MATHEMATICA, Tome 50 (73), N° 2, 2008, pp. 187–190

A NOTE ON FLAT COVERS OF COMODULES

SEPTIMIU CRIVEI

Abstract. We present another proof for the existence of a flat cover for every comodule over a semiperfect coalgebra.
MSC 2000. 16D90.
Key words. Flat cover, comodule, semiperfect coalgebra.

1. INTRODUCTION

The Flat Cover Conjecture has played a special part in the theory of module approximation, originated in the work of Auslander and Smalø for finitely generated modules over finite dimensional algebras [2] and Enochs for modules over arbitrary rings [9]. After Bican, El Bashir and Enochs gave a positive answer to it, showing that every module has a flat cover [3, Theorem 3], the problem of the existence of flat covers for any object was raised and solved in more general contexts, such as Grothendieck categories. Thus, Aldrich, Enochs, García Rozas and Oyonarte [1] proved the existence of a flat cover for any object in a Grothendieck category with a flat generator. El Bashir [8] showed that if $\mathcal{F}$ is a class of objects of a Grothendieck category $\mathcal{G}$ closed under coproducts and directed colimits and there exists a subset $\mathcal{S}$ of $\mathcal{F}$ such that each object in $\mathcal{F}$ is a directed colimit of objects in $\mathcal{S}$, then each object of $\mathcal{G}$ has an $\mathcal{F}$-cover. Cuadra and Simson [6] showed that if $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projectives, then every object of $\mathcal{A}$ has a flat cover. As a consequence for comodules, if C is a right semiperfect coalgebra over a field k, then every right C-comodule has a flat cover [6].

In the present note, we present a different proof of this last result, that uses the idea of showing that a certain cotorsion theory is cogenerated by a set, and follows the same steps as one of the corresponding proofs of the existence of flat covers of modules. The existence of enough projectives in our category of comodules will be exploited. A key step is to involve solvability of systems of equations in comodules, which was characterized in [5].

Let us recall the concepts of purity and flatness in a locally finitely presented category. An additive category $\mathcal{C}$ is called *locally finitely presented* (or *finitely accessible*) if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects [4]. A short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$ is called *pure* if the induced sequence $0 \to \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y) \to \mathrm{Hom}(P, Z) \to 0$ is exact for every finitely presented object P of $\mathcal{C}$. An object Z of $\mathcal{C}$ is called *flat* if every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$ is pure. Now these apply to the category of comodules over a coalgebra over a field, which is known to be locally finite Grothendieck. If C is a right semiperfect coalgebra over a field k, then the category of right comodules over C has enough projectives. For terminology and further information on coalgebras and comodules the reader is referred to [7].

2. THE RESULT

Let C be a coalgebra over a field k and M a right C-comodule with structure map $\rho_M : M \to M \otimes C$. Recall from [5] that an *equation* in M is an expression of the form

$$\sum_i \rho_M(v_i)\alpha_i + \sum_i v_i \otimes c_i = 0,$$

where each $\alpha_i \in k$, $c_i \in C$. A comodule monomorphism $K \to M$ is called *pure* if every system of equations in M which has a solution in M already has a solution in K. This definition of purity is similar to the well known characterization of purity for modules in terms of solvability of certain systems of equations (e.g., see [12, 34.5]), and agrees with the categorical definition of purity (see [5, Proposition 5] and the final remarks from the same paper).

The following lemma is the key ingredient of our proof for the existence of flat covers of comodules over a semiperfect coalgebra.

