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BOUNDS ON THE COEFFICIENTS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract. For the real number $\alpha > 1$, we use a technique due to Nehari and Netanyahu and an application of certain integral iteration of Caratheodory functions to find the best-possible upper bounds on the coefficients of functions of the class $T_n^\alpha(\beta)$ introduced in [4] by Opoola.

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1. INTRODUCTION

Let A denote the class of functions:

$$f(z) = z + a_2 z^2 + \cdots . \tag{1}$$

which are analytic in the unit disk $E = \{z \in \mathcal{C} \colon |z| < 1\}$. In [4], Opoola introduced the subclass $T_n^\alpha(\beta)$ consisting of functions $f \in A$ which satisfy:

$$\mathrm{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta . \tag{2}$$

where $\alpha > 0$ is real, $0 \le \beta < 1$, $D^n (n \in N_0 = \{0, 1, 2, \dots\})$ is the Salagean derivative operator defined as: $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$ and powers in (2) meaning principal determinations only. The geometric condition (2) slightly modifies the one given originally in [4] (see [1]).

The object of the present work is the extension of some earlier results regarding the bounds on the coefficients, a_k, of functions belonging to the class $T_n^\alpha(\beta)$. Babalola and Opoola have begun to solve this problem in [2]. They determined sharp bounds on a_k for some $\alpha > 0$ and gave some rough estimate of the general coefficient bounds using the logarithmic coefficient approach. The sharp bounds were stated as follows:

THEOREM 1. *Let $f \in T_n^\alpha(\beta)$. Define*

$$B_m = \frac{2^m (1-\beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1 - j\alpha)}{m!},$$

$$A_k(n, \alpha, \beta) = \sum_{m=1}^{k-1} B_m Q_{k-1}^{(m)},$$

$$A_{k-1}(n,\alpha,\beta)=\sum_{m=1}^{k-2}B_mQ_{k-1}^{(m)},$$

where for each $m=1,2,\dots,Q_{k-1}^{(m)}$ *is defined by the power series*

$$\left(\sum_{k=1}^{\infty}\frac{z^k}{(\alpha+k)^n}\right)^m=Q_m^{(m)}z^m+Q_{m+1}^{(m)}z^{m+1}+Q_{m+2}^{(m)}z^{m+2}+\cdots.$$

Also let

$$\Omega_1=\{\alpha|0<\alpha<(k-2)^{-1},k=2,3,\dots\},$$
$$\Omega_2=\{\alpha|(k-2)^{-1}\le\alpha\le(k-3)^{-1},k=4,6,\dots\},$$
$$\Omega_3=\{\alpha|(k-2)^{-1}\le\alpha<(k-3)^{-1},k=3,5,\dots\}.$$

Then

$$|a_k|\le\begin{cases}A_k & \text{if } \alpha\in\Omega_1\cup\Omega_2,\\ A_{k-1} & \text{if } \alpha\in\Omega_3.\end{cases}$$

The inequalities are sharp. Equalities are attained for $f(z)$ *satisfying*

$$\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}=\begin{cases}\frac{1+(1-2\beta)z}{1-z} & \text{if } \alpha\in\Omega_1\cup\Omega_2,\\ \frac{1+(1-2\beta)z^2}{1-z^2} & \text{if } \alpha\in\Omega_3.\end{cases}$$

The rough estimate was also given as:

THEOREM 2. *Let* $f\in T_n^\alpha(\beta), n\in N$. *Suppose*

$$\frac{f(z)^\alpha}{z^\alpha}=\sum_{k=0}^{\infty}A_{k+1}(\alpha)z^k, A_1(\alpha)=1.$$

Then

$$|A_{k+1}(\alpha)|<\exp\left\{0.624\alpha^2+\left(2\alpha^2-\frac{1}{2}\right)\sum_{j=1}^{k}\frac{1}{j}\right\}.$$

From Theorem 1, which is best-possible, it is obvious that the problem has only been solved completely for a_2 and a_3 for all values of the index α. For $k\ge4$, the problem has remained open for all $\alpha>(k-3)^{-1}$. In this article we proceed with the proof of the sharp bounds on the coefficients of functions in the class $T_n^\alpha(\beta)$, $\alpha>1$. Our result is the following:

THEOREM 3. *Let* $f\in T_n^\alpha(\beta)$. *If* $\alpha>1$, *we have the sharp inequalities*

$$|a_k|\le\frac{2(1-\beta)\alpha^{n-1}}{(\alpha+k-1)^n},\quad k=2,\ 3,\ 4,\dots. \tag{3}$$

Equalities are attained for $f(z)$ *satisfying*

$$\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}=\frac{1+(1-2\beta)z^{k-1}}{1-z^{k-1}}.$$

For $k = 2,\ 3$, the above result is contained in Theorem 1 above. The proof here is however new. Also Singh [6] proved the same result for $k = 2,\ 3,\ 4$ for the particular case $n = 1$ and $\beta = 0$. In our proof we combine a method of classical analysis due to Nehari and Netanyau [3] (also used by Singh [6]) with an application of certain integral iteration of the Caratheodory functions [1]. That is presented in Section 3. In the next section we state and prove some preliminary lemmas.

