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MULTIPLICITY RESULTS FOR NONLINEAR EIGENVALUE PROBLEMS ON UNBOUNDED DOMAINS

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Abstract. In this paper we prove a multiplicity result for a class of eigenvalue problems with nonlinear boundary conditions on an unbounded domain. Many results have been obtained by Cârstea and Rădulescu [3], Chabrowski [5], [6], Kandilakis and Lyberopoulos [10], Lisei, Varga and Horváth [13] and Pflüger [16].

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1. INTRODUCTION AND PRELIMINARY RESULTS

This paper is motivated by recent advances in elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids") where some processes are modeled by nonhomogeneous quasilinear operators (see Acerbi and Mingione [1], Diening [8], Halsey [9], Kristály, Lisei and Varga [11], Kristály and Varga [12], Ruzicka [20], Zhikov [21, 22], and the references therein). We refer mainly to the $p(x)$-Laplace operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. Recent qualitative properties of solutions to quasilinear problems in Sobolev spaces with variable exponent have been obtained by Alves and Souto [2], Chabrowski and Fu [4], Mihăilescu and Rădulescu [14] and Rădulescu [18].

Let $\Omega \subset \mathbf{R}^N$ be an unbounded domain with smooth boundary Γ. Set

$$C_+(\overline{\Omega}) = \{h :\ h \in C(\overline{\Omega}),\ h(x) > 1,\ \text{for all } x \in \overline{\Omega}\}.$$

For $h \in C_+(\overline{\Omega})$ let

$$h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x), \quad h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x).$$

We assume throughout this paper that $p \in C_+(\overline{\Omega})$, m, q and α_1, α_2 are real numbers satisfying

$$(1) \qquad 1 < p^+ < N, \quad p^+ \le q \le \frac{p^+ N}{N - p^+}, \quad -N < \alpha_1 \le q \cdot \frac{N - p^+}{p^+} - N,$$

$$(2)\qquad p^+ \leq m \leq p^+ \cdot \frac{N-1}{N-p^+} \quad \text{and} \quad -N < \alpha_2 \leq m \cdot \frac{N-p^+}{N} - N + 1.$$

We define the weighted Sobolev space E as the completion of $C_0^\infty(\Omega)$ with the norm

$$||u||_E = \left(\int_\Omega (|\nabla u(x)|^{p^+} + \frac{1}{(1+|x|)^{p^+}} |u(x)|^{p^+}) \mathrm{d}x \right)^{\frac{1}{p^+}}.$$

We denote by $L^q(\Omega; w_1)$ and by $L^m(\Gamma; w_2)$ the weighted Lebesgue spaces with respect to

$$(3)\qquad \omega_i(x) = (1+|x|)^{\alpha_i}, i = 1, 2,$$

and norms

$$||u||_{q,w_1}^q = \int_\Omega w_1(x)|u(x)|^q \mathrm{d}x, \qquad ||u||_{m,w_2}^m = \int_\Gamma w_2(x)|u(x)|^m \mathrm{d}\Gamma.$$

We have

PROPOSITION 1. (Pflüger [15]) *Assume that* (1) *holds. Then the embedding* $E \hookrightarrow L^q(\Omega; w_1)$ *is continuous. If the upper bound for* q *in* (1) *is strict, then the embedding is compact. Suppose that the inequalities in* (2) *are satisfied. Then the trace operator* $E \hookrightarrow L^m(\Gamma; w_2)$ *is continuous. If the upper bound for* m *in* (2) *is strict, then the trace operator is compact.*

The best embedding constant of $E \hookrightarrow L^q(\Omega; w_1)$ will be denoted by C_{q,w_1} and that of $E \hookrightarrow L^m(\Gamma; w_2)$ by C_{m,w_2}.

