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VOLTERRA-FREDHOLM NONLINEAR INTEGRAL EQUATIONS VIA PICARD OPERATORS THEORY

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Abstract. In the present paper we study existence and uniqueness of the solution, data dependence of the solution, comparison theorems, lower and upper subsolutions and differentiability of the solution with respect to a parameter of the following Volterra-Fredholm nonlinear integral equation

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s.$$

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Key words. Volterra-Fredholm integral equation, fixed point, Picard operator, data dependence, subsolution, differentiability of the solution.

1. INTRODUCTION

Let $(X, \|\cdot\|_X)$ be a Banach space. In this paper we consider the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \tag{1}$$

for all $(t,x) \in [0,T] \times \Omega := \overline{D}$, where $T > 0$ and $\Omega \subset \mathbb{R}^m$ is bounded and closed.

Volterra-Fredholm integral equations (VF for short) often arise from the mathematical modeling of the spreading (in space and time) of some contagious diseases; they also come up in the theory of nonlinear parabolic boundary value problems and in many physical and biological models. There are many results for the VF equation (1) which establish numerical approximation of the solutions; see, e.g., [6], [7], [18], [1], [8], [2].

In [17] H. R. Thieme considered a model for the spatial spread of an epidemic consisting of a nonlinear integral equation of Volterra-Fredholm type having an unique solution. The author showed that this solution has a temporally asymptotic limit which describes the final state of the epidemic and is the minimal solution of another nonlinear integral equation.

In [3] O. Diekmann described, derived and analysed a model of the spatial and temporal development of an epidemic. The model considered leads (see [10]) to the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = f(t,x) + \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)\mathrm{d}\xi\mathrm{d}\tau, \tag{2}$$

for all $(t,x) \in [0,\infty] \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$.

In [10] B. G. Pachpatte considered the integral equation (1). Using the Contraction Principle, the author proved that, under appropriate assumptions, (1) has a unique solution in a subset S of $C(D, \mathbb{R}^n)$. The result was then applied to show the existence and uniqueness of the solution of certain nonlinear parabolic differential equations and mixed Volterra-Fredholm integral equations occurring in specific physical and biological problems (for example, a reliable treatment of the Diekmann's model mentioned above is given).

In this paper we will present an extended treatment for equation (1). While in [10] the author gives a local existence and uniqueness theorem with the contraction condition required, we will give a global existence and uniqueness theorem without the contraction condition required (i. e., the assumption $L_K m(\Omega) < 1$ is not necessary). Moreover, we will study the data dependence of the solution, comparison theorems, lower and upper subsolutions and the differentiability of the solution with respect to a parameter. For the last problem we will consider the following VF equation with a parameter λ:

$$u(t,x) = g(t,x) + \int_0^t \int_a^b K(t,x,s,y,u(s,y),\lambda)\mathrm{d}y\mathrm{d}s, \tag{3}$$

for all $(t,x) \in [0,T] \times [\alpha,\beta]$, with $[a,b] \subset [\alpha,\beta] \subset \mathbb{R}_+$, where $\lambda \in J \subset \mathbb{R}$, J being a compact interval. Also, we will prove the differentiability of the solution with respect to λ.

Since the main tool used in the present paper are the Picard operators, we start by presenting some basic notions and results concerning this important class of operators.

2. PICARD OPERATORS

Let (X,d) be a metric space and $A\colon X \to X$ an operator. We will use the following notations:

$$F_A := \{x \in X : \ A(x) = x\};$$
$$A^0 := 1_X\,,\ A^{n+1} := A \circ A^n,\ \forall n \in \mathbb{N}.$$

If $(X,d,\leq)$ is an ordered metric space, let

$$(LF)_A := \big\{x \in X : \ x \leq A(x)\big\},$$
$$(UF)_A := \big\{x \in X : \ x \geq A(x)\big\}.$$

Definition 1. (Rus [11]) The operator A is said to be:

(i) a *weakly Picard operator* (wPo) if $A^n(x_0) \to x_0^*$, for every $x_0 \in X$, and the limit x_0^* is a fixed point of A, which may depend on x_0.

