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INDECOMPOSABLE MODULES OVER GROUP GRADED SKEW ALGEBRAS

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Abstract. G-graded skew algebras over a G-acted commutative ring have been introduced by E. Dade as a framework to combine Clifford theory and Galois theory. In this note we consider indecomposable graded modules over such algebras and their endomorphism rings.

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1. INTRODUCTION AND PRELIMINARIES

1.1. Let G be a group, $R = \bigoplus_{g\in G} R_g$ a G-graded ring (not necessarily strongly graded, and let $\mathcal{O}$ be a commutative G-ring. We assume that rings have identity elements, and that actions are on the left.

The following concept was introduced by E. Dade [1].

Definition 1.2. *The G-graded ring R is called a G-graded skew algebra over $\mathcal{O}$ if there is an identity preserving ring homomorphism*

$$\chi : \mathcal{O} \to R$$

satisfying

$$a\chi(r) = \chi({}^g r)a \in R_g$$

for all $r \in \mathcal{O}$, $g \in G$ and $a \in R$.

1.3. Note that R_1 becomes an $\mathcal{O}$-algebra since χ induces a ring homomorphism $\mathcal{O} \to Z(R_1)$, and moreover, R becomes an $(\mathcal{O}, \mathcal{O})$-bimodule, where by definition

$$ras = \chi(r)a\chi(s)$$

for all $r, s \in \mathcal{O}$ and $a \in R$. By 1.2, we have that

$$ar = {}^g r \cdot a$$

for all $g \in G$, $r \in \mathcal{O}$ and $a \in R_g$.

1.4. Let us consider the case when A is strongly graded. Then there is a well-known action of G on the centralizer $C_R(R_1)$ of R_1 in R, defined as

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follows. Let $g \in G$. Then, since $R_g R_{g^{-1}} = R_1$, there are elements $u_{g,i} \in R_g$ and $u'_{g,i} \in R_{g^{-1}}$ such that

$$\sum_i u_{g,i} u'_{g,i} = 1.$$

If $c \in C_R(R_1)$, define

$${}^g c = \sum_i u_{g,i} c u'_{g,i}.$$

This is independent on the choice of the elements $u_{g,i}$ and $u'_{g,i}$, and in fact, ${}^g c$ is the unique element of $C_R(R_1)$ satisfying

$$ac = {}^g c \cdot a$$

for all $a \in R_g$. Note that if $c \in C_R(R_1)_h$, then ${}^g c \in C_R(R_1)_{ghg^{-1}}$ for all $h \in G$.

In particular, the center $Z(R_1)$ of R_1 becomes a G-ring, and moreover, R is a G-graded skew algebra over $Z(R_1)$. In this situation, R is a G-graded skew algebra over $\mathcal{O}$ if and only if there is a homomorphism $\mathcal{O} \to Z(R_1)$ of G-rings.

1.5. The introduction of G-graded skew algebras is motivated in [1] by the new strengthenings of the Alperin-McKay conjecture due to G. Navarro involving Galois actions on characters of finite groups. Schur indices are also included in a conjecture formulated by A. Turull [6].

Dade showed in [1] that starting with a crossed product which is a G-graded skew algebra over a G-field, then the Clifford theoretical constructions performed with a split simple module lead to a crossed product skew algebra over the same G-field. In this note we extend this result to the situation of Clifford theory for indecomposable modules as considered in [3].

1.6. All the unexplained concepts and facts can be found in [4]. We recall here one more notion needed in the next section. By definition, the graded Jacobson radical $J_{\mathrm{gr}}(R)$ of R, is the intersection of the maximal graded left ideals of R. This is a graded ideal of R, with 1-component $J_{\mathrm{gr}}(R)_1$ coinciding with the Jacobson radical $J(R_1)$ of R_1. Moreover, we have that $J_{\mathrm{gr}}(R)_1 \subseteq J(R)$.

2. ENDOMORPHISM RINGS OF G-GRADED INDECOMPOSABLE MODULES

2.1. Let R be a G-graded skew algebra over $\mathcal{O}$, and let $M = \bigoplus_{x \in G} M_x$ be a G-graded (left) R-module. Then, by [1, Proposition 4.1], M has a structure of an $(\mathcal{O}, \mathcal{O})$-bimodule by letting

$$mr = {}^g r \cdot m$$

for all $g \in G$, $m \in M_g$ and $r \in \mathcal{O}$.

