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EPIC ENVELOPES BY GENERALIZED FLAT MODULES

SEPTIMIU CRIVEI

Abstract. We establish equivalent conditions under which every right R-module has an epic $\mathcal{X}$-envelope, when $\mathcal{X}$ is a Tor-orthogonal class of left-R-modules. As particular cases, one may recover and complete known results on various generalizations of flatness.

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Key words. Envelope, flat module.

1. INTRODUCTION

Except for some very special cases, for instance when the ring is of finite representation type, it is virtually impossible to describe all the modules over a ring. That is why one approximates modules with modules in special classes and uses the properties of those classes in order to study the entire category of modules. The research by Auslander and Smalø [1] in the case of finitely generated modules over finite dimensional algebras, and that by Enochs [4] for arbitrary modules set the base for a modern general theory of (pre)envelopes and (pre)covers. The class of flat modules has received a special attention. The Flat Cover Conjecture, which states that every module has a flat cover, has resisted until the early 2000s and was affirmatively solved by Bican, El Bashir and Enochs [2]. In the meantime, the situation when every left R-module has a flat envelope was much easier and was clarified by Enochs [4], who characterize it by the condition that R is right coherent. Relative covers and envelopes which are monomorphisms or epimorphisms respectively are of particular interest. In this sense, Rada and Saorín proved that every module has an epic $\mathcal{X}$-(pre)envelope (i.e., $\mathcal{X}$-(pre)envelope which is an epimorphism) if and only if the class $\mathcal{X}$ is closed under direct products and submodules [12, Proposition 4.1].

Such results can be further developed for a class

$$\mathcal{A}^{\top_n} = \{B \mid \mathrm{Tor}_n^R(A, B) = 0\,, \forall A \in \mathcal{A}\}$$

of left R-modules, where $\mathcal{A}$ is a class of right R-modules and $n \geq 1$ is a positive integer. In a series of recent articles, Mao and Ding have studied various classes of type $\mathcal{A}^{\top_n}$, considering the classes of min-flat modules [7], (k,l)-flat modules [8], m-copure flat modules [9], FI-flat modules [10] or D-flat modules [11], which are respectively obtained for $n = 1$ and $\mathcal{A}$ the classes

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of right R-modules R/I with I a simple right ideal of R, (k,l)-presented right R-modules, right R-modules of injective dimension at most m, FP-injective right R-modules and divisible right R-modules respectively. Among other results, equivalent conditions under which every module has an epic envelope relative to some of the above classes have been established. Our goal is to unify and complete all these results in the setting of a class of the form $\mathcal{A}^{\top_n}$. The paper also complements the results from [3].

Throughout R is an associative ring with identity, and modules are unitary. The categories of left R-modules and of right R-modules are denoted by R-Mod and Mod-R respectively. For a left or right R-module M we use the notation $M^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$. By a class of modules we mean a class of modules closed under isomorphisms. Also, $\mathcal{A}$ is a class of right R-modules and $n \geq 1$ is an integer. We denote

$$\mathcal{A}^{\perp_n} = \{B \in \text{Mod-}R \mid \mathrm{Ext}^n_R(A, B) = 0\,, \forall A \in \mathcal{A}\}.$$

We recall some terminology on preenvelopes, mainly following [14]. Let M be a left R-module and $\mathcal{X}$ a class of modules in R-Mod. A homomorphism $f \in \mathrm{Hom}(M, X)$, with $X \in \mathcal{X}$, is called an *$\mathcal{X}$-preenvelope* of M if the induced abelian group homomorphism $\mathrm{Hom}(X, X') \to \mathrm{Hom}(M, X')$ is surjective for every $X' \in \mathcal{X}$. An $\mathcal{X}$-preenvelope $f \in \mathrm{Hom}(M, X)$ of M is called an *$\mathcal{X}$-envelope* if every endomorphism $g : X \to X$ with $gf = f$ is an automorphism. An $\mathcal{X}$-(pre)envelope $f \in \mathrm{Hom}(M, X)$ of M is said to *have the unique mapping property* if for every $f' \in \mathrm{Hom}(M, X')$ with $X' \in \mathcal{X}$, there exists a unique $g \in \mathrm{Hom}(X, X')$ such that $gf = f'$. The notions of $\mathcal{X}$-(pre)cover and $\mathcal{X}$-(pre)cover with the unique mapping property are defined in a dual manner.

