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A MULTIPLICITY RESULT FOR A DOUBLE EIGENVALUE P-LAPLACIAN EQUATION ON UNBOUNDED DOMAIN

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Abstract. We present a multiplicity result concerning a class of quasilinear eigenvalue problems with nonlinear boundary conditions on unbounded domain. The proofs are based on the Mountain Pass theorem applied to weighted Sobolev spaces. Our paper completes the results obtained in this direction (see for instance [1], [3], [5], [6], [8]).

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1. PRELIMINARIES

The eigenvalue problems involving the p-Laplacian operator have been intensively studied by many authors in the last decade. This is motivated by the importance of their applications to mathematical physics.

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ. We assume throughout this paper that m, p, q and α_1, α_2 are real numbers satisfying

$$1 < p < N, \quad p \leq q \leq \frac{pN}{N-p}, \quad -N < \alpha_1 \leq q \cdot \frac{N-p}{p} - N, \tag{1}$$

$$p \leq m \leq p \cdot \frac{N-1}{N-p} \quad \text{and} \quad -N < \alpha_2 \leq m \cdot \frac{N-p}{N} - N + 1. \tag{2}$$

We define the weighted Sobolev space E as the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||_E = \left(\int_\Omega (|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p) \mathrm{d}x \right)^{\frac{1}{p}}.$$

We denote by $L^q(\Omega; w_1)$ and by $L^m(\Gamma; w_2)$ the weighted Lebesgue spaces with respect to

$$w_i(x) = (1+|x|)^{\alpha_i}, i = 1, 2 \tag{3}$$

and norms

$$||u||_{q,w_1}^q = \int_\Omega w_1(x)|u(x)|^q \mathrm{d}x \quad \text{and} \quad ||u||_{m,w_2}^m = \int_\Gamma w_2(x)|u(x)|^m \mathrm{d}\Gamma.$$

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We shall use in our paper the following embedding result from [7].

PROPOSITION 1. *Assume that* (1) *holds.*

Then the embedding $E \hookrightarrow L^q(\Omega; w_1)$ *is continuous. If the upper bound for* q *in* (1) *is strict, then the embedding is compact. Suppose that the inequalities in* (2) *are satisfied. Then the trace operator* $E \to L^m(\Gamma; w_2)$ *is continuous. If the upper bound for* m *in* (2) *is strict, then the trace operator is compact.*

Notations: The best embedding constant of $E \hookrightarrow L^q(\Omega; w_1)$ will be denoted by C_{q,w_1} and that of $E \hookrightarrow L^m(\Gamma; w_2)$ by C_{m,w_2}.

We assume throughout this paper that $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$ such that

$$a(x) \geq a_0 > 0 \quad \text{for a.e. } x \in \Omega \tag{4}$$

and

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}} \quad \text{for a.e. } x \in \Gamma, \tag{5}$$

where c, $C > 0$ are constants.

REMARK 1. Note, that

$$||u||_b^p = \int_\Omega a(x)|\nabla u(x)|^p \mathrm{d}x + \int_\Gamma b(x)|u(x)|^p \mathrm{d}\Gamma$$

defines an equivalent norm on E, see [6], Lemma 2.

The main result of this paper is based on the following Mountain Pass Theorem.

THEOREM 1. *Let* $f : E \to \mathbb{R}$ *be a* C^1 *functional satisfying the Palais-Smale condition and verifying the hypotheses*

(a) *there exist constants* $\alpha > 0$ *and* $\rho > 0$ *such that* $f(u) \geq \alpha$, *for every* $||u|| = \rho$;

(b) *there is* $e \in E$ *with* $||e|| > \rho$ *and* $f(e) \leq \alpha$.

Then the number

$$c = \inf_{g \in \Gamma} \max_{v \in [0,e]} f(g(v)),$$

where $[0, e]$ *is the closed line segment in* E *joining* 0 *and* e *and*

$$\Gamma = \{c \in C([0, e], X) : g(0) = 0, g(e) = e\},$$

is a critical value of f *with* $c \geq \alpha$.