(ii) a *Picard operator* (Po) if $F_A = \{x^*\}$ and $A^n(x_0) \to x^*$, for every $x_0 \in X$.

For a weakly Picard operator A one defines the operator A^∞ as follows:

$$A^\infty\colon X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Note that $A^\infty(X) = F_A$. If A is Picard operator, then $A^\infty(x) = x^*$, for every $x \in X$, where x^* is the unique fixed point of A.

The following abstract theorem is needed to study the data dependence of the solution:

THEOREM 1. (Rus [13]) *For a complete metric space* (X, d) *and the operators* $A, B\colon X \to X$ *assume that:*
(i) *there exists* $\alpha \in [0, 1[$ *such that* A *is an* α*-contraction; let* $F_A = \{x_A^*\}$*;*
(ii) $F_B \neq \emptyset$*; let* $x_B^* \in F_B$*;*
(iii) *there exists* $\eta > 0$ *such that* $d\,(A(x), B(x)) \leq \eta$*, for all* $x \in X$*. Then*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

The following lemma will be applied in order to prove a comparison theorem for VF equations:

LEMMA 1. (The Abstract Comparison Lemma; Rus [13]) *Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C\colon X \to X$ *operators satisfying the following conditions:*
(i) $A \leq B \leq C$*;*
(ii) A, B, C *are weakly Picard;*
(iii) B *is increasing.*
If $x, y, z \in X$ *are so that* $x \leq y \leq z$*, then* $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$*.*

In order to study lower and upper subsolutions the following abstract lemma in ordered metric spaces is required:

LEMMA 2. (The Abstract Gronwall Lemma; Rus [13]) *Let* $(X, d, \leq)$ *be an ordered metric space and* $A\colon X \to X$ *an operator such that:*
(i) A *is increasing;*
(ii) A *is Picard; let* $F_A = \{x_A^*\}$*.*
If $x \in (LF)_A$ *and* $y \in (UF)_A$ *then* $x \leq x_A^* \leq y$*.*

In order to study the differentiability of the solution with respect to a parameter we need the following theorem:

THEOREM 2. (The Fiber Contraction Principle, Rus [12]) *Let* (X, d)*,* (Y, ρ) *be two metric spaces and* $B\colon X \to X$*,* $C\colon X \times Y \to Y$ *operators such that:*
(i) (Y, ρ) *is complete;*
(ii) B *is a Picard operator,* $F_B = \{x^*\}$*;*
(iii) $C(\cdot, y)\colon X \to Y$ *is continuous, for all* $y \in Y$*;*
(iv) *there exists* $\alpha \in]0, 1[$ *such that the operator* $C(x, \cdot)\colon Y \to Y$ *is* α*-contraction for all* $x \in X$*; let* y^* *be the unique fixed point of* $C(x^*, \cdot)$*.*
Then

$$A\colon X \times Y \to X \times Y, \quad A(x, y) := (B(x), C(x, y))$$

is a Picard operator and $F_A = \{(x^*, y^*)\}$*.*

For Picard operators applied in the study of differential or integral equations we refer to [15], [14], [13], [9], [16], [5], [4].

3. THE EXISTENCE AND UNIQUENESS THEOREM

Consider the equation (1).

THEOREM 3. *If the following conditions are satisfied:*
(i) $g \in C(\overline{D}, X)$ *and* $K \in C(\overline{D} \times \overline{D} \times X, X)$*;*
(ii) *there exists a real constant* $L_K > 0$ *such that for all* $(t, x, s, y) \in \overline{D} \times \overline{D}$ *and for all* $u, v \in X$

$$\|K(t,x,s,y,u) - K(t,x,s,y,v)\|_X \le L_K \|u - v\|_X, \tag{4}$$

then (1) *has an unique solution* $u^* \in C(\overline{D}, X)$.