2. CHARACTERIZATIONS

Let us first recall a couple of general results given by Rada and Saorín.

Proposition 2.1. [12, Corollary 3.5] *Let $\mathcal{X}$ be a class of left R-modules closed under pure submodules. Then every left R-module has an $\mathcal{X}$-preenvelope if and only if $\mathcal{X}$ is closed under direct products.*

Proposition 2.2. [12, Proposition 4.1] *Let $\mathcal{X}$ be a class of left R-modules. Then every left R-module has an epic $\mathcal{X}$-(pre)envelope if and only if $\mathcal{X}$ is closed under direct products and submodules.*

Then we have the following consequence for our class $\mathcal{A}^{\top_n}$.

Corollary 2.3. *Every left R-module has an $\mathcal{A}^{\top_n}$-preenvelope if and only if $\mathcal{A}^{\top_n}$ is closed under direct products.*

Proof. In view of Proposition 2.1, it is enough to show that the class $\mathcal{A}^{\top_n}$ is closed under pure submodules. To this end, let $M \in \mathcal{A}^{\top_n}$ and let K be a pure submodule of M. The pure exact sequence $0 \to K \to M \to M/K \to 0$ induces a split exact sequence

$$0 \to (M/K)^* \to M^* \to K^* \to 0$$

Using the canonical isomorphism

$$\mathrm{Ext}_R^n(A, M^*) \cong (\mathrm{Tor}_n^R(A, M))^*$$

for every $A \in \mathcal{A}$, we have $M^* \in \mathcal{A}^{\perp_n}$, hence $K^* \in \mathcal{A}^{\perp_n}$ as a direct summand of M^*. By the same canonical isomorphism, it follows that $K \in \mathcal{A}^{\top_n}$. □

Now we may state our main result, which collects several characterizations of the existence of an epic $\mathcal{A}^{\top_n}$-envelope for every left R-module.

THEOREM 2.4. *The following are equivalent:*

(i) *Every left R-module has an epic $\mathcal{A}^{\top_n}$-envelope.*
(ii) *$\mathcal{A}^{\top_n}$ is closed under direct products and submodules.*
(iii) *$\mathcal{A}^{\top_n}$ is closed under direct products and every right R-module in $\mathcal{A}$ has flat dimension at most n.*
(iv) *$\mathcal{A}^{\top_n}$ is closed under direct products and every left R-module is in $\mathcal{A}^{\top_{n+1}}$.*
(v) *$\mathcal{A}^{\top_n}$ is closed under direct products and every submodule of a flat left R-module is in $\mathcal{A}^{\top_n}$.*
(vi) *$\mathcal{A}^{\top_n}$ is closed under direct products and every submodule of a flat left R-module has an epic $\mathcal{A}^{\top_n}$-envelope.*
(vii) *$\mathcal{A}^{\top_n}$ is closed under direct products and every cotorsion left R-module has an epic $\mathcal{A}^{\top_n}$-envelope.*
(viii) *$\mathcal{A}^{\top_n}$ is closed under direct products and every left R-module has an $\mathcal{A}^{\top_{n+1}}$-precover with the unique mapping property.*
(ix) *For every left R-module M there is a short exact sequence*

$$0 \to K \to M \to N \to 0$$

of left R-modules such that $N \in \mathcal{A}^{\top_n}$ and K has no non-zero homomorphic image which is a submodule of a left R-module in $\mathcal{A}^{\top_n}$.

Proof. (i)⇔(ii) By Proposition 2.2.

(ii)⇔(iii)⇔(iv) This is straightforward homological algebra.

(i)⇒(v)⇒(vi) Clear.

