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INTEGRAL CHARACTERIZATIONS OF WEIGHTED BLOCH SPACES AND $Q_{K,\omega}(p,q)$ SPACES

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Abstract. In this paper we introduce a new space of functions, the so called $Q_{K,\omega}(p,q)$ space of holomorphic functions on the unit disk in terms of nondecreasing functions. The relation between the integral norm of the $Q_{K,\omega}(p,q)$ space and the integral norm of the weighted Bloch space $\mathcal{B}^{\alpha}_{\omega}$ is also given. Further, we obtain similar integral criteria for the little weighted Bloch functions of analytic functions and meromorphic functions.

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1. INTRODUCTION

Let $\Delta = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [2]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty\};$$

the little Bloch space $\mathcal{B}_0$ (cf. [2]) is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

The Dirichlet space is defined by

$$\mathcal{D} = \{f : f \text{ analytic in } \Delta \text{ and } \int_{\Delta}|f'(z)|^2 \mathrm{d}\sigma_z < \infty\},$$

where $\mathrm{d}\sigma_z$ is the Euclidean area element $\mathrm{d}x\mathrm{d}y$. Let $0 < q < \infty$. The Besov-type spaces

$$\mathbf{B^q} = \left\{f : f \text{ analytic in } \Delta,\ \sup_{a\in\Delta}\int_{\Delta}|f'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^2\mathrm{d}\sigma_z < \infty\right\}$$

have been introduced and studied intensively by Stroethoff (cf. [13]). Here, $\varphi_a(z)$ stands for the Möbius transformation of Δ given by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{where } a \in \Delta.$$

In 1994, Aulaskari and Lappan [2] introduced a class of holomorphic functions, the so called Q_p-spaces, as follows:

$$Q_p = \left\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_{\Delta}|f'(z)|^2 g^p(z,a)\mathrm{d}\sigma_z < \infty\right\},$$

where $0 < p < \infty$ and the weight function

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$$

is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. The weight function $g(z,a)$ is actually Green's function in Δ with pole at $a \in \Delta$.

For $0 < p < \infty$, $-2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ (cf. [18]), if

$$\|f\|_{K,p,q} \quad = \quad \sup_{a\in\Delta}\int_{\Delta}\left|f'(z)\right|^p\left(1-|z|^2\right)^q K(g(z,a))\mathrm{d}\sigma_z < \infty.$$

Recall that the analytic function

$$f(z) = \sum_{k}^{\infty} a_k z^{n_k} \quad (\text{with } \ n_k \in \mathbb{N}\ ; \text{ for all } \ k \in \mathbb{N} = \{1,2,3,\dots\}\,)$$

is said to belong to the Hadamard gap class (also known as lacunary series) if there exists a constant $c > 1$ such that $\frac{n_{k+1}}{n_k} \geq c$ for all $k \in \mathbb{N}$ (see e.g. [20]).

Two quantities A_f and B_f, both depending on an analytic function f on Δ, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on Δ we have $\frac{1}{C}B_f \leq A_f \leq CB_f$. If the quantities A_f and B_f are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

Now, given a reasonable function $\omega\colon (0,1] \to [0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ (see [5]) is defined as the set of all analytic functions f on Δ satisfying

$$(1-|z|)|f'(z)| \leq C\omega(1-|z|), \quad z \in \Delta,$$

for some fixed $C = C_f > 0$. In the special case when $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here the word "reasonable" is a non-mathematical term; it means that the function is "not too bad" and that it satisfies some natural conditions.

We introduce now the following definitions:

Definition 1.1. Let $\omega\colon (0,1] \to [0,\infty)$ be a given reasonable function and $0 < \alpha < \infty$. An analytic function f on Δ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}^\alpha_\omega$ if

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\Delta}\frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| < \infty.$$

Definition 1.2. For a given reasonable function $\omega : (0,1] \to [0,\infty)$ and for $0 < \alpha < \infty$ an analytic function f on Δ is said to belong to the little weighted Bloch space $\mathcal{B}^\alpha_{\omega,0}$ if

$$\|f\|_{\mathcal{B}^\alpha_{\omega,0}} = \lim_{|z|\to 1^-}\frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| = 0.$$

Throughout this paper and for some techniqual reasons we assume that $\omega \not\equiv 0$. Now, we introduce the following new definition:

DEFINITION 1.3. For a nondecreasing function $K : [0,\infty) \to [0,\infty)$, for $0 < p < \infty$, $-2 < q < \infty$, and for a given reasonable function $\omega : (0,1] \to (0,\infty)$ an analytic function f in Δ is said to belong to the space $Q_{K,\omega}(p,q)$ if

$$\|f\|^p_{K,\omega,p,q} = \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, \mathrm{d}\sigma_z < \infty.$$

