and (37), with z_1 and z_2 being the values $z_n^{(j)}$ and $z_n^{(j+k)}$, respectively.

A similar Quasi-Monte Carlo Algorithm can be written for estimating the solution $V(u,b)$ of equation (28).

Following an idea from Albrecher and Kainhofer [1], one can combine the two double integrals from operator (30) into one integral, by a suitable change of variables. We denote

$$y_{\min}(u,b,t) = \min\Big(\big(u+\frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i}, \ln(\mathrm{e}^b + at)\Big), \tag{44}$$

and consider the following change of variables

$$t = -\frac{\ln(1-v_1)}{\lambda+i}, \tag{45}$$

$$y = F^{-1}\Big(v_2 \cdot F\Big(y_{\min}(u,b,t)\Big)\Big). \tag{46}$$

Then, the operator A, defined in (30), can be expressed as

(47)

$$Ag(u,b) = \frac{\lambda}{\lambda+i}\int_{[0,1]^2} F(y_{\min}(u,b,t))g(y_{\min}(u,b,t)-y,\ \ln(\mathrm{e}^b+at))\mathrm{d}v_1\mathrm{d}v_2 + p(u,b).$$

Like in the previous procedure, we start with an initial bounded function $g^{(0)}(u,b) = f(u,b)$ and apply k times the integral operator A onto $g^{(0)}(u,b) = f(u,b)$, $g^{(k)}(u,b) = A^k g^{(0)}(u,b)$. Thus, we obtain an approximation of the fixed point of the operator A, which is the solution of the integro-differential equation (28).

The MC estimator for $V(u,b)$ is

$$\widehat{V}_{MC}(u,b) = \frac{1}{N}\sum_{n=1}^{N} g_n^{(k)}(u,b), \tag{48}$$

where each $g_n^{(k)}(u,b)$ $(n = 1,\dots N)$ is calculated recursively by

$$g_n^{(0)}(u,b) = f(u,b), \tag{49}$$

$$g_n^{(j)}(u,b) = \frac{\lambda}{\lambda+i}F(y_{\min}(u,b,t_n^{(j)})) \cdot g_n^{(j-1)}(y_{\min}(u,b,t_n^{(j)}) - y_n^{(j)},\ \ln(\mathrm{e}^b + at_n^{(j)})) + p(u,b), \tag{50}$$

for $j = 1,\dots,k$. The values $t_n^{(j)}$, $y_n^{(j)}$, $j = 1,\dots,k$, are determined as described below. We first consider the random point $(v_n^{(1)},\dots,v_n^{(2k)})$ uniformly distributed on $[0,1]^{2k}$. Next the values $t_n^{(j)}$ and $y_n^{(j)}$, $j = 1, \dots, k$, are determined according to relations (45) and (46), with x_1 and x_2 being the values $v_n^{(j)}$ and $v_n^{(j+k)}$, respectively.

This procedure is formulated in the following algorithm:

ALGORITHM 8. *Monte Carlo Algorithm for estimating the solution $V(u,b)$ of equation (28), in the case of modified operator*

Input data:

- the initial reserve u of the insurance company, and the initial value b of the logarithmic barrier;
- the parameters λ, δ, μ, a, c;
- the integers N, k;
- the bounded function f;

FOR $n = 1, \dots, N$ DO

Generate the random point $(v_n^{(1)}, \dots, v_n^{(2k)})$ uniformly distributed on $[0,1]^{2k}$.

Determine the values $t_n^{(j)}$ and $y_n^{(j)}$, $j = 1, \dots, k$, according to relations (45) and (46), with x_1 and x_2 being the values $v_n^{(j)}$ and $v_n^{(j+k)}$, respectively.

Calculate recursively $g_n^{(k)}(u,b)$, using formulas (49) and (50).

END FOR

Compute the MC estimate

$$\widehat{V}_{MC}(u,b) = \frac{1}{N} \sum_{n=1}^{N} g_n^{(k)}(u,b).$$

Output data: The value $\widehat{V}(u,b)$, which approximates the expected sum of discounted dividend payments $V(u,b)$.

The QMC estimator for $V(u,b)$ is

$$\widehat{V}_{QMC}(u,b) = \frac{1}{N} \sum_{n=1}^{N} g_n^{(k)}(u,b), \tag{51}$$

where each $g_n^{(k)}(u,b)$ $(n = 1, \dots N)$ is calculated recursively, using formulas (49) and (50).