(vi)⇒(ii) Let $M \in \mathcal{A}^{\top_n}$, let K be a submodule of M and consider the induced exact sequence $0 \to K \to M \to N \to 0$. Consider a short exact sequence $0 \to X \xrightarrow{g} F \to N \to 0$ with F flat. By hypothesis, X has an epic $\mathcal{A}^{\top_n}$-envelope, say $f : X \to D$. Now g factors through f, whence it follows that $X \cong D \in \mathcal{A}^{\top_n}$. Now let $A \in \mathcal{A}$. In the exact sequence

$$\mathrm{Tor}_{n+1}^R(A, F) \to \mathrm{Tor}_{n+1}^R(A, N) \to \mathrm{Tor}_n^R(A, X)$$

the first and the last terms are zero, hence $\mathrm{Tor}_{n+1}^R(A, N) = 0$. On the other hand, in the induced exact sequence

$$\mathrm{Tor}_{n+1}^R(A, N) \to \mathrm{Tor}_n^R(A, K) \to \mathrm{Tor}_n^R(A, M)$$

the first and the last terms are zero, hence $\mathrm{Tor}_n^R(A, K) = 0$. Therefore, $K \in \mathcal{A}^{\top_n}$, and so $\mathcal{A}^{\top_n}$ is closed under submodules.

(i)$\Rightarrow$(vii) Clear.

(vii)$\Rightarrow$(ii) Let $M \in \mathcal{A}^{\top_n}$ and let K be a submodule of M. Consider a flat cover $F \to M/K$ of M/K and the induced short exact sequence $0 \to C \to F \to M/K \to 0$, where C is cotorsion by Wakamatsu's Lemma [14, Lemma 2.1.1]. Now the conclusion follows by using the same argument as in (vi)$\Rightarrow$(ii) with C instead of X.

(iv)$\Rightarrow$(viii) Clear.

(viii)$\Rightarrow$(iv) Let M be an left R-module and $g : F \to M$ an $\mathcal{A}^{\top_{n+1}}$-precover of M with the unique mapping property. Then the natural epimorphism from a flat cover of M to M factors through g, hence g is an epimorphism. So we have an exact sequence $0 \to K \stackrel{f}{\to} F \stackrel{g}{\to} M \to 0$. Let $h : F' \to K$ be an $\mathcal{A}^{\top_{n+1}}$-precover of K, which is again an epimorphism. We have $gfh = 0$, so $fh = 0$ by the unique mapping property of g. Then $K \subseteq \operatorname{Ker}(f) = 0$, hence $M \cong F \in \mathcal{A}^{\top_{n+1}}$.

(i)$\Rightarrow$(ix) Let M be an left R-module and let $f : M \to N$ be an epic $\mathcal{A}^{\top_n}$-envelope of M. Let L be a homomorphic image of K such that $L \in \mathcal{A}^{\top_n}$. By a pushout we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \stackrel{u}{\longrightarrow} & M & \stackrel{f}{\longrightarrow} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & \swarrow \alpha & \| & & \\
0 & \longrightarrow & L & \longrightarrow & Z & \stackrel{p}{\longrightarrow} & N & \longrightarrow & 0
\end{array}$$

Since $L, N \in \mathcal{A}^{\top_n}$, we have $Z \in \mathcal{A}^{\top_n}$. Then there is a homomorphism $\alpha : N \to Z$ such that $\alpha f = g$. Note that α is an epimorphism. We have $p\alpha f = pg = f$, and so $p\alpha = 1_N$. Thus α is an isomorphism with inverse p, whence $L = 0$. Noting that $\mathcal{A}^{\top_n}$ is closed under submodules by the equivalence (i)$\Leftrightarrow$(ii), the conclusion follows.

(ix)$\Rightarrow$(i) Let M be an left R-module and consider a short exact sequence $0 \to K \to M \to N \to 0$ satisfying the conditions from hypothesis. Let $h : M \to N'$ be a homomorphism with $N' \in \mathcal{A}^{\top_n}$. With the above notation, we have $hu : K \to N'$. Now by hypothesis we must have $hu = 0$, whence $\operatorname{Ker}(f) \subseteq \operatorname{Ker}(h)$. Then h factors through f, showing that $f : M \to N$ is an epic $\mathcal{A}^{\top_n}$-(pre)envelope of M. $\square$

3. APPLICATIONS

We consider some particular cases to illustrate Theorem 2.4. For $\mathcal{A} = \text{Mod-}R$ and $n = 1$ we obtain the following corollary, which completes several known results (e.g., see [10, Theorem 3.7]).

Corollary 3.1. *The following are equivalent:*

(i) *Every left R-module has an epic flat envelope.*
(ii) *R is right semihereditary.*
(iii) *R is right coherent and has flat dimension at most* 1.