We assume throughout this paper that $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$ such that

$$(4)\qquad a(x) \geq a > 0, \text{ for a.e. } x \in \Omega,$$

and

$$(5)\qquad \frac{c}{(1+|x|)^{p^+-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p^+-1}}, \text{ for a.e } x \in \Gamma,$$

where c, $C > 0$ are constants.

REMARK 1. (see [16]) Note that

$$||u||_b^{p^+} = \int_\Omega a(x)|\nabla u(x)|^{p^+} \mathrm{d}x + \int_\Gamma b(x)|u(x)|^{p^+} \mathrm{d}\Gamma$$

defines an equivalent norm on E.

We are concerned in this paper with elliptic problems of the following type

$$(P_{\lambda,\mu})\qquad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x, u(x)) & \text{in } \Omega, \\ a(x)|\nabla u|^{p(x)-2}\nabla u \cdot n + b(x)|u|^{p(x)-2}u = \mu g(x, u(x)) & \text{on } \Gamma, \\ u \neq 0 & \text{in } \Omega, \end{cases}$$

where n denotes the unit outward normal on Γ, and λ, $\mu > 0$.

We also consider the following assumptions:

$(F1)$ $f\colon \Omega\times\mathbf{R}\to\mathbf{R}$ is a Carathéodory function such that $f(\cdot,0)=0$ and

$$|f(x,s)|\le f_0(x)+f_1(x)|s|^{r-1},$$

where $p^+<r<\frac{p^+N}{N-p^+}$, and f_0, f_1 are measurable functions which satisfy

$$0<f_0(x)\le C_f w_1(x),\quad \text{and}\quad 0\le f_1(x)\le C_f w_1(x),\quad \text{a.e.}\quad x\in\Omega,$$

$$f_0\in L^{\frac{r}{r-1}}\left(\Omega; w_1^{\frac{1}{1-r}}\right);$$

$(F2)$ $\displaystyle\lim_{s\to 0}\frac{f(x,s)}{f_0(x)|s|^{p^+-1}}=0$, uniformly in $x\in\Omega$;

$(F3)$ $\displaystyle\limsup_{s\to+\infty}\frac{1}{f_0(x)|s^{p^+}|}F(x,s)\le 0$, uniformly for all $x\in\Omega$, and

$$\max_{|s|\le M}F(\cdot,s)\in L^1(\Omega),\ \text{for all } M>0,$$

where F denotes the primitive function of f with respect to the second variable, that is, $F(x,u)=\int_0^u f(x,s)\mathrm{d}s$;

$(F4)$ there exists $u_0\in E$ such that $\int_\Omega F(x,u_0(x))\mathrm{d}x>0$.

$(G1)$ $g\colon \Gamma\times\mathbf{R}\to\mathbf{R}$ is a Carathéodory function such that $g(\cdot,0)=0$ and

$$|g(x,s)|\le g_0(x)+g_1(x)|s|^{m-1},$$

where $p^+\le m<p^+\cdot\frac{N-1}{N-p^+}$, and g_0, g_1 are measurable functions satisfying

$$0<g_0(x)\le C_g w_2(x)\quad \text{and}\quad 0\le g_1(x)\le C_g w_2(x),\quad \textit{a.e.}\quad x\in\Gamma,$$

$$g_0\in L^{\frac{q}{q-1}}\left(\Gamma; w_2^{\frac{1}{1-q}}\right);$$

$(G2)$ $\displaystyle\lim_{s\to 0}\frac{g(x,s)}{g_0(x)|s|^{p^+-1}}=0$, uniformly in $x\in\Gamma$;

$(G3)$ $\displaystyle\limsup_{s\to+\infty}\frac{1}{g_0(x)|s^{p^+}|}G(x,s)<+\infty$, uniformly for all $x\in\Gamma$, and

$$\max_{|s|\le M}G(\cdot,s)\in L^1(\Gamma),\ \text{for all } M>0,$$

where G is the primitive function of g with respect to the second variable, that is $G(x,u)=\int_0^u g(x,s)\mathrm{d}s$.