REMARK 1.1. It should be remarked that our $Q_{K,\omega}(p,q)$ classes are more general than many classes of analytic functions. If $\omega \equiv 1$, we obtain $Q_K(p,q)$ type spaces (cf. [18] and [17]). If $q = p = 2$, and $\omega(t) = t$, we obtain Q_K spaces as studied recently in [6, 7, 11, 15, 16, 19] and others. If $q = p = 2$, $\omega(t) = t$ and $K(t) = t^p$, we obtain Q_p spaces as studied in [2, 3, 20] and others. If $\omega \equiv 1$ and $K(t) = t^s$, then $Q_{K,\omega} = F(p,q,s)$ classes (cf. [1, 21]).

In this paper, we characterize the weighted Bloch space $\mathcal{B}^\alpha_\omega$ by mean of our $Q_{K,\omega}(p,q)$ spaces. One of the main results is a general Besov-type characterization for $\mathcal{B}^\alpha_\omega$ functions that extends and generalizes the Stroethoff's theorem in [13]. Also, we extend and improve some results due to Essén et. al ([7]) using our new definitions.

2. HOLOMORPHIC $Q_{K,\omega}$ CLASSES

In this paper we show some relations between the $Q_{K,\omega}(p,q)$ norms and the $\mathcal{B}^\alpha_\omega$ norms for a nondecreasing function K. We also give a general way to construct different spaces $Q_{K,\omega_1}(p,q)$ and $Q_{K_2,\omega}(p,q)$ by using some functions K_1 and K_2.

Before proving the main theorems we recall a few facts about the Möbius function φ_a. First, the function φ_a is easily seen to be its own inverse under composition: $(\varphi_a \circ \varphi_a)(z) = z$ for all $z \in \Delta$. The following identity can be obtained by straightforward computation:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\ , \quad (a, z \in \Delta).$$

A slightly different form in which we will apply the above identity is:

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi'_a(z)|\ , \quad (a, z \in \Delta). \tag{1}$$

For $a \in \Delta$ consider the substitution $z = \varphi_a(w)$. Then $\mathrm{d}\sigma_w = |\varphi'_a(z)|^2 \mathrm{d}\sigma_z$ by the Jacobian change in measure. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)} h(\varphi_a(w))\mathrm{d}\sigma_w = \int_{\Delta(a,r)} h(z) \left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 \mathrm{d}\sigma_z\ . \tag{2}$$

We assume throughout this paper that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,\mathrm{d}r<\infty\,. \tag{3}$$

We need the following lemmas in the sequel.

LEMMA 2.1. ([20]) *Let* $\alpha\in(0,\infty)$ *and suppose that* $f(z)=\sum_{j=1}^{\infty}a_j z^{n_j}$ *belongs to Hadamard gap class. Then* $f\in\mathcal{B}^{\alpha}$ *if and only if* $\sup_{j\in\mathbb{N}}|a_j|n_j^{1-\alpha}<\infty$, *where* $\mathbb{N}=\{1,2,3,\dots\}$.

LEMMA 2.2. ([10]) *Let* $\omega:(0,1]\to(0,\infty)$ *and let* $1\le\alpha<\infty$. *Then there are two functions* f_1 , $f_2\in\mathcal{B}^{\alpha}_{\omega}$ *such that*

$$|f_1'(z)|+|f_2'(z)|\approx\frac{\omega(1-|z|)}{(1-|z|)^{\alpha}}\,,\qquad z\in\Delta. \tag{4}$$

THEOREM 2.1. *Let* $0<p<\infty$, $-2<q<\infty$. *Then, for each non-decreasing function* $K:[0,\infty)\to[0,\infty)$ *and for a given reasonable non-decreasing function* $\omega:(0,1]\to(0,\infty)$ *with* $\omega(k\,t)\approx\omega(t)$, $k>0$, *we have*

(i) $Q_{K,\omega}(p,q)\subset\mathcal{B}_{\omega}^{\frac{q+2}{p}}$ *and*

(ii) $Q_{K,\omega}(p,q)=\mathcal{B}_{\omega}^{\frac{q+2}{p}}$ *if and only if*

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,\mathrm{d}r<\infty.$$

Proof. For fixed $r\in(0,1)$ and $a\in\Delta$ let

$$E(a,r)=\left\{z\in\Delta:\ |z-a|<r(1-|a|)\right\}.$$

We know that $E(a,r)\subset\Delta(a,r)$. Also, for any $z\in E(a,r)$ we have

$$(1-r)(1-|a|)\le 1-|z|\le(1+r)(1-|a|),$$

which means that $1-|z|^2\simeq 1-|a|^2$ for any $z\in E(a,r)$. Denote

$$F_{\omega,p,q}(f)(z)=\left|f'(z)\right|^p\,\frac{(1-|z|)^q}{\omega^p(1-|z|)}.$$