(iv) *R is right coherent and every submodule of a flat left R-module has an epic flat envelope.*
(v) *R is right coherent and every cotorsion left R-module has an epic flat envelope.*
(vi) *R is right coherent and every left R-module has a precover with the unique mapping property by left R-modules of flat dimension at most 1.*
(vii) *For every left R-module M there is a short exact sequence*

$$0 \to K \to M \to N \to 0$$

of left R-modules such that N is flat and K has no non-zero homomorphic image which is a submodule of a flat left R-module.

Let $k, l \geq 0$ be integers. Recall that a left R-module M is called *(k,l)-flat* if $M \in \mathcal{A}^{\top_1}$ with $\mathcal{A}$ the class of (k,l)-presented right R-modules [8]. A right R-module X is called *(k,l)-injective* if $M \in \mathcal{A}^{\perp_1}$ with $\mathcal{A}$ the class of (k,l)-presented right R-modules [15]. A ring R is called *right (k,l)-coherent* if every l-generated submodule of R^k is finitely presented. A ring R is right (k,l)-coherent if and only if the class of (k,l)-flat left R-modules is closed under direct products [8, Theorem 3.1]. Now for $\mathcal{A}$ the class of (k,l)-presented right R-modules and $n = 1$ we obtain the following corollary, which completes [8, Theorem 5.1].

Corollary 3.2. *The following are equivalent:*

(i) *Every left R-module has an epic (k,l)-flat envelope.*
(ii) *R is right (k,l)-coherent and the class of (k,l)-flat left R-modules is closed under submodules.*
(iii) *R is right (k,l)-coherent and every (k,l)-presented right R-module has flat dimension at most n.*
(iv) *R is right (k,l)-coherent and every submodule of a flat left R-module is (k,l)-flat.*
(v) *R is right (k,l)-coherent and every submodule of a flat left R-module has an epic (k,l)-flat envelope.*
(vi) *R is right (k,l)-coherent and every cotorsion left R-module has an epic (k,l)-flat envelope.*
(vii) *For every left R-module M there is a short exact sequence*

$$0 \to K \to M \to N \to 0$$

of left R-modules such that N is (k,l)-flat and K has no non-zero homomorphic image which is a submodule of a (k,l)-flat left R-module.

Let $m \geq 0$ be an integer. Recall that a left R-module M is called *m-copure flat* if $M \in \mathcal{A}^{\top_1}$ with $\mathcal{A}$ the class of right R-modules of injective dimension at most m [9]. A right R-module X is called *m-copure injective* if $M \in \mathcal{A}^{\perp_1}$ with $\mathcal{A}$ the class of right R-modules of injective dimension at most m [9]. By [9, Remark 2.3], a left R-module is m-copure flat if and only if it is Gorenstein flat, and a right R-module is m-copure injective if and only if it is Gorenstein injective. Over an m-Gorenstein ring, every left R-module has a Gorenstein flat preenvelope [5, Theorem 11.8.2], and so the class of Gorenstein flat left R-modules is closed under direct products by Proposition 2.1. Now for $\mathcal{A}$ the class of right R-modules of injective dimension at most m and $n = 1$ we obtain the following corollary, which generalizes and completes [9, Theorem 4.2].

Corollary 3.3. *The following are equivalent for an m-Gorenstein ring R:*

(i) *Every left R-module has an epic Gorenstein flat envelope.*
(ii) *The class of Gorenstein flat left R-modules is closed under submodules.*
(iii) *Every right R-module of injective dimension at most m has flat dimension at most* 1.
(iv) *Every submodule of a flat left R-module is Gorenstein flat.*
(v) *Every submodule of a flat left R-module has an epic Gorenstein flat envelope.*
(vi) *Every cotorsion left R-module has an epic Gorenstein flat envelope.*
(vii) *For every left R-module M there is a short exact sequence*

$$0 \to K \to M \to N \to 0$$

of left R-modules such that N is Gorenstein flat and K has no nonzero homomorphic image which is a submodule of a Gorenstein flat left R-module.

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"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu Nr. 1
400084 Cluj-Napoca, Romania
E-mail: crivei@math.ubbcluj.ro