The energy functional corresponding to $(P_{\lambda,\mu})$ is given by $\mathcal{E}_{\lambda,\mu}\colon E\to\mathbf{R}$

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(u)=&\int_\Omega\frac{1}{p(x)}a(x)|\nabla u(x)|^{p(x)}\mathrm{d}x\\&+\int_\Gamma\frac{1}{p(x)}b(x)|u(x)|^{p(x)}\mathrm{d}\Gamma-\lambda J_F(u)-\mu J_G(u),\end{aligned}$$

where $J_F, J_G\colon E\to\mathbf{R}$ are defined by

$$J_F(u)=\int_\Omega F(x,u(x))\mathrm{d}x\quad\text{and}\quad J_G(u)=\int_\Gamma G(x,u(x))\mathrm{d}\Gamma.$$

Proposition 1 implies that $\mathcal{E}_{\lambda,\mu}$ is well defined. The solutions of problem $(P_{\lambda,\mu})$ will be found as critical points of $\mathcal{E}_{\lambda,\mu}$. Therefore, a function $u \in E$ is a solution of problem $(P_{\lambda,\mu})$ provided that, for any $v \in E$,

$$\int_\Omega a(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)\mathrm{d}x + \int_\Gamma b(x)|u(x)|^{p(x)-2}u(x)v(x)\mathrm{d}\Gamma$$

$$= \lambda \int_\Omega f(x,u(x))u(x)v(x)\mathrm{d}x + \mu \int_\Gamma g(x,u(x))u(x)v(x)\mathrm{d}\Gamma.$$

We recall a result of Ricceri; for the reader's convenience, we state it in a slightly modified form, suitable for our purposes:

Theorem 1. ([19], Theorem 4) *Let X be a real, reflexive, separable Banach space, let $\Lambda \subseteq \mathbf{R}$ be an interval, let $\Psi : X \times \Lambda \to \mathbf{R}$ be a function satisfying the following conditions:*

(1) *$\Psi(x,\cdot)$ is concave in Λ for all $x \in X$;*

(2) *$\Psi(\cdot,\lambda)$ is continuous, coercive and sequentially weakly l.s.c. (lower semicontinuous) in X for all $\lambda \in \Lambda$;*

(3) *$\beta_1 := \sup_{\lambda\in\Lambda} \inf_{x\in X} \Psi(x,\lambda) < \inf_{x\in X} \sup_{\lambda\in\Lambda} \Psi(x,\lambda) =: \beta_2$.*

Then, for each $\delta > \beta_1$, there exists a non-empty open set $\Lambda_0 \subset \Lambda$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly l.s.c. function $\Phi : X \to \mathbf{R}$, there exists $\mu_0 > 0$ such that, for each $\mu \in [0,\mu_0]$, the function $\Psi(\cdot,\lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X : \Psi(x,\lambda) < \delta\}$.

2. THE MAIN RESULT

Our main result is given by the following theorem.

Theorem 2. *Let $f\colon \Omega \times \mathbf{R} \to \mathbf{R}$ be a function satisfying the conditions $(F1) - (F4)$. Then there exists a non-degenerate compact interval $[a,b] \subset [0,+\infty]$ with the following properties:*

1. *there exists a number $\sigma_0 > 0$ such that, for every $\lambda \in [a,b]$ and for every function $g\colon \Gamma \times \mathbf{R} \to \mathbf{R}$ satisfying the conditions $(G1) - (G2)$, there exists $\mu_0 > 0$ such that, for each $\mu \in [0,\mu_0]$, the functional $\mathcal{E}_{\lambda,\mu}$ has at least two critical points with norms less than σ_0;*
2. *there exists a number $\sigma_1 > 0$ such that, for every $\lambda \in [a,b]$ and for every function $g\colon \Gamma \times \mathbf{R} \to \mathbf{R}$ satisfying the conditions $(G1) - (G3)$, there exists $\mu_1 > 0$ such that, for each $\mu \in [0,\mu_1]$, the functional $\mathcal{E}_{\lambda,\mu}$ has at least three critical points with norms less than σ_1.*

We need the following auxiliary results for the proof of the main theorem.