Proof. Endow the space $C(\overline{D}, X)$ with a Bielecki-Chebysev suitable norm

$$\|u\|_{BC} := \sup\{\|u(t,x)\|_X \, \mathrm{e}^{-\tau t} : \ t \in [0,T], \ x \in \Omega\}, \quad \tau > 0. \tag{5}$$

Consider the operator $A \colon C(\overline{D}, X) \to C(\overline{D}, X)$ defined by

$$A(u)(t,x) := g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \quad \forall\ (t,x) \in \overline{D}. \tag{6}$$

For all $u, v \in C(\overline{D}, X)$ we have

$$\begin{aligned}
\|A(u)(t,x) \ &- \ A(v)(t,x)\|_X \\
&\le \int_0^t \int_\Omega \|K(t,x,s,y,u(s,y)) - K(t,x,s,y,v(s,y))\|_X \mathrm{d}y\mathrm{d}s \\
&\le L_K \int_0^t \int_\Omega \|u(s,y) - v(s,y)\|_X \mathrm{d}y\mathrm{d}s \\
&\le L_K \int_0^t m(\Omega) \sup_{y \in \Omega} \|u(s,y) - v(s,y)\|_X \mathrm{d}s \\
&\le \frac{L_K m(\Omega)}{\tau} \|u - v\|_{BC} \, \mathrm{e}^{\tau t},
\end{aligned}$$

hence

$$\|A(u) - A(v)\|_{BC} \le \frac{L_K m(\Omega)}{\tau} \|u - v\|_{BC} \, .$$

Choosing τ such that $\alpha := \dfrac{L_K m(\Omega)}{\tau} < 1$, it follows that $A \colon C(\overline{D}, X) \to C(\overline{D}, X)$ is an α-contraction. Thus, by the Contraction Principle, the conclusion follows. Note that A is a Picard operator. □

4. DATA DEPENDENCE OF THE SOLUTION

In order to prove the dependence of the solution of (1) on g and K let us consider the following VF equation

$$u(t,x) = h(t,x) + \int_0^t \int_\Omega N(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \tag{7}$$

for all $(t,x) \in \overline{D}$, with $h \in C(\overline{D}, X)$ and $N \in C(\overline{D} \times \overline{D} \times X, X)$.

Theorem 4. *Consider equation* (1) *and assume that conditions* (i) *and* (ii) *from Theorem* 3 *are satisfied. Let* u^* *be the unique solution of* (1)*. Furthermore, assume that the equation* (7) *has at least one solution and let* v^* *be such a solution. If there exist* $\eta_1, \eta_2 > 0$ *such that*

$$\|g(t,x) - h(t,x)\|_X \le \eta_1, \quad \forall\ (t,x) \in \overline{D},$$

and

$$\|K(t,x,s,y,u) - N(t,x,s,y,u)\|_X \le \eta_2, \quad \forall\ (t,x,s,y,u) \in \overline{D} \times \overline{D} \times X,$$

then

$$\|u^* - v^*\|_{BC} \le \frac{\eta_1 + Tm(\Omega)\eta_2}{1 - \dfrac{L_K m(\Omega)}{\tau}},$$

where $\tau > 0$ *is suitably selected.*

Proof. In the space $C(\overline{D}, X)$, endowed with Bielecki-Chebysev norm (5), we define the operators $A_i \colon C(\overline{D}, X) \to C(\overline{D}, X)$, $i = 1, 2$, by

$$\begin{aligned} A_1(u)(t,x) &:= g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \\ A_2(u)(t,x) &:= h(t,x) + \int_0^t \int_\Omega N(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \end{aligned}$$

for all $(t,x) \in \overline{D}$, $i = 1, 2$.

Let L_K be the Lipschitz constant of K. Then, for a suitable $\tau > 0$, A_1 is an α-contraction, with $\alpha := \dfrac{L_K m(\Omega)}{\tau} < 1$. We have $F_{A_1} = \{u^*\}$ and $v^* \in F_{A_2} \neq \emptyset$. Relation (iii) implies that $\|A_1(u) - A_2(u)\|_{BC} \le \eta_1 + Tm(\Omega)\eta_2$, for all $u \in C(\overline{D})$. The conclusion follows now from Theorem 1. □

5. DATA DEPENDENCE: MONOTONICITY

Consider the equation (1).

Theorem 5. (A Gronwall-type theorem) *Assume that the conditions of Theorem* 3 *are satisfied and let* u^* *be the unique solution of* (1)*. Moreover, assume that* $K(t,x,s,y,\cdot)$ *is increasing for all* $(t,x,s,y) \in \overline{D} \times \overline{D}$*. If* $v \in C(\overline{D}, X)$ *is a subsolution of* (1) *and* $w \in C(\overline{D}, X)$ *is a suprasolution of* (1)*, then* $v \le u^* \le w$.