Then, we obtain

$$\begin{aligned}\int_{\Delta}F_{\omega,p,q}(f)(z)K\big(g(z,a)\big)\,\mathrm{d}\sigma_z\ &\ge\ \int_{\Delta(a,r)}F_{\omega,p,q}(f)(z)K\big(g(z,a)\big)\,\mathrm{d}\sigma_z\\ &\ge\ K\left(\log\frac{1}{r}\right)\int_{\Delta(a,r)}F_{\omega,p,q}(f)(z)\,\mathrm{d}\,\sigma_z\\ &\ge\ K\left(\log\frac{1}{r}\right)\int_{E(a,r)}F_{\omega,p,q}(f)(z)\,\mathrm{d}\sigma_z.\end{aligned}$$

For every $z\in E(a,r)$ we have

$$(1-r)(1-|a|)\le 1-|z|\le(1+r)(1-|a|),$$

hence

$$(1-|z|)^p \geq (1-r)^p(1-|a|)^p\ ,\ \ \forall\ \ p>0.$$

Now, since ω is non-decreasing, we obtain that

$$\int_{E(a,r)} F_{\omega,p,q}(f)(z)\ \mathrm{d}\sigma_z \geq \frac{(1-r)^p(1-|a|)^q}{\omega^p((1-r)(1-|a|))}\int_{E(a,r)} \big|f'(z)\big|^p\ \mathrm{d}\sigma_z.$$

Since $|f'(z)|^p$ is a subharmonic function, it follows that

$$\int_{E(a,r)} \big|f'(z)\big|^p\ \mathrm{d}\sigma_z \geq |E(a,r)|\ \cdot\ |f'(a)\big|^p = r^2(1-|a|)^2|f'(a)\big|^p.$$

Then we obtain

$$\begin{aligned}\int_\Delta F_{\omega,p}(f)(z)K\big(g(z,a)\big)\ \mathrm{d}\sigma_z\ &\geq\ K\bigg(\log\frac{1}{r}\bigg)\frac{(1-r)^p(1-|a|)^{q+2}}{\omega^p((1-r)(1-|a|))}|f'(a)\big|^p\\ &\geq\ \lambda K\bigg(\log\frac{1}{r}\bigg)\frac{(1-r)^p(1-|a|)^{q+2}}{\omega^p(1-|a|)}\big|f'(a)\big|^p,\end{aligned}$$

where λ is a constant. If $f\in Q_{K,\omega}(p,q)$, then by the above estimate we have

$$\sup_{a\in\Delta}\frac{(1-|a|)^{q+2}|f'(z)|^p}{\omega^p(1-|a|)} < \infty.$$

The proof of (i) is therefore completed.

Now, we show that $\mathcal{B}_\omega^{\frac{q+2}{p}} \subset Q_{K,\omega}(p,q)$ provided that K satisfies condition (3). For $f\in\mathcal{B}_\omega^{\frac{q+2}{p}}$, we have

$$\begin{aligned}\int_\Delta F_{\omega,p,q}(f)(z)K\big(g(z,a)\big)\ \mathrm{d}\sigma_z\ \ &\leq \big\|f\big\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\ \int_\Delta(1-|z|^2)^{-2}K\big(g(z,a)\big)\,\mathrm{d}\,\sigma_z\\ &= 2\pi\big\|f\big\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\ \int_0^1 K\bigg(\log\tfrac{1}{r}\bigg)\ \tfrac{r}{(1-r^2)^2}\,\mathrm{d}r < \infty,\end{aligned}$$

which shows that

$$\mathcal{B}_\omega^{\frac{q+2}{p}} \subset Q_{K,\omega}(p,q).$$

Now we assume that $\mathcal{B}_\omega^{\frac{p+2}{p}} = Q_{K,\omega}(p,q)$ and we show that (3) holds. From Lemma 2.3, for f_1 and f_2 in $\mathcal{B}_\omega^{\frac{q+2}{p}}$, we have

$$|f_1'(z)|+|f_2'(z)| \geq \frac{\omega(1-|z|)}{(1-|z|)^{\frac{q+2}{p}}}. \tag{5}$$

Then $f_1, f_2\in Q_{K,\omega}(p,q)$ and

$$\begin{aligned}\infty\ \ &>\ \ \sup_{a\in\Delta}\int_\Delta\bigg(\big|f_1'(z)\big|^p+\big|f_2'(z)\big|^p\bigg)(1-|z|)^q\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\ \mathrm{d}\sigma_z\\ &\geq\ \ \int_\Delta\bigg(\big|f_1'(z)\big|+\big|f_2'(z)\big|\bigg)^p\ (1-|z|)^q\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\ \mathrm{d}\sigma_z. \qquad (6)\end{aligned}$$