2. MAIN RESULT

For $\lambda > 0$ and $\mu \in \mathbb{R}$ we consider the problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) \text{ in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda\mu g(x,u) \text{ on } \Gamma \\ u \neq 0 \text{ in } \Omega, \end{cases}$$

where n denotes the unit outward normal on Γ, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

We consider the following assumptions:

(F1) $f(\cdot,0) = 0$ and $|f(x,s)| \le f_0(x) + f_1(x)|s|^{r-1}$, where $p < r < \dfrac{pN}{N-p}$, and f_0, f_1 are measurable functions which satisfy

$$0 < f_0(x) \le C_f w_1(x), \text{ and } 0 \le f_1(x) \le C_f w_1(x) \quad \text{a.e. } x \in \Omega,$$

$$f_0 \in L^{\frac{r}{r-1}}(\Omega; w_1^{\frac{1}{1-r}}).$$

(F2) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{f_0(x)|s|^{p-1}} = 0$, uniformly in $x \in \Omega$;

(F3) $\displaystyle\limsup_{s\to+\infty} \frac{1}{f_0(x)|s^p|} F(x,s) \le 0$, uniformly for $x \in \Omega$, $\displaystyle\max_{|s|\le M} F(\cdot,s) \in L^1(\Omega)$ for all $M > 0$, where F denotes the primitive function of f with respect to the second variable, i.e. $F(x,u) = \displaystyle\int_0^u f(x,s)\mathrm{d}s$;

(F4) there exist $x_0 \in \Omega$, $R_0 > 0$ and $s_0 \in \mathbb{R}$ such that $\displaystyle\min_{|x-x_0|<R_0} F(x,s_0) > 0$.

(G1) Let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a a Carathéodory function such that $g(\cdot,0) = 0$ and

$$|g(x,s)| \le g_0(x) + g_1(x)|s|^{m-1},$$

where $p \le m < p \cdot \dfrac{N-1}{N-p}$, and g_0, g_1 are measurable functions satisfying

$$0 < g_0(x) \le C_g w_2(x) \text{ and } 0 \le g_1(x) \le C_g w_2(x), \quad \text{a.e. } x \in \Gamma,$$

$$g_0 \in L^{\frac{q}{q-1}}(\Gamma; w_2^{\frac{1}{1-q}});$$

(G2)

$$\lim_{s\to 0} \frac{g(x,s)}{g_0(x)|s|^{p-1}} = 0, \quad \text{uniformly in } x \in \Gamma;$$

(G3) $\displaystyle\limsup_{s\to+\infty} \frac{1}{g_0(x)|s^p|} G(x,s) \le 0$, uniformly for $x \in \Gamma$, $\displaystyle\max_{|s|\le M} G(\cdot,s) \in L^1(\Gamma)$ for all $M > 0$, where G is the primitive function of g with respect to the second variable, i.e. $G(x,u) = \displaystyle\int_0^u g(x,s)\mathrm{d}s$.

We introduce the functionals $J_F, J_G, J_\mu : E \to \mathbb{R}$, defined by

$$J_F(u) = \int_\Omega F(x, u(x))\mathrm{d}x, \qquad J_G(u) = \int_\Gamma G(x, u(x))\mathrm{d}\Gamma,$$

$$J_\mu(u) = J_F(u) + \mu J_G(u).$$

A standard argument, which is based on the embedding results from Proposition 1 and the assumptions (F1), (G1) shows that the functionals J_F, J_G hence J_μ too, are well defined, are of class C^1 and their directional derivative in direction $v \in E$ are

$$\langle J_F(u), v\rangle = \int_\Omega F(x, u(x))v(x)\mathrm{d}x, \qquad \langle J_G(u), v\rangle = \int_\Gamma G(x, u(x))v(x)\mathrm{d}\Gamma,$$

$$\langle J_\mu(u), v\rangle = \int_\Omega F(x, u(x))v(x)\mathrm{d}x + \mu \int_\Gamma G(x, u(x))v(x)\mathrm{d}\Gamma,$$

for each $u \in E$ (see for instance [10], Lemma 3.10).