Proof. Consider the operator $A \colon C(\overline{D}, X) \to C(\overline{D}, X)$ defined by (6). From Theorem 3 we know that A is a Picard operator and that $F_A = \{u^*\}$. But $v \in (LF)_A$, $w \in (UF)_A$, so the conditions of Lemma 2 are fulfilled and the conclusion follows. □

Now consider the following three equations:

$$u(t,x) = g_i(t,x) + \int_0^t \int_\Omega K_i(t,x,s,y,u(s,y))\mathrm{d}y\mathrm{d}s, \quad i = 1, 2, 3. \tag{8}$$

THEOREM 6. *Assume that conditions* (i) *and* (ii) *from Theorem* 3 *are fulfilled for each equation from* (8) *and let* u_i^* *be their solutions,* $i = 1, 2, 3$. *If, in addition, we have that*

(iii) $K_2(t, x, s, y, \cdot)$ *is increasing for all* $(t, x, s, y) \in \overline{D} \times \overline{D}$,

(iv) $g_1 \leq g_2 \leq g_3$ *and* $K_1 \leq K_2 \leq K_3$,

then $u_1^* \leq u_2^* \leq u_3^*$.

Proof. Let $A_i \colon C(\overline{D}, X) \to C(\overline{D}, X)$ be defined by

$$A_i(u)(t, x) := g_i(t, x) + \int_0^t \int_\Omega K_i(t, x, s, y, u(s, y))\mathrm{d}y\mathrm{d}s, \quad i = 1, 2, 3.$$

From the proof of Theorem 3 we know that A_i is a Picard operator, so

$$A_i^\infty(u) = u_i^*, \quad \forall\, u \in C(\overline{D}, X), \quad \forall\, i = 1, 2, 3. \tag{9}$$

Condition (iii) yields that A_2 is increasing, and from (iv) it follows that $A_1 \leq A_2 \leq A_3$. Thus the conditions of Lemma 1 are satisfied, so $A_1^\infty \leq A_2^\infty \leq A_3^\infty$. By (9) it follows that $u_1^* \leq u_2^* \leq u_3^*$. □

6. DIFFERENTIABILITY OF THE SOLUTION WITH RESPECT TO PARAMETERS

In this section we consider equation (3) under certain assumptions and we will prove the differentiability of its solution with respect to the parameter λ.

THEOREM 7. *Let* $J \subset \mathbb{R}$ *be a compact interval and* $\lambda \in J$. *Assume that:*

(i) $g \in C(\overline{D})$ *and* $K \in C(\overline{D} \times \overline{D} \times \mathbb{R} \times J)$;

(ii) *there exists* $L_K > 0$ *such that*

$$|K(t, x, s, y, u, \lambda) - K(t, x, s, y, v, \lambda)| \leq L_K |u - v| \tag{10}$$

for all $(t, x, s, y) \in \overline{D} \times \overline{D}$, $u, v \in \mathbb{R}$, *and* $\lambda \in J$.

Then the following assertions hold:

a) *Equation* (3) *has in* $C(\overline{D})$ *a unique solution* $u^*(\cdot, \cdot, \lambda)$, *for every* $\lambda \in J$.

b) *The sequence* $(u_n)_{n \geq 0}$, *defined for each* $u_0 \in C(\overline{D})$ *by*

$$u_n(t, x, \lambda) = g(t, x) + \int_0^t \int_\Omega K(t, x, s, y, u_{n-1}(s, y, \lambda), \lambda)\mathrm{d}y\mathrm{d}s$$

converges uniformly to u^*, *for every* $(t, x, \lambda) \in \overline{D} \times J$.

c) *The function* u^* *which assigns to each* $(t, x, \lambda) \mapsto u^*(t, x, \lambda)$ *is continuous, i.e.,* $u^* \in C(\overline{D} \times J)$.

d) *If* $K(t, x, s, y, \cdot, \cdot) \in C^1(\mathbb{R} \times J)$, *for each* $(t, x, s, y) \in \overline{D} \times \overline{D}$, *then* $u^*(t, x, \cdot) \in C^1(J)$, *for every* $(t, x) \in \overline{D}$.