From (5) and (6), we obtain

$$\int_{\Delta}\left(\left|f_1'(z)\right|^p+\left|f_2'(z)\right|^p\right)(1-|z|)^q\,\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z \approx 2\pi\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,\mathrm{d}r.$$

Thus (3) holds, and this completes the proof. □

3. THE CLASSES $Q_{K,\omega,0}$ AND $\mathcal{B}^{\alpha}_{\omega,0}$

We say that $f\in Q_{K,\omega,0}(p,q)$ if

$$\lim_{|a|\to 1^-}\int_{\Delta}\left|f'(z)\right|^p(1-|z|)^q\,\frac{K(g(z,a))}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z=0. \tag{7}$$

Also, as a subspace of $\mathcal{B}^{\alpha}_{\omega}$, we define the little weighted Bloch space $\mathcal{B}^{\alpha}_{\omega,0}$ as the space which consists of analytic functions f on Δ such that

$$\lim_{|z|\to 1^-}\frac{(1-|z|)^{\alpha}|f'(z)|}{\omega(1-|z|)}=0,$$

where $0<\alpha<\infty$. Then we obtain the following theorem:

THEOREM 3.1. *Let* $0<p<\infty$, $-2<q<\infty$. *Then, for each non-decreasing function* $K:[0,\infty)\to[0,\infty)$ *and for a given reasonable non-decreasing function* $\omega:(0,1]\to(0,\infty)$ *with* $\omega(k\,t)\approx\omega(t)$, $k>0$, *we have*

(i) $Q_{K,\omega,0}(p,q)\subset\mathcal{B}^{\frac{q+2}{p}}_{\omega,0}$ *and*

(ii) $Q_{K,\omega,0}(p,q)=\mathcal{B}^{\frac{q+2}{p}}_{\omega,0}$ *if and only if* (3) *holds.*

Proof. Without loss of generality we assume that $K(1)>0$. From the proof of Theorem 2.1, we have

$$\begin{aligned}\pi(\tfrac{1}{\mathrm{e}})^2\,K(1)\tfrac{(1-|a|)^{q+2}}{\omega^p(1-|a|)}|f'(a)|^p &\le K(1)\int_{E(a)}F_{\omega,p,q}(f)(z)\,\mathrm{d}\sigma_z\\ &\le K(1)\int_{\Delta(a,\frac{1}{\mathrm{e}})}F_{\omega,p,q}(f)(z)\,\mathrm{d}\sigma_z\\ &\le\int_{\Delta}F_{\omega,p,q}(f)(z)K\big(g(z,a)\big)\ \mathrm{d}\sigma_z,\end{aligned}$$

where

$$E(a)=\left\{z\in\Delta:\ \ |z-a|<\frac{1}{\mathrm{e}}(1-|a|)\right\}.$$

If $f\in Q_{K,\omega,0}$, we obtain that

$$\lim_{|a|\to 1^-}\frac{(1-|a|)^{q+2}|f'(a)|^p}{\omega^p(1-|a|)}=0.$$

(ii) We only have to prove that $\mathcal{B}_{\omega,0}^{\frac{q+2}{p}} \subset Q_{K,w,0}(p,q)$. Assume that

$$A = \int_0^1 K\left(\log \frac{1}{r}\right) \frac{r}{(1-r^2)^2}\, \mathrm{d}r < \infty.$$

For a given $\epsilon > 0$ there exists a real number r_1 with $0 < r_1 < 1$ such that

$$\int_{r_1}^1 K\left(\log \frac{1}{r}\right) \frac{r}{(1-r^2)^2}\, \mathrm{d}r < \epsilon. \tag{8}$$

Then we have

$$\begin{aligned} \int_{\Delta\setminus\Delta(a,r_1)} & \left|f'(z)\right|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, \mathrm{d}\sigma_z \\ \leq & \left\|f\right\|^p_{\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}} \int_{\Delta\setminus\Delta(a,r_1)} \frac{K(g(z,a))}{(1-|z|^2)^2}\, \mathrm{d}\sigma_z \\ = & \left\|f\right\|^p_{\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}} \int_{r_1<|w|<1} K\left(\log \frac{1}{|w|}\right) \frac{1}{(1-|w|^2)^2}\, \mathrm{d}\sigma_w \\ = & \left\|f\right\|^p_{\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}} \int_{r_1}^1 K\left(\log \frac{1}{r}\right) \frac{r}{(1-r^2)^2}\, \mathrm{d}r \\ \leq & \ 2\pi\,\epsilon \left\|f\right\|^p_{\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}}. \end{aligned} \tag{9}$$