Lemma 1. *We assume that the conditions $(F1)$ and $(F2)$ are satisfied. Then the functional J_F is sequentially weakly continuous.*

Proof. First we observe that, from assumption $(F1)$ and $(F2)$, for every $\varepsilon > 0$, there exists C_ε such that

$$|F(x,u)| \le \varepsilon f_0(x)|u(x)|^{p^+} + C_\varepsilon(f_0(x)+f_1(x))|u(x)|^r. \tag{6}$$

Now we argue by contradiction and assume that there exist a sequence $\{u_n\}$ in E weakly convergent to $u \in E$, and $d > 0$ such that

$$|J_F(u_n) - J_F(u)| \ge d, \text{ for all } n \in \mathbb{N}.$$

Without loss of generality, we can assume that there is a positive constant M such that

$$||u||_b \le M, \quad ||u_n||_b \le M, \text{ and } \quad ||u_n - u||_b \le M, \text{ for all } n \in \mathbb{N}.$$

Since the embedding $E \hookrightarrow L^r(\Omega; w_1)$ is compact, we have $||u_n - u||_{r,w_1} \to 0$. By (6), Proposition 1 and the Hölder inequality we get

$$\begin{aligned}
|J_F(u_n) - J_F(u)| &\le \int_\Omega |F(x,u_n(x)) - F(x,u(x))|\mathrm{d}x \\
&\le \varepsilon\hat{c}\int_\Omega f_0(x)|u_n(x)-u(x)|(|u_n(x)|^{p^+-1} + |u(x)|^{p^+-1})\mathrm{d}x \\
&+ \hat{c}C_\varepsilon \int_\Omega (f_0(x)+f_1(x))|u_n(x)-u(x)|(|u_n(x)|^{r-1} + |u(x)|^{r-1})\mathrm{d}x \\
&\le \varepsilon\hat{c}C_f \int_\Omega w_1(x)|u_n(x)-u(x)|(|u_n(x)|^{p^+-1} + |u(x)^{p^+-1}|)\mathrm{d}x \\
&+ 2\hat{c}C_\varepsilon C_f \int_\Omega w_1(x)|u_n(x)-u(x)|(u_n(x)|^{r-1} + |u(x)|^{r-1})\mathrm{d}x \\
&\le \varepsilon\hat{c}C_f||u_n - u||_{p^+,w_1}(||u_n||_{p^+,w_1}^{\frac{p^+}{p'}} + ||u||_{p^+,w_1}^{\frac{p^+}{p'}}) \\
&+ 2\hat{c}C_\varepsilon C_f||u_n - u||_{r,w_1}(||u_n||_{r,w_1}^{\frac{r}{r'}} + ||u||_{r,w_1}^{\frac{r}{r'}}),
\end{aligned}$$

where $\hat{c} > 0$ is a constant, $\frac{1}{p^+} + \frac{1}{p'} = 1$, and $\frac{1}{r} + \frac{1}{r'} = 1$. Using the embedding results from Proposition 1 it folows that

$$d \le |J_F(u_n) - J_F(u)| \le 2\varepsilon\hat{c}C_f C_{p^+,w_1}^{p^+} M^{p^+} + 4\hat{c}C_\varepsilon C_f C_{r,w_1}^{\frac{r}{r'}} M^{\frac{r}{r'}}||u_n - u||_{r,w_1}.$$

Therefore, if $\varepsilon > 0$ is sufficiently small and $n \in \mathbb{N}$ is large enough, we have

$$d \le |J_F(u_n) - J_F(u)| < d,$$

which is a contradiction. □

Similar to Lemma 1 we have the next result.