In the case of QMC method, the values $t_n^{(j)}$, $y_n^{(j)}$, $j = 1, \dots, k$, in formulas (49) and (50), are determined as follows. We first consider the low-discrepancy sequence of points on $[0,1]^{2k}$, $v = (v_1, \dots, v_N)$. Each point of the sequence v is of the form and $v_n = (v_n^{(1)}, \dots, v_n^{(2k)})$ $(n = 1, \dots, N)$. Next the values $t_n^{(j)}$ and $y_n^{(j)}$, $j = 1, \dots, k$, are determined according to relations (45) and (46), with x_1 and x_2 being the values $v_n^{(j)}$ and $v_n^{(j+k)}$, respectively.

A similar Quasi-Monte Carlo Algorithm can be written for estimating the solution $V(u,b)$ of equation (28).

4.2. The direct simulation approach. As we know, there are no analytical solutions for the survival probability $\phi(u,b)$ or for the expected sum of discounted dividend payments $V(u,b)$, which are the most important quantities in risk theory models. That is why estimates obtained through the direct simulation of the risk process (see [14]) are needed to measure the accuracy of the estimates calculated using the MC and QMC algorithms that we developed in the previous paragraph.

We simulate N paths of the risk process (1) as follows:

- Start with the initial values $t_0 = 0$, $r_0 = u$, $b_0 = b$, where u is the initial reserve of the insurance company and b is the initial value of the logarithmic barrier b_t.
- As the number of claims from the interval $[0,t]$ are Poisson distributed with parameter λ, it follows that the time between successive claims (inter-arrival time) is exponentially distributed with intensity λ and independent of the past. For $k \geq 0$, we generate an exponentially distributed random variable $\bar{t}_k$ with parameter λ and set the time $t_{k+1} := t_k + \bar{t}_k$, which is the time when the $(k+1)$-th claim occurs.
- For $k \geq 0$ we generate the claim amount y_k, from a random variable with distribution function $F(y)$. In order to do this, one can use the inversion method (see [16] and [17]).
- We set the new reserve after a claim $r_{k+1} := \min\{\big(r_k + \frac{c}{i}\big)\mathrm{e}^{i\bar{t}_k} - \frac{c}{i},\ \ln(\mathrm{e}^{b_k} + a\bar{t}_k)\} - y_k$. Due to the dividend barrier's structure, we can reset the origin to time t_{k+1} in every step, by setting $b_{k+1} = \ln(\mathrm{e}^{b_k} + a\bar{t}_k)$.

Counting the trajectories that survive and dividing this number by the total number N of simulated trajectories, we obtain an unbiased estimator for the survival probability $\phi(u,b)$

$$\widehat{\phi}(u,b) = \frac{1}{N}\sum_{j=1}^{N} 1_A(T_j), \tag{52}$$

where A is the set of all trajectories T_j, for which ruin does not occur (i. e. $r_k > 0,\ \forall k$).

It can happen that $R_t \to \infty$, as $t \to \infty$, without the reserve R_t ever becoming negative. Hence, the reserve process stops with probability less than 1. Therefore it is necessary to stop the process at some time instance. This can be done at time $t_{\max}$, for suitably large $t_{\max}$. Stopping the process at this time, we actually overestimate the true survival probability. However, if the stopping time $t_{\max}$ is chosen large enough, this bias becomes negligible. In our simulations, we have increased $t_{\max}$ until practically no difference in the value of (52) was observable by further increasing $t_{\max}$.

Instead of considering as stopping criterion a finite time $t_{\max}$ one can employ an upper absorbing horizontal barrier at $r_{\max} > u$, as stopping criteria, for trajectories not leading to ruin. Hence, we stop every realization of a path at time $T = \min\{T_{\text{ruin}},\ \inf\{t|R_t \geq r_{\max}\}\}$, where T_{ruin} is the time of ruin. Using this stopping criterion we again overestimate the survival probability $\phi(u, b)$. However, if the threshold $r_{\max}$ is chosen large enough, this effect becomes negligible.