Similarly, if $f \in \mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$, we obtain that

$$|f'(\varphi_a(w))|^p \frac{(1-|\varphi_a(w)|^2)^{\frac{q+2}{p}}}{\omega^p(1-|\varphi_a(w)|)} \quad \longrightarrow 0$$

converges uniformly for $|w| \leq r$ if $|a| \to 1^-$, where r is fixed and $0 < r < 1$. Then we obtain that

$$\begin{aligned} & \lim_{|a|\to 1^-} \int_\Delta \left|f'(z)\right|^p (1-|z|)^q \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\, \mathrm{d}\sigma_z \\ = & \lim_{|a|\to 1^-} \int_{|w|<r} \frac{\left|f'(\varphi_a(w))\right|^p (1-|\varphi_a(w)|)^q\, K\big(\log \frac{1}{|w|}\big)}{\omega^p(1-|\varphi_a(w)|)(1-|w|^2)^2}\, \mathrm{d}\sigma_w \\ \leq & \ A \lim_{|a|\to 1^-} \sup_{|w|\leq r_1} \left|f'(\varphi_a(w))\right|^p \frac{(1-|\varphi_a(w)|)^{q+2}}{\omega^p(1-|\varphi_a(w)|)} = 0. \end{aligned} \tag{10}$$

By (9) and (10) it is easy to obtain that

$$\lim_{|a|\to 1^-} \int_\Delta \left|f'(z)\right|^p (1-|z|)^q \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\, \mathrm{d}\sigma_z = 0. \tag{11}$$

Conversely, suppose that (3) does not hold, that is

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,\mathrm{d}r=\infty.$$

Thus we find a continuous strictly decreasing function $g\colon[0,1)\longrightarrow[0,\infty)$ tending to zero at 1 such that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{g(r)}{(1-r^2)^2}\,r\,\mathrm{d}r=\infty. \tag{12}$$

It is easy to see that

$$r^{2^{k+1}-2}\geq\exp\{-2^{k+2}(1+r)\},\quad r\in[0.5,1). \tag{13}$$

We know that $t^{2\beta}\exp\{-4t\}_{t=\frac{\beta}{2}}=\left(\frac{\beta}{2}\right)^{2\beta}\exp\{-2\beta\}$ for $\beta>0$. Then there exists an integer k for $\frac{3}{4}\leq r<1$ such that $\frac{\beta}{2}\leq 2^k(1-r)<\frac{\beta+1}{2}$ and

$$\begin{aligned}2^{\beta k}\exp\{-2^{k+2}(1-r)\} &= (1-r)^{-2\beta}\Big(2^k(1-r)\Big)^{2\beta}\exp\{-2^{k+2}(1-r)\}\\ &> \left(\frac{1+\beta}{2}\right)^{2\beta}(1-r)^{-2\beta}\exp\{-2(\beta+1)\}. \qquad (14)\end{aligned}$$

For $\frac{3}{4}\leq r<1$ we define

$$f_0(z)=\sum_{k=0}^{\infty}a_k\,2^{\frac{2k}{p}}\,z^{2^k},$$

where $a_k=g\big(1-\frac{(p+1)}{p}\,2^k\big),\ k=0,1,2,\ldots$. By (13) and (14) we deduce that

$$\begin{aligned}M_2^2(r,f_0') &= \int_0^{2\pi}|f'_0(r\,\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta=2\pi\sum_{k=0}^{\infty}a_k^2\,2^{\frac{2k(p+2)}{p}}\,z^{2^k-2}\\ &\geq 2\pi\big(g(r)\big)^{\frac{2}{p}}\,2^{\frac{2k(q+2)}{p}}\exp\{-2^{k+2}(1-r)\}\\ &\geq \lambda\,\big(g(r)\big)^{\frac{2}{p}}\,(1-r)^{\frac{-2(q+2)}{p}}, \qquad (15)\end{aligned}$$

where λ is a constant. Since f_0 is defined by a gap series with Hadamard condition, we have

$$M_2(r,f_0')\approx M_p(r,f_0'),\ \text{ where }\quad M_p(r,f_0')=\left(\int_0^{2\pi}|f'_0(r\,\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}}.$$

Therefore

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{\Delta}\left|f_0'(z)\right|^p(1-|z|)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z\\
&\geq\int_0^1 M_p^p(r,f_0')(1-r^2)^q K\left(\log\frac{1}{r}\right)r\,\mathrm{d}r\\
&\approx\int_0^1 M_2^p(r,f_0')(1-r^2)^q K\left(\log\frac{1}{r}\right)r\,\mathrm{d}r\\
&\geq\int_{\frac{3}{4}}^1 K\left(\log\frac{1}{r}\right)\frac{g(r)}{(1-r^2)^2}\,r\,\mathrm{d}r=\infty.
\end{aligned}$$

This means that $f_0\in\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}\setminus Q_{K,w,0}(p,q)$, which is a contraction. Hence (3) holds. This completes the proof of our theorem. □

4. MORE RESULTS ON $Q_{K,\omega}$-SPACES

The following result means that the kernel function K can be chosen as bounded.