LEMMA 2. *If the conditions* $(G1)$ *and* $(G2)$ *are satisfied, then the functional* J_G *is sequentially weakly continuous.*

Lemma 3. *Suppose that the conditions* $(F1)$ *and* $(F3)$ *are satisfied. Then, for every* $\lambda \geq 0$, *the functional*

$$u \mapsto \frac{||u||_b^{p^+}}{p^+} - \lambda J_F(u)$$

is coercive on E.

Proof. If $\lambda = 0$, the statement is trivial.

Now fix $\lambda > 0$ and $a \in \left[0, \frac{1}{\lambda p^+ C_{p^+,w_1}^{p^+}}\right]$. There exists a positive function $h_a \in L^1(\Omega; w_1)$ such that

$$F(x,s) \leq a f_0(x)|s|^{p^+} + h_a(x)w_1(x), \text{ for all } (x,s) \in \Omega \times \mathbf{R}.$$

It follows that

$$\begin{aligned} \frac{||u||_b^{p^+}}{p^+} - \lambda J_F(u) &\geq \frac{||u||_b^{p^+}}{p^+} - \lambda a \int_\Omega w_1(x)|u(x)|^{p^+} dx - \lambda \int_\Omega h_a(x)w_1(x)\mathrm{d}x \\ &\geq ||u||_b^{p^+} \left(\frac{1}{p^+} - \lambda a C_{p^+,w_1}^{p^+}\right) - \lambda ||h_a||_{1,w_1}, \end{aligned}$$

that goes to ∞ as $||u||_b \to \infty$. □

Proof. (Theorem 2). We define the function $h\colon [0, +\infty] \to \mathbf{R}$ by setting

$$h(t) = \sup\left\{J_F(u) : \frac{||u||_b^{p^+}}{p^+} \leq t\right\}, \text{ for all } t > 0.$$

Using (6), we have that

$$0 \leq h(t) \leq \varepsilon p^+ C_f C_{p^+,w_1}^{p^+} t + 2(p^+)^{\frac{r}{p^+}} C_\varepsilon C_f C_{r,w_1}^r t^{\frac{r}{p^+}}, \text{ for all } t > 0;$$

since $p^+ < r$, it follows that

$$\lim_{t \to 0^+} \frac{h(t)}{t} = 0.$$

By $(F4)$ it is clear that $u_0 \neq 0$ (since $J_F(0) = 0$). Thus, due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \frac{||u_0||_b^{p^+}}{p^+}$ and

$$\frac{h(t_0)}{t_0} < \frac{p^+}{||u_0||_b^{p^+}} \cdot J_F(u_0).$$

We choose $\rho_0 > 0$ such that

$$h(t_0) < \rho_0 < \frac{p^+}{||u_0||_b^{p^+}} \cdot J_F(u_0) t_0. \tag{7}$$

In particular, we have $\rho_0 < J_F(u_0)$.

Now we are going to apply Theorem 1 to the space E, the interval $\Lambda = [0,+\infty]$ and the function $\Psi\colon E\times\Lambda\to\mathbf{R}$ defined by

$$\Psi(u,\lambda)=\frac{||u||_b^{p^+}}{p^+}+\lambda(\rho_0-J_F(u)),\ \text{for all } (u,\lambda)\in E\times\Lambda,$$

and $\Phi\colon E\to\mathbf{R}$ by

$$\Phi(u)=-J_G(u),\ \text{for all } u\in E.$$

Clearly, condition (1) from Theorem 1 is fulfilled.

In order to check condition (2) from Theorem 1, let us fix $\lambda\in[0,+\infty]$. Using Lemma 3, it follows that the functional $\Psi(\cdot,\lambda)$ is coercive; moreover, $\Psi(\cdot,\lambda)$ is the sum of $u\mapsto\frac{||u||_b^{p^+}}{p^+}$, which is sequentially weakly l.s.c., and of $u\mapsto\lambda(\rho_0-J_F(u))$, which is sequentially weakly continuous (see Lemma 1).