Now, we can define the energy functional $\mathcal{E}_{\lambda,\mu} : E \to \mathbb{R}$ corresponding to $(P_{\lambda,\mu})$ as follows

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u(x)|^p \mathrm{d}x + \frac{1}{p}\int_\Gamma b(x)|u(x)|^p \mathrm{d}\Gamma - \lambda J_F(u) - \lambda\mu J_G(u).$$

$\mathcal{E}_{\lambda,\mu}$ is well defined and the solutions of problem $(P_{\lambda,\mu})$ will be found as critical points of $\mathcal{E}_{\lambda,\mu}$. Therefore, a function $u \in E$ is a solution of problem $(P_{\lambda,\mu})$ provided that, for any $v \in E$,

$$\begin{aligned}\langle \mathcal{E}'_{\lambda,\mu}(u), v\rangle &= \int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x \\ &+ \int_\Gamma b(x)|u(x)|^{p-2}u(x)v(x)\mathrm{d}\Gamma \\ &- \lambda\int_\Omega f(x, u(x))v(x)\mathrm{d}x + \lambda\mu\int_\Gamma g(x, u(x))v(x)\mathrm{d}\Gamma = 0.\end{aligned}$$

The main result of this paper is the following:

Theorem 2. *We suppose that the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively. Then there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ the problem $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$, with the property*

$$\mathcal{E}_{\lambda,\mu}(u_\lambda, \mu) < 0 < \mathcal{E}_{\lambda,\mu}(v_\lambda, \mu).$$

3. PROOF OF THE MAIN THEOREM

We start with some auxiliary results.

LEMMA 1. [5, Lemma 3.3] *Suppose that the conditions* (F1)*,* (F3)*,* (G1) *and* (G3) *are satisfied. Then, for every* $\lambda \geq 0$ *and* $\mu \in \mathbb{R}$ *the functional* $\mathcal{E}_{\lambda,\mu}$ *is coercive on* E*.*

LEMMA 2. $\mathcal{E}_{\lambda,\mu} : E \to \mathbb{R}$ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset E$ be a (PS)-sequence for the function $\mathcal{E}_{\lambda,\mu}$, i.e.
(1) $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded;
(2) $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$.

Since $\mathcal{E}_{\lambda,\mu}$ is coercive, we have that $\{u_n\}$ is bounded. The reflexivity of the Banach space E implies the existence of a subsequence notated also by $\{u_n\}$ such that $\{u_n\}$ is weakly convergent to an element $u \in E$. Because the inclusion $E \hookrightarrow L^p(\Omega, w_1)$ is compact, we have that $u_n \to u$ strongly in $L^p(\Omega, w_1)$. We want to prove, that u_n converges strongly to u in E. For this end, we will use the following inequalities from ([2], Lemma 4.10)

$$|\xi - \zeta|^p \leq M_1(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{6}$$

$$|\xi - \zeta|^2 \leq M_2(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } p \in]1, 2[, \tag{7}$$

where M_1 and M_2 are some positive constants. We separate two cases. In the first case let $p \geq 2$. Then we have:

$$\begin{aligned}
&||u_n - u||_b^p = \int_\Omega a(x)|\nabla u_n(x) - \nabla u(x)|^p \mathrm{d}x + \int_\Gamma b(x)|u_n(x) - u(x)|^p \mathrm{d}\Gamma \\
&\leq M_1 \int_\Omega a(x)\left[|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right](\nabla u_n(x) - \nabla u(x))\mathrm{d}x \\
&+ M_1 \int_\Gamma b(x)\left[|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\right](u_n(x) - u(x))\mathrm{d}\Gamma \\
&= M_1(\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u\rangle - \langle \mathcal{E}'_{\lambda,\mu}(u), u_n - u\rangle + \lambda\langle J'_F(u_n) - J'_F(u), u_n - u\rangle \\
&+ \lambda\mu\langle J'_G(u_n) - J'_G(u), u_n - u\rangle) \\
&\leq M_1(||\mathcal{E}'_{\lambda,\mu}(u_n)|| + \lambda||J'_F(u_n) - J'_F(u)|| + \lambda\mu||J'_G(u_n) - J'_G(u)||) \cdot ||u_n - u||_b \\
&- M_1\langle \mathcal{E}'_{\lambda,\mu}(u), u_n - u\rangle.
\end{aligned}$$

Since $u_n \to u$ weakly in E and J'_F, J'_G are compact (see [6]), we have that $||J'_F(u_n) - J'_F(u)|| \to 0$ and $||J'_G(u_n) - J'_G(u)|| \to 0$. Moreover $\mathcal{E}'_{\lambda,\mu}(u) = 0$ and $||\mathcal{E}'_{\lambda,\mu}(u_n)|| \to 0$, so $||u_n - u||_b \to 0$, i.e. u_n converges strongly to u in E.