THEOREM 4.1. *Assume that $K(1)>0$ and let $K_1(r)=\inf\{K(r),K(1)\}$. Then, for $0<p<\infty$, $-2<q<\infty$, we have $Q_{K,w}=Q_{K_1,w}$.*

Proof. Since $K_1\leq K$ and K_1 is nondecreasing, it is clear that $Q_{K,\omega}(p,q)\subset Q_{K_1,w}(p,q)$. It remains to prove that $Q_{K_1,\omega}(p,q)\subset Q_{K,\omega}(p,q)$.

We note that $g(z,a)>1$, $z\in\Delta(a,\frac{1}{\mathrm{e}})$ and $g(z,a)\leq 1$, $z\in\Delta\setminus\Delta(a,\frac{1}{\mathrm{e}})$. Thus $K(g(z,a))=K_1(g(z,a))$ in $\Delta\setminus\Delta(a,\frac{1}{\mathrm{e}})$. It suffices to deal with integrals over $\Delta(a,\frac{1}{\mathrm{e}})$. If $f\in Q_{K_1,\omega}(p,q)$ and f is a weighted Bloch function, i.e., $f\in\mathcal{B}_\omega$, then, by Theorem 2.1, it follows that

$$\begin{aligned}
&\int_{\Delta(a,\frac{1}{\mathrm{e}})}|f'(z)|^p\,(1-|z|)^q\,\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z\\
&\leq\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\int_{\Delta(a,\frac{1}{\mathrm{e}})}K\big(g(z,a)\big)\frac{1}{(1-|z|^2)^2}\,\mathrm{d}\sigma_z\\
&=\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\int_{\Delta(0,\frac{1}{\mathrm{e}})}K\left(\log\frac{1}{|w|}\right)\frac{1}{(1-|z|^2)^2}\,\mathrm{d}\sigma_w\leq C\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}.
\end{aligned}$$

Thus $f\in Q_{K,\omega}(p,q)$, which finishes the proof. □

COROLLARY 4.1. *Let $0<p<\infty$, $-2<q<\infty$ and $\omega\colon(0,1]\to(0,\infty)$. Then $f\in Q_{K,w}(p,q)$ if and only if*

$$\sup_{a\in\Delta}\int_{\Delta}|f'(z)|^p\,(1-|z|)^q\,\frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z<\infty.$$

For later use we state the following lemma which is needed for the application of the above results.

LEMMA 4.1. *Let* $K\colon [0,\infty) \to [0,\infty)$, $0 < p < \infty$, $-2 < q < \infty$ *and* $\omega : (0,1] \to (0,\infty)$. *Then the following assertions hold:*

(i) $f \in \mathcal{B}_{\omega}^{\frac{q+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\sup_{a\in\Delta} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \, d\sigma_z < \infty. \tag{16}$$

(ii) $f \in \mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\lim_{|a|\to 1^-} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \, d\sigma_z = 0. \tag{17}$$

Proof. (i) Assume that $f \in \mathcal{B}_{\omega}^{\frac{q+2}{p}}$. Then, for any $R \in (0,1)$ and $a \in \Delta$, we have

$$\begin{aligned}
&\int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \, d\sigma_z \\
&= \int_{\Delta(0,R)} |f'(\varphi_a(z))|^p \frac{(1-|\varphi_a(z)|^2)^{q+2}}{(1+|\varphi_a(z)|)^{q+2}} \frac{K\left(\frac{1}{|z|}\right)}{(1-|z|^2)^2\omega^p(1-|z|)} \, d\sigma_z \\
&\le \|f\|^p_{\mathcal{B}_{\omega}^{\frac{q+2}{p}}} \int_{\Delta(0,R)} K\left(\log\frac{1}{|z|}\right) \frac{1}{(1-|z|^2)^2} \, d\sigma_z \\
&\le \lambda_1 \|f\|^p_{\mathcal{B}_{\omega}^{\frac{q+2}{p}}} ,
\end{aligned}$$