For simulating the expected sum $V(u, b)$ of discounted dividend payments, we proceed as for the generation of N paths of the risk process (1). Whenever the process hits the dividend barrier, i.e., $\big(r_k + \frac{c}{i}\big)\mathrm{e}^{i\bar{t}_k} - \frac{c}{i} > \ln(\mathrm{e}^{b_k} + a\bar{t}_k)$, we need to calculate the amount of dividends that are paid out until the $(k+1)$-th claim occurs:

$$\begin{aligned} v_{k+1} &:= v_k + \mathrm{e}^{-it_{k+1}} \int_{t^*}^{\bar{t}_k} \mathrm{e}^{-is}\Big((c + ir_k)\mathrm{e}^{is} - \frac{a}{\mathrm{e}^{b_k} + as}\Big)\mathrm{d}s, \quad k \geq 0, \\ v_0 &:= 0, \end{aligned}$$

where t^* is the unique positive solution of the equation $\big(r_k + \frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i} = \ln(\mathrm{e}^{b_k} + at)$. The process is stopped whenever the ruin occurs (i.e., $\exists k > 0$ such that $r_k < 0$) or at sufficiently large time instance $t_{\max}$, after which the expected value of discounted dividend payments is negligible, due to the discount factor. Then the expected value of discounted dividend payments is estimated by

$$\overline{V}(u, b) = \frac{1}{N}\sum_{l=1}^{N} v(l), \tag{53}$$

where $v(l)$ is the final value of v_i for trajectory l.

4.3. Numerical results. In this section we present the numerical results that are obtained using exponentially distributed claim amounts ($F(y) = 1 - \mathrm{e}^{-\mu y}$). Note that in this case the solution t^* of the equation

$$\big(u + \frac{c}{i}\big)\mathrm{e}^{it} - \frac{c}{i} = \ln(\mathrm{e}^{b} + at) \tag{54}$$

needs to be calculated numerically.

The parameters are set to $c = 1.5$, $\delta = 0.1$, $a = 1$ and $\lambda = \mu = 1$. The values for u and b will be specified later.

The MC estimates of $\phi(u, b)$ are obtained from the direct simulation, using $N = 1000$ and $N = 5000$ paths. The corresponding "true" or "exact" value of $\phi(u, b)$ is obtained, in the lack of an analytical solution, by a long direct simulation of 100 000 paths, for each choice of the pair (u, b) that we analyze. As stopping criteria for the direct simulation of the survival probability $\phi(u, b)$, we employed a finite time $t_{\max} = 100$.

The estimates $\widetilde{V}_{MC}(u, b)$ and $\widetilde{V}_{QMC}(u, b)$, given by formulas (40) and (43), respectively, are obtained using $N = 200$ and $N = 400$ paths. For the estimates $\widehat{V}_{MC}(u, b)$ and $\widehat{V}_{QMC}(u, b)$, given by formulas (48) and (51), respectively, we used a double number of paths N. In the lack of an analytical solution, the corresponding "exact" value for $V(u,b)$ is obtained by a long direct simulation of 10000 paths, for each choice of the pair (u,b) that we analyze.

In the calculations of $\widetilde{V}_{MC}(u,b)$, $\widetilde{V}_{QMC}(u,b)$, $\widehat{V}_{MC}(u,b)$ and $\widehat{V}_{QMC}(u,b)$, we considered a recursion depth of $k=5$, leading us to a 10-dimensional integral. We choose the function $f(u,b)$ to be constant, the value of the function being equal to the expected value of discounted dividend payments, obtained in a short direct simulation of the risk process, with $N=10$.

We employed Halton sequences and SQRT sequences (see [13]), for our QMC calculations.

As stopping criteria for the direct simulation of the expected value of discounted dividend payments, it turned out that a choice of $t_{\max}=200$ is sufficient for our estimations. As the inter-arrival times are exponentially distributed with mean 1, there are necessary, on average, 200 exponentially distributed times and 200 claim occurrences until the process is stopped.

The exact values for the survival probability $\phi(u,b)$ and for the expected sum of discounted dividend payments $V(u,b)$, along with the absolute value of the errors for the corresponding estimates $\widehat{\phi}(u,b)$, $\widetilde{V}_{MC}(u,b)$, $\widetilde{V}_{QMC}(u,b)$, $\widehat{V}_{MC}(u,b)$ and $\widehat{V}_{QMC}(u,b)$ are given for each choice of u and b.