The g-conjugate ${}^g M$ (also denoted by $M(g)$) of M coincides with M as an R-module, but has components

$$({}^g M)_x = M_{xg}$$

for all $x \in G$.

Let $r \in \mathcal{O}$, and let ${}^g m \in {}^g M_x$ denote the element $m \in M_{xg}$ regarded in ${}^g M$. Then mr belongs to M_{xg}, ${}^g(mr) \in {}^g M_x$, and we have that

$$ {}^g(mr) = {}^g m \cdot {}^g r. $$

The stabilizer of M in G is, by definition, the subgroup

$$ G_M = \{g \in G \mid M \simeq {}^g M \text{ as } G\text{-graded } R\text{-modules}\} $$

Finally, let $E := \mathrm{End}_R(M)^{\mathrm{op}}$, and for $f, f' \in E$ and $m \in M$, $mf = f(m)$ and $ff' = f' \circ f$. Then E is a G-graded ring such that M is a G-graded (R, E)-bimodule. The g-component of E is

$$ \begin{aligned} E_g &= \{f \in \mathrm{End}_R(M) \mid f(M_x) \subseteq M_{xg} \text{ for all } x \in G\} \\ &= \mathrm{Hom}_{R\text{-Gr}}(M, {}^g M). \end{aligned} $$

2.2. From now on we assume that G is a finite group acting on the commutative noetherian ring $\mathcal{O}$. Then the residue field $k = \mathcal{O}/J(\mathcal{O})$ is a G-field in a natural way. Moreover, we assume that $R/J(R)$ is finite dimensional over k.

We say that the G-graded R-module M is gr-indecomposable if it is not a direct sum of two nontrivial graded submodules.

Theorem 2.3. *Let M be a gr-indecomposable R-module, free of finite rank over $\mathcal{O}$, and let $D := E/J_{\mathrm{gr}}(E)$. Then D is a k-skew crossed product of the division k-algebra $D_1 \simeq E_1/J(E_1)$ and G_M. The action of G_M on k coming from (1.3) is the same as the action coming from the G-graded skew $\mathcal{O}$-algebra structure of R.*

Proof. Consider the map

$$ \chi' : \mathcal{O} \to E_1 = \mathrm{End}_{R\text{-Gr}}(M)^{\mathrm{op}}, \quad \chi'(r)(m) = mr $$

for all $r \in \mathcal{O}$ and $m \in M$. Then for all $f \in E_g$ we have that

$$ f\chi'(r) = \chi'({}^g r)f $$

(see the proof of [1, Proposition 5.1]. It follows that E becomes a G-graded skew algebra over the given G-ring $\mathcal{O}$.

Since $E_1 \cap J_{\mathrm{gr}}(E) = J_{\mathrm{gr}}(E)_1 = J(E_1)$, we have that $D_1 \simeq E_1/J(E_1)$. Moreover, $E_1 = \mathrm{End}_{R\text{-Gr}}(M)^{\mathrm{op}}$ is a local ring since M is gr-indecomposable, and we get that D_1 is a division k-algebra.

Let $g \in G \setminus G_M$. Then M and ${}^g M$ are non-isomorphic gr-indecomposable R-modules, hence every grade-preserving map $f : M \to {}^g M$ generates a graded ideal of E. It follows that $E_g \subseteq J_{\mathrm{gr}}(E)$.

If $g \in G_M$, then there exists an isomorphism $f : M \to {}^g M$, which gives an invertible element $\bar{f} \in U(D) \cap D_g$. Consequently, D is a crossed product of D_1 and G_M. Moreover, for any $r \in \mathcal{O}$, denoting by $\bar{r}$ the image of r in k, we have

$$ fr = {}^g r \cdot f, $$

hence

$$ \bar{f}\bar{r} = {}^g \bar{r} \cdot \bar{f}. $$

It follows that the action of G_M on k induced by the crossed product D coincides with the initial action. □

Example 2.4. Here is a situation which motivates our assumptions. Let $\mathcal{O}$ be a complete local principal ideal domain with residue field k of characteristic $p > 0$. Let b be a block with defect group D of the block algebra $\mathcal{O}G$, and let c be the Brauer corresponding block of $\mathcal{O}N_G(D)$. Fix a root e of b, so e is a block of $kC_G(D)$ with defect group $Z(D)$. Let

$$N_G(D, e) = \{g \in N_G(D) \mid {}^g e = e\}$$

let

$$E_G(D, e) = N_G(D, e)/DC_G(D),$$

so $E_G(D, e)$ acts naturally on D.