where $1 < (1+|\varphi_a(z)|)^{q+2} < 2^{q+2}$ and λ_1 is a constant. Conversely, suppose that (16) holds for some R with $0 < R < 1$. By the proof of Theorem 2.1 (i) with $1-|a| \approx 1-|z|$ on $E(a,R)$, $a, z \in \Delta$, we obtain

$$\begin{aligned}
&\int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \, d\sigma_z \\
&\ge K(\log\frac{1}{R}) \int_{\Delta(a,R)} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} \, d\sigma_z \\
&\ge \lambda_2 K\left(\log\frac{1}{R}\right) \omega^{-p}(1-|a|) \int_{E(a,R)} |f'(z)|^p (1-|z|)^q \, d\sigma_z \\
&\ge \pi\lambda_2 R^2 K\left(\log\frac{1}{R}\right) \frac{(1-|a|)^q}{\omega^p(1-|a|)} |f'(a)|^p ,
\end{aligned} \tag{18}$$

where λ_2 is a constant. The last inequality shows that $f \in \mathcal{B}_{\omega}^{\frac{q+2}{p}}$.

The proof of (ii) is similar to that of (i) (one takes the limit when $|a| \longrightarrow 1^-$ in (i)), hence it can be omitted. □

THEOREM 4.2. *Let* $0 < p < \infty$, $-2 < q < \infty$ *and* $\omega\colon (0,1] \to (0,\infty)$. *Assume that* $K_1(r) \le K_2(r)$ *for* $r \in (0,1)$ *and* $\frac{K_1(r)}{K_2(r)} \to 0$ *as* $r \to 0$. *If the integral in* (3) *is divergent for* K_2, *then* $Q_{K_2,\omega}(p,q) \subsetneqq Q_{K_1,\omega}(p,q)$.

Proof. It is clear that $Q_{K_2,\omega}(p,q) \subset Q_{K_1,\omega}(p,q)$. Suppose that $Q_{K_2,\omega}(p,q) = Q_{K_1,\omega}(p,q)$. By the open mapping theorem (see [9]) we know that the identity map from one of these spaces into the other one is continuous. Thus there exists a constant C such that

$$\|f\|_{K_{2,\omega}(p,q)} \le C\|f\|_{K_{1,\omega}(p,q)}\ .$$

Since $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$, there exists $r_0 \in (0,1)$ such that

$$K_1(r) \le (2C)^{-1} K_2(r) \quad \text{for } 0 < r \le r_0.$$

Put $t_0 = \mathrm{e}^{-r_0}$. If $f \in Q_{K_2,\omega}$, then

$$\begin{aligned}
&\sup_{a\in\Delta} \int_\Delta |f'(z)|^p\,(1-|z|)^q\ \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z \\
&\le C \sup_{a\in\Delta} \int_{\Delta(a,t_0)} |f'(z)|^p\,(1-|z|)^q\ \frac{K_1\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z \\
&+\frac{1}{2} \sup_{a\in\Delta} \int_\Delta |f'(z)|^p\,(1-|z|)^q\ \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{a\in\Delta} \int_\Delta |f'(z)|^p\,(1-|z|)^q\ \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z \\
&\le 2C \sup_{a\in\Delta} \int_{\Delta(a,t_0)} |f'(z)|^p\,(1-|z|)^q\ \frac{K_1\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z\ .
\end{aligned}$$

For $f \in Q_{K_2,\omega}(p,q)$ there exists, by Lemma 4.1, a constant C_1 such that

$$\sup_{a\in\Delta} \int_\Delta |f'(z)|^p\,(1-|z|)^q\ \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z \le C_1\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\,. \tag{19}$$

If $g \in \mathcal{B}_\omega^{\frac{q+2}{p}}$ and $g_r(z) = g(rz)$, $0 < r < 1$, then

$$\big\|g_r\big\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \le \big\|g\big\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\ .$$

Since $g_r \in Q_{K_2,\omega}(p,q)$, $0 < r < 1$, we can choose $f = g_r$ in the inequality (19). Using Fatou's lemma (see [12]), we deduce that

$$\sup_{a\in\Delta} \int_\Delta |g'(z)|^p\,(1-|z|)^q\ \frac{K_2(g(z,a))}{\omega^p(1-|z|)}\mathrm{d}\sigma_z\ < C_1\big\|g\big\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\ .$$

Thus $g \in Q_{K_2,\omega}(p,q)$, which means that $Q_{K_2,\omega}(p,q) = \mathcal{B}_\omega^{\frac{q+2}{p}}$. It follows from Theorem 2.1 that the integral in (3) with $K = K_2$ must be convergent, a contradiction. We obtain $Q_{K_2,\omega}(p,q) \subsetneqq Q_{K_1,\omega}(p,q)$, finishing the proof. □

5. MEROMORPHIC CLASSES $Q^{\#}_{K,\omega}$

For a meromorphic function f a natural analogue of $|f'(z)|$ is the spherical derivative

$$f^{\#}(z) = \frac{|f'(z)|}{(1+|f(z)|^2)}.$$

In an analogous way to the analytic case we define the meromorphic classes $Q^{\#}_{K,\omega}$ as follows.