In the second case, when $1 < p < 2$, we use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{8}$$

Then we obtain

(9)

$$||u_n-u||_b^2=\left(\int_\Omega a(x)|\nabla u_n(x)-\nabla u(x)|^p\mathrm{d}x+\int_\Gamma b(x)|u_n(x)-u(x)|^p\mathrm{d}\Gamma\right)^{\frac{2}{p}}$$
$$\leq C_p\left[\left(\int_\Omega a(x)|\nabla u_n(x)-\nabla u(x)|^p\mathrm{d}x\right)^{\frac{2}{p}}+\left(\int_\Gamma b(x)|u_n(x)-u(x)|^p\mathrm{d}\Gamma\right)^{\frac{2}{p}}\right].$$

Now, using the (7) and the Hölder inequalities we get

$$\int_\Omega a(x)|\nabla u_n(x)-\nabla u(x)|^p\mathrm{d}x=\int_\Omega a(x)(|\nabla u_n(x)-\nabla u(x)|^2)^{\frac{p}{2}}\mathrm{d}x$$
$$\leq M_2\int_\Omega a(x)\left((|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)\right.$$
$$\left.-\nabla u(x))\right)^{\frac{p}{2}}(|\nabla u_n(x)|+|\nabla u(x)|)^{\frac{p(2-p)}{2}}\mathrm{d}x$$
$$=M_2\int_\Omega\left[a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\right]^{\frac{p}{2}}$$
$$\cdot[a(x)(|\nabla u_n(x)|+|\nabla u(x)|)^p]^{\frac{2-p}{2}}\mathrm{d}x$$
$$\leq\widetilde{M}_2\left(\int_\Omega a(x)|\nabla u_n(x)|^p\mathrm{d}x+\int_\Omega a(x)|\nabla u(x)|^p\mathrm{d}x\right)^{\frac{2-p}{2}}$$
$$\cdot\left(\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\mathrm{d}x\right)^{\frac{p}{2}}$$
$$\leq\overline{M}_2\left[\left(\int_\Omega a(x)|\nabla u_n(x)|^p\mathrm{d}x\right)^{\frac{2-p}{2}}+\left(\int_\Omega a(x)|\nabla u(x)|^p\mathrm{d}x\right)^{\frac{2-p}{2}}\right]$$
$$\cdot\left(\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\mathrm{d}x\right)^{\frac{p}{2}}$$
$$\leq\widehat{M}_2\left(||u_n||_b^{\frac{(2-p)p}{2}}+||u||_b^{\frac{(2-p)p}{2}}\right)$$
$$\cdot\left[\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))\cdot(\nabla u_n(x)-\nabla u(x))\mathrm{d}x\right]^{\frac{p}{2}}.$$

Then using again the relation (8) and the above inequality we have:

(10)
$$\left(\int_\Omega a(x)|\nabla u_n(x)-\nabla u(x)|^p\mathrm{d}x\right)^{\frac{2}{p}}$$
$$\leq M_2'\left(||u_n||_b^{2-p}+||u||_b^{2-p}\right)$$
$$\cdot\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\mathrm{d}x.$$

In a similar way we obtain the following estimate

(11)
$$\left(\int_\Gamma b(x)|u_n(x)-u(x)|^p\mathrm{d}\Gamma\right)^{\frac{2}{p}}$$
$$\leq\ M_2'\cdot\left(\int_\Gamma b(x)(|u_n(x)|^{p-2}u_n(x)-|u(x)|^{p-2}u(x))(u_n(x)-u(x))\mathrm{d}\Gamma\right)\cdot$$
$$\left(||u_n||_b^{2-p}+||u||_b^{2-p}\right).$$

We introduce the following notation: $I(u)=\frac{1}{p}||u||_b^p$. As we used before, the directional derivative of I, in the direction $v\in E$ is

$$\langle I'(u),v\rangle=\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x+\int_\Gamma b(x)|u(x)|^{p-2}u(x)v(x)\mathrm{d}\Gamma.$$

Using the inequalities (9), (10), (11) we have

$$||u_n - u||_b^2 < M_2' \cdot \langle I'(u_n) - I'(u), u_n - u\rangle \cdot (||u_n||_b^{p-2} + ||u||_b^{2-p}).$$

Since u_n is bounded, the same argument as in the first case (when $p \geq 2$) shows that u_n converges to u strongly in E.