Proof. Let $Y := C(\overline{D} \times J)$. Consider the operator $B \colon Y \to Y$ defined by

$$B(u)(t, x, \lambda) := g(t, x) + \int_0^t \int_\Omega K(t, x, s, y, u(s, y, \lambda), \lambda)\mathrm{d}y\mathrm{d}s.$$

The operator B satisfies the hypotheses of Theorem 3, so assertions a), b) and c) hold.

For any $\lambda \in J$ there is a unique solution $u^*(\cdot,\cdot,\lambda) \in C(\overline{D})$ and we have

$$u^*(t,x,\lambda) = g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u^*(s,y,\lambda),\lambda)\mathrm{d}y\mathrm{d}s. \tag{11}$$

We will prove that $\dfrac{\partial u^*(t,x,\lambda)}{\partial \lambda}$ exists and is continuous. Supposing that $\dfrac{\partial u^*(t,x,\lambda)}{\partial \lambda}$ exists, we obtain from (11) that

$$\begin{aligned}\frac{\partial u^*(t,x,\lambda)}{\partial \lambda} &= \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u^*(s,y,\lambda),\lambda)}{\partial u} \cdot \frac{\partial u^*(s,y,\lambda)}{\partial \lambda}\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u^*(s,y,\lambda),\lambda)}{\partial \lambda}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

This suggests us to consider the operator $C\colon Y \times Y \to Y$, defined by

$$\begin{aligned}C(u,v)(t,x,\lambda) &:= \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u(s,y,\lambda),\lambda)}{\partial u} \cdot v(s,y,\lambda)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u(s,y,\lambda),\lambda)}{\partial \lambda}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

The operator $C(u,\cdot)$ is a contraction for all $u \in Y$; let v^* be the unique fixed point of $C(u^*,\cdot)$. If we define the operator $A\colon Y \times Y \to Y \times Y$,

$$A(u,v)(t,x,\lambda) := \left(B(u)(t,x,\lambda), C(u,v)(t,x,\lambda)\right),$$

then the conditions of the Theorem 2 are fulfilled. It follows that A is a Picard operator and $F_A = \{(u^*,v^*)\}$.

Consider now the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined by

$$\begin{aligned}u_n(t,x,\lambda) &:= B(u_{n-1}(t,x,\lambda)) \\ &= g(t,x) + \int_0^t \int_\Omega K(t,x,s,y,u_{n-1}(s,y,\lambda),\lambda)\mathrm{d}y\mathrm{d}s, \quad \forall\, n \geq 1, \\ v_n(t,x,\lambda) &:= C(u_{n-1}(t,x,\lambda), v_{n-1}(t,x,\lambda)) \\ &= \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u_{n-1}(s,y,\lambda),\lambda)}{\partial u} \cdot v_{n-1}(s,y,\lambda)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_\Omega \frac{\partial K(t,x,s,y,u_{n-1}(s,y,\lambda),\lambda)}{\partial \lambda}\mathrm{d}y\mathrm{d}s, \quad \forall\, n \geq 1.\end{aligned}$$

We have

$$u_n \rightrightarrows u^* \ (n \to \infty) \quad \text{and} \quad v_n \rightrightarrows v^* \ (n \to \infty) \tag{12}$$

for $(t,x,\lambda) \in \overline{D} \times J$, and for each $u_0,\, v_0 \in C(\overline{D} \times J)$. We take $u_0 = v_0 := 0$, so $v_1 = \dfrac{\partial u_1}{\partial \lambda}$. An induction argument yields that $v_n = \dfrac{\partial u_n}{\partial \lambda}$, for all $n \geq 1$, and (12) implies that $\frac{\partial u_n}{\partial \lambda} \rightrightarrows v^*$ $(n \to \infty)$. Applying the theorem of Weierstrass, it follows that $\dfrac{\partial u^*}{\partial \lambda}$ exists and $\frac{\partial u^*(t,x,\lambda)}{\partial \lambda} = v^*(t,x,\lambda)$. $\square$

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