Next we prove that Ψ satisfies the minimax inequality (3) from Theorem 1. The function

$$\lambda\mapsto\inf_{u\in E}\Psi(u,\lambda)$$

is upper semicontinuous on Λ. Using

$$\inf_{u\in E}\Psi(u,\lambda)\le\Psi(u_0,\lambda)=\frac{||u_0||_b^{p^+}}{p^+}+\lambda(\rho_0-J_F(u_0))$$

and $\rho_0<J_F(u_0)$, we obtain that

$$\lim_{\lambda\to+\infty}\inf_{u\in E}\Psi(u,\lambda)=-\infty.$$

Therefore we can find $\overline{\lambda}\in\Lambda$ such that

$$\beta_1=\sup_{\lambda\in\Lambda}\inf_{u\in E}\Psi(u,\lambda)=\inf_{u\in E}\Psi(u,\overline{\lambda}).$$

In order to prove that $\beta_1<t_0$, we distinguish two cases:

I. If $0\le\overline{\lambda}<\frac{t_0}{\rho_0}$, we have

$$\beta_1\le\Psi(0,\overline{\lambda})=\overline{\lambda}\rho_0<t_0.$$

II. If $\overline{\lambda}\ge\frac{t_0}{\rho_0}$, then we use $\rho_0<J_F(u_0)$ and the inequality (7) to get

$$\beta_1\le\Psi(u_0,\overline{\lambda})\le\frac{||u_0||_b^{p^+}}{p^+}+\frac{t_0}{\rho_0}(\rho_0-J_F(u_0))<t_0.$$

Let us focus next on the right hand side of the inequality (3) of Theorem 1. Clearly

$$\beta_2=\inf_{u\in E}\sup_{\lambda\in\Lambda}\Psi(u,\lambda)=\inf\left\{\frac{||u||_b^{p^+}}{p^+}:J_F(u)\ge\rho_0\right\}.$$

On the other hand, using again (7), we easily get

$$t_0\le\inf\left\{\frac{||u||_b^{p^+}}{p^+}:J_F(u)\ge\rho_0\right\}.$$

Thus

$$\beta_1 < t_0 \leq \beta_2,$$

that is, condition (3) from Theorem 1 holds.

Next, we can apply Theorem 1. Fix $\delta > \beta_1$, and for every $\lambda \in \Lambda$ denote

$$S_\lambda = \{u \in E : \Psi(u, \lambda) < \delta\}.$$

There exists a non-empty open set $\Lambda_0 \subset [0, +\infty]$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly l.s.c. $\Phi\colon E \to \mathbf{R}$, there exists $\mu_0 > 0$, such that for each $\mu \in [0, \mu_0]$, the functional

$$u \mapsto \Psi(u, \lambda) + \mu\Phi(u)$$

has at least two local minima lying in the set S_λ. Let $[a, b] \subset \Lambda_0$ be a non-degenerate compact interval.

We prove now the two assertions of our theorem:

1. Let $\lambda \in [a, b]$ be a real number, and let $g\colon \Gamma \times \mathbf{R} \to \mathbf{R}$ satisfying the conditions $(G1)$ and $(G2)$, and let $\Phi = -J_G$. Then, by Lemma 2, Φ is sequentially weakly continuous. From what we have stated above it follows that there exists $\mu_0 > 0$ such that for all $\mu \in [0, \mu_0]$ the functional $\mathcal{E}_{\lambda,\mu}$ admits at least two local minima $u^1_{\lambda,\mu}, u^2_{\lambda,\mu} \in S_\lambda$, therefore these are critical points of $\mathcal{E}_{\lambda,\mu}$.