Lemma 2.1. *Let C be a coalgebra over a field k, $\aleph$ an infinite cardinal such that $|k| \leq \aleph$ and $|C| \leq \aleph$, and let M be a right C-comodule. Then for any subcomodule A of M with $|A| \leq \aleph$, there exists a pure subcomodule S of M containing A with $|S| \leq \aleph$.*

Proof. Put $S_0 = A$ and consider the set of all finite systems of equations

$$\sum_i \rho_M(a_i)\alpha_{i\lambda} + \sum_i a_i \otimes c_{i\lambda} = 0 \tag{1}$$

solvable in M with each $a_i \in S_0$, $\alpha_{i\lambda} \in k$, $c_{i\lambda} \in C$. Consider the subcomodule S_1 of M generated by S_0 and a single solution of each of the systems above. The number of systems considered is less than or equal to $\aleph_0 \cdot |k| \cdot |C| \cdot |A| \leq \aleph$, so the number of solutions that we take to generate S_1 is less than or equal to $\aleph_0 \cdot \aleph = \aleph$. Hence

$$|S_1| \leq |k| \cdot |S_0| \cdot \aleph \leq \aleph.$$

Now take the set of all finite systems of equations (1) solvable in M with each $a_i \in S_1$. Repeat the process above in order to construct a subcomodule S_2 of M with $|S_2| \leq \aleph$. By induction we construct for every natural number $n \in \mathbb{N}$ a subcomodule S_n of M with $|S_n| \leq \aleph$.

Let $S = \bigcup_{n \in \mathbb{N}} S_n$ and let (1) be a finite system of equations solvable in M with each $a_i \in S$. Then there is a natural number t such that $a_i \in S_t$. Hence our system has a solution in S_{t+1}, and so in S by the construction of the S_n's. Therefore, S is a pure subcomodule of M containing A with $|S| \leq \aleph$. $\square$

Let us recall some needed terminology, following [10]. Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. Recall that a *continuous chain* of subobjects of a given object X is a set of subobjects $\{X_\alpha \mid \alpha < \lambda\}$ of X for some ordinal λ such that X_α is a subobject of X_β for all $\alpha \leq \beta < \lambda$, and that $X_\gamma = \sum_{\alpha<\gamma} X_\alpha$ whenever $\gamma < \lambda$ is a limit ordinal. For a class $\mathcal{F}$ of objects of $\mathcal{A}$, denote

$$\mathcal{F}^\perp = \{X \in \mathcal{A} \mid \mathrm{Ext}^1(F, X) = 0 \text{ for all } F \in \mathcal{F}\},$$

$$^\perp\mathcal{F} = \{X \in \mathcal{A} \mid \mathrm{Ext}^1(X, F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

A *cotorsion theory* in $\mathcal{A}$ is a pair of classes $(\mathcal{F}, \mathcal{C})$ of objects of $\mathcal{A}$ such that $\mathcal{F}^\perp = \mathcal{C}$ and $^\perp\mathcal{C} = \mathcal{F}$. If $\mathcal{F}$ is a class of objects of $\mathcal{A}$, then the pair of classes $(\mathcal{F}, \mathcal{F}^\perp)$ is said to be *cogenerated by a set* $\mathcal{S}$ of objects of $\mathcal{A}$ provided $X \in \mathcal{F}^\perp$ if and only if $\mathrm{Ext}^1(F, X) = 0$ for every $F \in \mathcal{S}$.

Also, recall the following results.

THEOREM 2.2. [10, Proposition 3.1.1] *Assume that $\mathcal{A}$ has direct limits and let A, B be objects of $\mathcal{A}$. If A is the direct union of a continuous chain of subobjects $\{A_\alpha \mid \alpha < \lambda\}$ for some ordinal λ such that* $\mathrm{Ext}^1(A_0, B) = 0$ *and* $\mathrm{Ext}^1(A_{\alpha+1}/A_\alpha, B) = 0$ *for all $\alpha < \lambda$, then* $\mathrm{Ext}^1(A, B) = 0$.