Thus $\mathcal{E}_{\lambda,\mu}$ satisfy the (PS) condition for all $\lambda > 0$ and $\mu \in \mathbb{R}$. □

Using the assumption (F4) one can prove the existence of an element $u_0 \in E$ such that $J_F(u_0) > 0$ (see [5], Lemma 3.2).

Let us define $m = \int_\Gamma |G(x, u_0(x))| \mathrm{d}\Gamma$,

$$\lambda_0 = \frac{I(u_0)}{J_F(u_0)} > 0 \quad \text{and} \quad \mu_\lambda^* = \frac{1}{\lambda(1+m)} \cdot (\lambda - \lambda_0) J_F(u_0) > 0.$$

Lemma 3. *For $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have*

$$\inf_{u \in E} \mathcal{E}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to prove, that, for $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$. Indeed,

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u_0) &= I(u_0) - \lambda J_f(u_0) - \lambda\mu J_G(u_0) \\
&\leq \lambda_0 J_F(u_0) - \lambda J_F(u_0) + \lambda|\mu| m \\
&= (\lambda_0 - \lambda) J_F(u_0) + \lambda|\mu| m \\
&= (\lambda_0 - \lambda)\frac{\lambda(1+m)\mu_\lambda^*}{\lambda - \lambda_0} + \lambda|\mu| m \\
&= -(1+m)\lambda\mu_\lambda^* + \lambda|\mu| m \\
&= -\lambda\mu_\lambda^* - m\lambda(\mu_\lambda^* - |\mu|) < 0.
\end{aligned}$$

for all $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$. □

Lemma 4. *For every $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*]$, the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry.*

Proof. From the assumptions (F1),(F2), (G1) and (G2) results the existence of $c_1(\varepsilon)$, $c_2(\varepsilon) > 0$ for every $\varepsilon > 0$, such that

$$|F(x, u(x))| \leq \varepsilon f_0(x)|u(x)|^p + c_1(\varepsilon)(f_0(x) + f_1(x))|u(x)|^r, \tag{12}$$

$$|G(x, u(x))| \leq \varepsilon g_0(x)|u(x)|^p + c_2(\varepsilon)(g_0(x) + g_1(x))|u(x)|^m, \tag{13}$$

where $r \in]p, p^*[$ and $m \in \left[p, p\frac{N-1}{N-p}\right]$. Using again the (F1) and (G1) assumptions, we get

$$|F(x, u(x))| \leq \varepsilon w_1(x) C_f |u(x)|^p + 2c_1(\varepsilon) C_f w_1(x))|u(x)|^r, \tag{14}$$

$$(15)\qquad |G(x,u(x))| \le \varepsilon w_2(x) C_g |u(x)|^p + 2c_2(\varepsilon) C_g w_2(x)|u(x)|^m.$$

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[$, then using the (14) and (15) inequalities for every $u \in E$ we have

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{p}||u||_b^p - \lambda J_\mu(u) \\
&\ge \frac{1}{p}||u||_b^p - \lambda \int_\Omega |F(x,u(x))| \mathrm{d}x - \lambda|\mu| \int_\Gamma |G(x,u(x))| \mathrm{d}\Gamma \\
&\ge \frac{1}{p}||u||_b^p - \lambda\varepsilon C_f \int_\Omega w_1(x)|u(x)|^p \mathrm{d}x - 2\lambda c_1(\varepsilon) C_f \int_\Omega w_1(x)|u(x)|^r \mathrm{d}x \\
&\quad -\lambda|\mu|\varepsilon C_g \int_\Gamma w_2(x)|u(x)|^p \mathrm{d}\Gamma - 2\lambda|\mu| c_2(\varepsilon) C_g \int_\Gamma w_2(x)|u(x)|^m \mathrm{d}\Gamma \\
&= \frac{1}{p}||u||_b^p - \lambda\varepsilon C_f ||u||^p_{p,w_1} - 2\lambda c_1(\varepsilon) C_f ||u||^r_{r,w_1} \\
&\quad -\lambda|\mu|\varepsilon C_g ||u||^p_{p,w_2} - 2\lambda|\mu| c_2(\varepsilon) C_g ||u||^m_{m,w_2} \\
&\ge \left(\frac{1}{p} - \lambda\varepsilon C_f C^p_{p,w_1} - \lambda|\mu|\varepsilon C_g C^p_{p,w_2}\right) ||u||_b^p \\
&\quad -2\lambda c_1(\varepsilon) C_f C^r_{r,w_1} ||u||_b^r - 2\lambda|\mu| c_2(\varepsilon) C_g C^m_{m,w_2} ||u||_b^m.
\end{aligned}$$