4.3.1. *Simulation results for survival probability.* The survival probability $\phi(u,b)$ is estimated using the direct simulation approach. Table 1 presents the exact values of $\phi(u,b)$ and the absolute error for the estimates $\widehat{\phi}(u,b)$, obtained through direct simulation, for $N=1000$ and $N=5000$, when b is fixed to value 20 and u is varying from 10 to 15.

Table 1: Exact values of $\phi(u,b)$ and absolute error $|\phi(u,b)-\widehat{\phi}(u,b)|$, for fixed $b=20$

(u, b)	Exact values of the survival probability $\phi(u,b)$	$\lvert\phi(u,b)-\widehat{\phi}(u,b)\rvert$ N=1000	$\lvert\phi(u,b)-\widehat{\phi}(u,b)\rvert$ N=5000
(10, 20)	0.99790	0.0019	0.0011
(11, 20)	0.99870	0.0017	0.0005
(12, 20)	0.99920	0.0012	0.0006
(13, 20)	0.99944	0.0004	0.0002
(14, 20)	0.99953	0.0005	0.0001
(15, 20)	0.99961	0.0006	0.0002

The numerical results show an important improvement as N increases from 1000 to 5000 paths. Even for small values of N, the absolute errors that we obtain are very small.

4.3.2. *Simulation results for the expected sum of discounted dividend payments.* The exact values of the expected sum of discounted dividend payments $V(u,b)$, for each choice of the pair (u,b), are presented in Table 2.

Table 2: Exact values of the expected discounted dividend payments $V(u, b)$

(u, b)	Exact values of the expected dividend payments
(0.5, 1)	2.4387
(0.5, 1.5)	2.5017
(1.5, 2)	3.6858
(1.5, 2.5)	3.8408
(1.5, 3)	3.9945

The numerical results for the estimators $\widetilde{V}_{MC}(u,b)$, $\widetilde{V}_{QMC}(u,b)$, $\widehat{V}_{MC}(u,b)$ and $\widehat{V}_{QMC}(u,b)$, in terms of their absolute error $|V(u,b)-\widetilde{V}_{MC}(u,b)|$, $|V(u,b)-\widetilde{V}_{QMC}(u,b)|$, $|V(u,b)-\widehat{V}_{MC}(u,b)|$ and $|V(u,b)-\widehat{V}_{QMC}(u,b)|$, respectively, are given in Table 3. We used a number of $N = 200$ and $N = 400$ paths for each estimate $\widetilde{V}_{MC}(u,b)$, $\widetilde{V}_{QMC}(u,b)$, $\widehat{V}_{MC}(u,b)$ and $\widehat{V}_{QMC}(u,b)$. In the case of QMC estimations we displayed the absolute errors obtained using Halton sequences and SQRT sequences (Err^{Halton} and Err^{SQRT}).

Table 3: Absolute errors $|V(u,b) - \widetilde{V}(u,b)|$ and $|V(u,b) - \widehat{V}(u,b)|$, obtained by MC and QMC methods

		Estimator $\widetilde{V}$			Estimator $\widehat{V}$		
(u, b)	N	Err^{MC}	Err^{Halton}	Err^{SQRT}	Err^{MC}	Err^{Halton}	Err^{SQRT}
(0.5, 1)	200	0.0165	0.0237	0.0210	0.0210	0.0279	0.0214
(0.5, 1)	400	0.0188	0.0220	0.0199	0.0222	0.0238	0.0202
(0.5, 1.5)	200	0.0162	0.0048	0.0125	0.0126	0.0022	0.0108
(0.5, 1.5)	400	0.0142	0.0077	0.0117	0.0111	0.0080	0.0120
(1.5, 2)	200	0.0083	0.0109	0.0100	0.0116	0.0124	0.0101
(1.5, 2)	400	0.0091	0.0104	0.0096	0.0114	0.0110	0.0096
(1.5, 2.5)	200	0.0126	0.0079	0.0111	0.0106	0.0067	0.0104
(1.5, 2.5)	400	0.0118	0.0091	0.0108	0.0105	0.0093	0.0109
(1.5, 3)	200	0.0045	0.0027	0.0028	0.0020	0.0031	0.0017
(1.3, 3)	400	0.0035	0.0009	0.0019	0.0011	0.0002	0.0022

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