Then the algebra $\mathcal{O}N_G(D)c$ is Morita equivalent to an $\hat{\mathcal{O}}$-skew crossed product of the group algebra $\hat{\mathcal{O}}D$ and $E_G(D, e)$, where $\hat{\mathcal{O}}$ is a separable cyclic extension of $\mathcal{O}$ on which $E_G(D, e)$ acts nontrivally in general.

This a generalization by Y. Fan and L. Puig of a result of B. Külshammer (see also [2] and the references given there).

Corollary 2.5. *Let R be an $\mathcal{O}$-skew crossed product, let N be anormal subgroup of G acting trivially on $\mathcal{O}$, and let U be an absolutely indecomposable R_N-module (i.e. $\mathrm{End}_{R_N}/J(\mathrm{End}_{R_N}(U) \simeq k)$. Denote $\bar{G} = G/N$, let $M = R \otimes_{R_N} U$ and let $\bar{D} = \bar{E}/J_{gr}(\bar{E})$, where $\bar{E} = \mathrm{End}_R(M)^{\mathrm{op}}$.*

Then $\bar{D}$ is a k-skew crossed product of k and $\bar{G}_M$, where the the action of $\bar{G}_M$ on k is induced by the action of G on $\mathcal{O}$.

Proof. Here we regard R and E as $\bar{G}$-graded skew algebras over $\mathcal{O}$. Since U is an indecomposable R_N-module and R is strongly graded, we have that M is a gr-indecomposable R-module,

$$\bar{G}_M = \{\bar{g} \in \bar{G} \mid R_{\bar{g}} \otimes_{R_N} U \simeq U \text{ in } R_N\text{-mod}\},$$

and the restriction to U of endomorphisms of M induce the isomorphisms

$$E_{\bar{1}} \simeq \mathrm{End}_{R_N}(U)^{\mathrm{op}}$$

and $D_{\bar{1}} \simeq k$. The statements now follow immediately from Theorem 2.3. □

2.6. We end by recalling the motivation for our interest in the algebra $D = E/J_{\mathrm{gr}}(E)$. Denote by $(R|M)$-mod the full subcategory of R-mod consisting of direct summands of finite direct copies of M, that is, $(R|M)$-mod is the smallest additive subcategory of R-mod containing M. By [4, Theorem 2.3.10], we have that the additive functor

$$D \otimes_E \mathrm{Hom}_R(R \otimes_{R_1} M, -) : (R|M)\text{-mod} \to D\text{-proj}$$

induces an isomorphism between the Grothendieck groups associated to these categories, and in particular, a bijection between the indecomposable summands of M and the indecomposable projective D-modules.

In the situation of the above corollary, the objects of the category $(R|R\otimes_{R_N} U)$-mod are called R-modules lying over U.

REFERENCES

[1] Dade, E.C., *Clifford theory and Galois theory. I*, J. Algebra, 319 (2008), 779–799.

[2] Dicu, C. and Marcus, A., *Source modules of blocks with normal defect groups*, Arch. Math. **88** (2007), 289-296.

[3] Marcus, A., *Static modules and Clifford theory for strongly graded rings*, Publ. Math. Debrecen, Tomus **42**, Fasc. 3-4, 303–314, 1993.

[4] Marcus, A., *Representation Theory of Group Graded Algebras*, Nova Science Publishers, Commack, NY, 1999.

[5] Navarro, G., *The McKay conjecture and Galois automorphisms*, Annals of Math. **160** (2004), 1129–1140.

[6] Turull, A., *Strengthening the McKay Conjecture to include local fields and local Schur indices*, J. Algebra, in press.

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