Definition 5.1. Let $K\colon [0,\infty) \to [0,\infty)$ be a nondecreasing function, $0 < p < \infty$, $-2 < q < \infty$, and $\omega : (0,1] \to (0,\infty)$ a reasonable function with $\omega(kt) \approx \omega(t)\ \ k > 0$. A meromorphic function f in Δ is said to belong to the classes $Q^{\#}_{K,\omega}(p,q)$ if

$$\sup_{a\in\Delta} \int_\Delta \big(f^{\#}(z)\big)^p (1-|z|)^q \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,\mathrm{d}\sigma_z < \infty. \tag{20}$$

Remark 5.1. Our $Q^{\#}_{K,\omega}(p,q)$ classes are more general than many other classes of meromorphic function spaces. If we take $\omega \equiv 1$, then we get the $Q^{\#}_K(p,q)$ type spaces (see [18]). If we take $q = p = 2$, and $\omega(t) = t$, we obtain the $Q^{\#}_K$ space (see [6, 7, 15]). If we take $q = p = 2$, $\omega(t) = t$, and $K(t) = t^p$, we obtain the $Q^{\#}_p$ spaces as studied in [2, 3, 4, 14]. If we take $\omega(t) \equiv 1$ and $K(t) = t^s$, then we obtain the $F^{\#}(p,q,s)$ classes (see [21]).

Definition 5.2. ([14]) A meromorphic function f on Δ is said to be a spherical Bloch function, denoted by $f \in \mathcal{B}^{\#}$, if there exists a real number r with $0 < r < 1$ such that

$$\sup_{a\in\Delta} \int_{\Delta(a,r)} (f^{\#}(z))^2\,\mathrm{d}\sigma_z < \infty.$$

Definition 5.3. ([14]) A meromorphic function f on Δ is said to be a spherical Dirichlet class if there exists a real number r with $0 < r < 1$ such that

$$\int_{\Delta(a,r)} (f^{\#}(z))^2\,\mathrm{d}\sigma_z < \infty.$$

The meromorphic counterpart of the spaces $\mathcal{B}^\alpha_\omega$ and $\mathcal{B}^\alpha_{\omega,0}$ are respectively the classes of the weighted normal and the little weighted normal functions defined below.

Definition 5.4. Let f be a meromorphic function in Δ, $0 < \alpha < \infty$ and $\omega\colon (0,1] \to (0,\infty)$. If

$$\|f\|_{N_{\omega,\alpha}} = \sup_{z\in\Delta} f^{\#}(z)\frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \infty,$$

then f belongs to the class $N_{\omega,\alpha}$ of weighted normal functions. Moreover, if

$$\lim_{|z|\to 1^-} f^{\#}(z)\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}=0,$$

then f belongs to the class $N_{\omega,\alpha,0}$ of little weighted normal functions.

The classes $N_{\omega,\alpha}$ and $N_{\omega,\alpha,0}$ are called respectively the class of weighted normal functions and the class of little weighted normal functions.

THEOREM 5.1. *For each nondecreasing function* $K\colon [0,\infty)\to[0,\infty)$*, for a given reasonable function* $\omega\colon (0,1]\to(0,\infty)$*, and for* $0<p<\infty,\ -2<q<\infty$ *the following assertions hold:*

(i) $Q^{\#}_{K,\omega}(p,q)\subset N_{\omega,\frac{q+2}{p}}$.

(ii) $Q^{\#}_{K,\omega}(p,q)=N_{\omega,\frac{q+2}{p}}$ *if and only if* (3) *holds.*

Proof. The proof of this theorem is much akin to that of Theorem 2.1 with some minor modifications, so it will be omitted. □

The little "oh" version of Theorem 5.1 can be obtained in view of Theorem 3.1 as follows:

THEOREM 5.2. *For each nondecreasing function* $K\colon [0,\infty)\to[0,\infty)$*, for a given reasonable function* $\omega\colon (0,1]\to(0,\infty)$*, and for* $0<p<\infty,\ -2<q<\infty$ *then following assertions hold:*

(i) $Q^{\#}_{K,\omega,0}(p,q)\subset N_{\omega,\frac{q+2}{p},0}$.

(ii) $Q^{\#}_{K,\omega,0}(p,q)=N_{\omega,\frac{q+2}{p},0}$ *if and only if* (3) *holds.*

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