Observe that

$$S := \bigcup_{\lambda \in [a,b]} S_\lambda \subseteq S_a \cup S_b.$$

Since $\Psi(\cdot, \lambda)$ is coercive for all $\lambda \geq 0$, the latter sets are bounded, hence S is bounded as well. Choosing $\sigma_0 > \sup_{u \in S} ||u||_b$, we get

$$||u^1_{\lambda,\mu}||_b, ||u^2_{\lambda,\mu}||_b < \sigma_0.$$

2. Let $\lambda \in [a, b]$ be a real number, and let $g\colon \Gamma \times \mathbf{R} \to \mathbf{R}$ satisfying the conditions $(G1) - (G3)$. As above, there exists $\mu_0 > 0$ such that for all $\mu \in [0, \mu_0]$ the functional $\mathcal{E}_{\lambda,\mu}$ has at least two local minima $u^1_{\lambda,\mu}, u^2_{\lambda,\mu} \in E$ with norms less than σ_0. To prove the existence of a third critical point for $\mathcal{E}_{\lambda,\mu}$, we are going to apply Corollary 1 of [17]. For this it is enough to prove that the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS) condition for $\mu > 0$ small enough. Since $(G3)$ holds, arguing as in Lemma 3, it is easy to prove that there exists $\mu_1 \in [0, \mu_0]$ such that $\mathcal{E}_{\lambda,\mu}$ is coercive in E for all $\mu \in [0, \mu_1]$. Let $\{u_n\}$ be a sequence such that $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded and $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$ holds. The coercivity of $\mathcal{E}_{\lambda,\mu}$ implies that $\{u_n\}$ is bounded in E. Because E is a reflexive Banach space we can find a subsequence, which we still denote by $\{u_n\}$, weakly convergent to a point $u_0 \in E$. We denote $I(u) = \frac{1}{p(x)}||u||_b^{p(x)}$. Then the directional derivative of $\mathcal{E}_{\lambda,\mu}$ in the direction $h \in E$ is

$$\langle \mathcal{E}'_{\lambda,\mu}(u), h\rangle = \langle I'(u), h\rangle - \lambda\langle J'_F(u), h\rangle - \mu\langle J'_G(u), h\rangle,$$

where

$$\langle I'(u),h\rangle = \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u(x)\nabla h(x)\mathrm{d}x+\int_\Gamma b(x)|u(x)|^{p(x)-2}u(x)h(x)\mathrm{d}\Gamma,$$

$$\langle J'_F(u),h\rangle = \int_\Omega f(x,u(x))h(x)\mathrm{d}x$$

and

$$\langle J'_G(u),h\rangle = \int_\Gamma g(x,u(x))h(x)\mathrm{d}\Gamma.$$

To show that $u_n \to u_0$ strongly in E we use the following inequalities for $\xi,\zeta \in \mathbf{R}^N$ (see [7], Lemma 4.10):

$$|\xi-\zeta|^{p^-} \le C^*(|\xi|^{p^--2}\xi - |\zeta|^{p^--2}\zeta)(\xi-\zeta),\ \text{ for } p^- \ge 2, \tag{8}$$

and, for $p^- \in [1,2]$,

$$|\xi-\zeta|^{p^-} \le C^*(|\xi|^{p^--2}\xi - |\zeta|^{p^--2}\zeta)(\xi-\zeta)^{\frac{p^-}{2}}(|\xi|^{p^-}+|\zeta|^{p^-})^{\frac{2-p^-}{2}}. \tag{9}$$

If $p^- \ge 2$ we obtain:

$$\begin{aligned}
||u_n-u_0||_b^{p^-} &= \int_\Omega a(x)|\nabla u_n(x)-\nabla u_0(x)|^{p^-}dx + \int_\Gamma b(x)|u_n(x)-u_0(x)|^{p^-}\mathrm{d}\Gamma\\
&\le C^*(\langle I'(u_n),u_n-u_0\rangle - \langle I'(u_0),u_n-u_0\rangle)\\
&= C^*(\langle \mathcal{E}'_{\lambda,\mu}(u_n),u_n-u_0\rangle - \langle \mathcal{E}'_{\lambda,\mu}(u_0),u_n-u_0\rangle + \langle \lambda J'_F(u_n)\\
&+\mu J'_G(u_n),u_n-u_0\rangle - \langle \lambda J'_F(u_0)+\mu J'_G(u_0),u_n-u_0\rangle)\\
&\le C^*(||\mathcal{E}'_{\lambda,\mu}(u_n)||_{E'} + \lambda||J'_F(u_n)-J'_F(u_0)||_{E'}\\
&+\mu||J'_G(u_n)-J'_G(u_0)||_{E'})||u_n-u_0||_b - C^*\langle \mathcal{E}'{}_{\lambda,\mu}(u_0),u_n-u_0\rangle.
\end{aligned}$$

Since $\mathcal{E}'{}_{\lambda,\mu}(u_n)\to 0$ and J'_F, J'_G are compact (see [16]), we have that $u_n \to u_0$ converges strongly in E.

If $1 < p^- < 2$, we use (9) and Hölder's inequality to obtain the estimate

$$||u_n-u_0||_b^{p^-} \le \hat{C}|\langle I'(u_n),u_n-u_0\rangle - \langle I'(u_0),u_n-u_0\rangle|(||u_n||_b^{p^-}+||u_0||_b^{p^-})^{\frac{2-p^-}{p^-}},$$

where $\hat{C}>0$ is a positive constant depending on p^- and C^*.

Thus, the condition (PS) is fulfilled for all $\mu \in [0,\mu_1]$. □

Corollary 1. *Let $f\colon \Omega\times\mathbf{R}\to\mathbf{R}$ be a function satisfying the conditions $(F1)$–$(F4)$. Then there exists a non-degenerate compact interval $[a,b]\subset[0,+\infty]$ with the following properties:*

I. *there exists a number $\sigma_0>0$ such that for every $\lambda\in[a,b]$ and for every function $g\colon\Gamma\times\mathbf{R}\to\mathbf{R}$ satisfying conditions $(G1)$–$(G2)$ there exists $\mu_0>0$ such that, for each $\mu\in[0,\mu_0]$, problem $(P_{\lambda,\mu})$ has at least one non-trivial solution in E with norm less than σ_0;*

II. *for every $\lambda\in[a,b]$ and for every function $g\colon\Gamma\times\mathbf{R}\to\mathbf{R}$ satisfying conditions $(G1)$–$(G3)$ there exists $\mu_1>0$ such that, for each $\mu\in[0,\mu_1]$, problem $(P_{\lambda,\mu})$ has at least two non-trivial solutions in E.*

Remark 2. Cârstea and Rădulescu studied in [3] the existence and multiplicity of the solutions of the following problem:

$$(I_{\lambda,\mu}) \qquad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) + h(x)|u|^{r-2}u = \lambda(1+|x|)^{\alpha_1}|u|^{q-2}u, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \mu g(x,u(x)), \\ u \geq 0, u \neq 0 \quad in \quad \Omega, \end{cases}$$

where $h\colon \Omega \to \mathbf{R}$ is a positive, continuous function satisfying

$$\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\mathrm{d}x < +\infty$$

and $\max\{p,2\} \leq q < r < p^* := \frac{pN}{N-p}, \quad -N < \alpha_1 \leq q \cdot \frac{N-p}{p} - N.$

If we consider the problem

$$(I'_{\lambda,\mu}) \quad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda[(1+|x|)^{\alpha_1}|u|^{q-2}u - h(x)|u|^{r-2}u], \\ a(x)|\nabla u|^{p(x)-2}\nabla u \cdot n + b(x)|u|^{p(x)-2}u = \mu g(x,u(x)), \\ u \neq 0 \quad in \quad \Omega, \end{cases}$$

with minor modifications, we can establish the same result as in Corollary 1, which completes the result obtained by Cârstea and Rădulescu in [3].

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