THEOREM 2.3. [10, Corollary 3.1.11] *Assume that $\mathcal{A}$ is Grothendieck and has enough projectives and let $\mathcal{F}$ be a class of objects of $\mathcal{A}$ closed under direct sums, extensions and continuous well ordered unions. If the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set, then every object of $\mathcal{A}$ has a special $\mathcal{F}$-precover. If moreover $\mathcal{F}$ is closed under well ordered direct limits, then every object of $\mathcal{A}$ has an $\mathcal{F}$-cover.*

Now we are ready to establish our result.

THEOREM 2.4. *Let C be a right semiperfect coalgebra over a field k and denote by $\mathcal{F}$ the class of flat right C-comodules. Then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.*

Proof. Let $\aleph$ be an infinite cardinal such that $|k| \leq \aleph$ and $|C| \leq \aleph$. Let F be a flat right C-comodule. Using Lemma 2.1, one can construct a continuous chain $\{S_\alpha \mid \alpha < \gamma\}$ of pure subcomodules of F such that $|S_0| \leq \aleph$ and $|S_{\alpha+1}/S_\alpha| \leq \aleph$ whenever $\alpha < \gamma$. Note that all S_α and $S_{\alpha+1}/S_\alpha$ are also flat (e.g., by [6, Proposition 2.2 and Remark 2.8] and [11, Proposition 5.9]).

Now let $\mathcal{S}$ be a representative set of flat right C-comodules G with $|G| \leq \aleph$. Using Theorem 2.2, we have

$$X \in \mathcal{F}^\perp \text{ if and only if } \mathrm{Ext}^1(G, X) = 0, \quad \forall G \in \mathcal{S}.$$

Hence $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set. □

Noting that flatness in categories of comodules is preserved under direct limits (e.g., see [6, Proposition 2.3]) and, if C is a right semiperfect coalgebra, then the category of right C-comodules has enough projectives, the following corollary is obtained by using Theorems 2.3 and 2.4.

COROLLARY 2.5. *Let C be a right semiperfect coalgebra over a field k. Then every right C-comodule has a flat cover.*

REFERENCES

[1] ALDRICH, S.T., ENOCHS, E.E., GARCÍA ROZAS, J.R. and OYONARTE, L., *Covers and envelopes in Grothendieck categories. Flat covers of complexes with applications*, J. Algebra, **243** (2001), 615–630.
[2] AUSLANDER, M. and SMALØ, S.O., *Preprojective modules over artin algebras*, J. Algebra, **66** (1980), 61–122.
[3] BICAN, L., EL BASHIR, R. and ENOCHS, E., *All modules have flat covers*, Bull. London Math. Soc., **33** (2001), 385–390.
[4] CRAWLEY-BOEVEY, W., *Locally finitely presented additive categories*, Comm. Algebra, **22** (1994), 1641–1674.
[5] CRIVEI, S., PREST, M. and REYNDERS, G., *Model theory of comodules*, J. Symbolic Logic, **69** (2004), 137–142.
[6] CUADRA, J. and SIMSON, D., *Flat comodules and perfect coalgebras*, Comm. Algebra, **35** (2007), 3164–3194.
[7] DĂSCĂLESCU, S., NĂSTĂSESCU, C. and RAIANU, Ş., *Hopf Algebras. An Introduction*, Marcel Dekker, New York, 2001.
[8] EL BASHIR, R., *Covers and directed colimits*, Algebr. Repres. Theory, **7** (2004), 423–430.
[9] ENOCHS, E.E., *Injective and flat covers, envelopes and resolvents*, Isr. J. Math., **39** (1981), 189–209.
[10] ENOCHS, E.E. and OYONARTE, L., *Covers, envelopes and cotorsion theories*, Nova Science, New York, 2002.
[11] PREST, M., *Definable additive categories: purity and model theory*, July 2006, MIMS EPrint: 2006.218.
[12] WISBAUER, R., *Foundations of module and ring theory*, Gordon and Breach, Reading, 1991.

Received March 28, 2008

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu Nr. 1
400084 Cluj-Napoca, Romania
E-mail: crivei@math.ubbcluj.ro