Using the notations

$$A = \left(\frac{1}{p} - \lambda\varepsilon C_f C^p_{p,w_1} - \lambda|\mu|\varepsilon C_g C^p_{p,w_2}\right),$$

$$B = 2\lambda c_1(\varepsilon) C_f C^r_{r,w_1}, \quad C = 2\lambda|\mu| c_2(\varepsilon) C_g C^m_{m,w_2},$$

we get

$$\mathcal{E}_{\lambda,\mu}(u) \ge (A - B||u||_b^{r-p} - C||u||_b^{m-p})||u||_b^p.$$

We choose $\varepsilon \in \left]0, \frac{1}{2p}\frac{1}{\lambda(\varepsilon C_f C^p_{p,w_1} + |\mu|\varepsilon C_g C^p_{p,w_2})}\right[$, so $A > 0$. Now, let $l : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by $l(t) = A - Bt^{r-p} - Ct^{m-p}$. We can see, that $l(0) = A > 0$, so because l is continuous, there exists an $\varepsilon^* > 0$ such that $l(t) > 0$, for every $t \in]0, \varepsilon^*[$. Then for every $u \in E$, with $||u|| = \varepsilon^{**} < \min\{\varepsilon^*, ||u_0||\}$, we have $\mathcal{E}_{\lambda,\mu}(u) \ge \eta(\lambda,\mu,\varepsilon^*) > 0$. From Lemma 3 we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$.

Therefore the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the hypotheses of the Mountain Pass theorem 1. □

Proof of theorem 2.

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[= I_\lambda$. From the lemma 2 we have that the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS)-condition, from the lemma 1 we have the coerciveness of $\mathcal{E}_{\lambda,\mu}$. Then there exists an element $u_{\lambda,\mu} \in E$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{v\in E} \mathcal{E}_{\lambda,\mu}(v)$ (see [9]). By using lemma 3 we have $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$. On the other hand by lemma 4 and the Mountain Pass Theorem 1, there exists an element $v_{\lambda,\mu} \in E$ such that $\mathcal{E}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}) \ge \eta(\lambda,\mu,\varepsilon^*) > 0$, which completes the proof. □

REFERENCES

[1] Cârstea, F.-C. Şt. and Rădulescu, V., *On a class of quasilinear eigenvalue problems on unbounded domains*, Arch. der Math., **77** (2001), 337–346.

[2] Diaz, J. I., *Nonlinear partial differential equations and free boundaries. Elliptic equations.* Pitman Adv. Publ., Boston , 1986.

[3] Lisei, H., Varga, Cs. and Horváth, A., *Multiplicity results for a class of quasilinear eigenvalue problems on unbounded domains*, Arch. der Math., in press

[4] de Nápoli, P. and Mariani, M. C., *Equations of p-Laplacian Type in Unbounded Domains* Adv. Nonlinear Studies **2** (2001), 237–250

[5] Mezei, I. I. and Varga, Cs., *Multiplicity result for a double eigenvalue quasilinear problem on unbounded domain*, Nonlinear Analysis, in press

[6] Pflüger, K., *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition*, Electronic J.Differential Equations **10** (1998), 1–13.

[7] Pflüger, K., *Compact traces in weighted Sobolev space.* Analysis **18** (1998), 65–83.

[8] Montefusco, E. and Radulescu, V., *Nonlinear eigenvalue problems for quasilinear operators on unbounded domains*, Nonlinear differ. equ. appl. **8** (2001) 481-497

[9] Rabinowitz, P. H., *Minimax Method in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Series in Math. **65**, AMS, Providence (1986)

[10] Willem, M., *Minimax theorems*, Birkhäuser, Boston, 1996

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