2. PRELIMINARY LEMMAS

Let P be the class of functions

$$p(z) = 1 + b_1 z + b_2 z^2 + \cdots \tag{4}$$

which are analytic in E and have positive real part. In [1] the following integral iteration of each $p \in P$ was identified:

Definition 1. [1] Let $p \in P$ and $\alpha > 0$ be real. The *nth* iterated integral transform of $p(z), z \in E$ is defined as

$$p_n(z) = \frac{\alpha}{z^\alpha} \int_0^z t^{\alpha-1} p_{n-1}(t)\mathrm{d}t, \quad n \geq 1,$$

with $p_0(z) = p(z)$. The family of the *nth* integral iteration of $p \in P$ was denoted by P_n. Functions $p_n(z)$ in P_n have series expansion:

$$p_n(z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha+k)^n} b_k z^k. \tag{5}$$

Furthermore if $Re\ p_n(z) > \beta,\ (0 \leq \beta < 1)$, we denote by $P_n(\beta)$ the family of such functions given by:

$$p_n(z) = 1 + (1-\beta) \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha+k)^n} b_k z^k.$$

In the proof of our main result we require the following lemmas:

Lemma 1. [3] *If* $p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ *and* $q(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *belong to* P*, then* $r(z) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} b_k c_k z^k$ *also belongs to* P*.*

Lemma 2. [3] *Let* $h(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ *and* $1 + G(z) = 1 + \sum_{k=1}^{\infty} b'_k z^k$ *be functions in* P*. Set*

$$\gamma_m = \frac{1}{2^m}\left[1 + \frac{1}{2}\sum_{\mu=1}^{m} \binom{m}{\mu} d_\mu\right], \quad \gamma_0 = 1. \tag{6}$$

If A_k *is defined by*

$$\sum_{m=1}^{\infty} (-1)^{m+1} \gamma_{m-1} G_1^m(z) = \sum_{k=1}^{\infty} A_k z^k.$$

Then

$$|A_k| \le 2, \quad k = 1,\ 2,\ \ldots.$$

If, in the proof of the above lemma (as contained in [3]), we define $h_n(z)$ as the *n-th* iterated integral transform of $h_0(z) = h(z)$ we immediately obtain the following corollary.

Corollary 1. *Let $h_n(z)$ be the nth integral iteration of $h_0(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ with Re $h_n(z) > \beta$, and $1 + G(z) = 1 + \sum_{k=1}^{\infty} b'_k z^k$ be functions in P. Define γ_m as in (6) and*

$$\eta_m = \frac{(1-\beta)\alpha^n}{(\alpha+m)^n}\gamma_m, \quad \eta_0 = 1-\beta. \tag{7}$$

If A_k is defined by

$$\sum_{m=1}^{\infty} (-1)^{m+1} \eta_{m-1} G_1^m(z) = \sum_{k=1}^{\infty} A_k z^k. \tag{8}$$

Then

$$|A_k| \le \frac{2(1-\beta)\alpha^n}{(\alpha+k)^n}, \quad k = 1,\ 2,\ \ldots. \tag{9}$$

Lemma 3. [2] *Let $G(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series. Then the mth integer product of G(z) is $G^m(z) = \sum_{k=0}^{\infty} c_k^{(m)} z^k$ where $c_k^{(1)} = c_k$ and $c_k^{(m)} = \sum_{j=0}^{k} c_j c_{k-j}^{(m-1)}$, $m \ge 2$.*

We now turn to the proof of the main result.

3. PROOF OF THE MAIN RESULT

Let $f \in T_n^{\alpha}(\beta)$. Then there exists an analytic functions $p_n \in P_n$ such that

$$f(z)^{\alpha} = z^{\alpha}[\beta + (1-\beta)p_n(z)], \tag{10}$$

where $p_n(z)$ given by (5) is the *nth* iterated integral transform of an analytic function $p \in P$ defined by (4) (see Lemma 4.2 of [1]). The first part of the proof involves obtaining expression for the coefficients, a_k, of $f(z)$ in terms of the coefficients, b_k, of the function $p \in P$. This is contained in [2] and adapted here for completeness. We have, using (5) in (10),

$$f(z) = z\left(1 + (1-\beta)\sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha+k)^n} b_k z^k\right)^{\frac{1}{\alpha}}. \tag{11}$$

Equation (11) expands binomially as

(12)
$$\frac{f(z)}{z} = 1+\frac{(1-\beta)\alpha^n}{\alpha}\sum_{k=1}^{\infty}\frac{b_k z^k}{(\alpha+k)^n}+\frac{(1-\beta)^2\alpha^{2n}(1-\alpha)}{2!\alpha^2}\left(\sum_{k=1}^{\infty}\frac{b_k z^k}{(\alpha+k)^n}\right)^2+ \cdots+\frac{(1-\beta)^m\alpha^{mn}\prod_{j=0}^{m-1}(1-j\alpha)}{m!\alpha^m}\left(\sum_{k=1}^{\infty}\frac{b_k z^k}{(\alpha+k)^n}\right)^m+\cdots .$$

Using Lemma 3 in (12) we get

$$\frac{f(z)}{z} = 1+\sum_{k=1}^{\infty}\widetilde{B}_1 C_k^{(1)} z^k+\cdots+\sum_{k=1}^{\infty}\widetilde{B}_m C_k^{(m)} z^k+\cdots ,$$

where

$$\widetilde{B}_m = \frac{(1-\beta)^m\alpha^{m(n-1)}\prod_{j=0}^{m-1}(1-j\alpha)}{m!}$$

and $C_k^{(m)}$, $m = 1,2,\ldots;\ k = m, m+1,\ldots$ is defined by

$$\left(\sum_{k=1}^{\infty}\frac{b_k z^k}{(\alpha+k)^n}\right)^m = \sum_{k=1}^{\infty} C_k^{(m)} z^k,$$

having the general form

$$C_k^{(m)} = \sum_{j=1}^{k} C_j \prod_{l=1}^{m}\frac{b_l^{\rho_l}}{(\alpha+l)^{n\rho_l}} \tag{13}$$

for some nonnegative constants C_j, $j = 1,2,\ldots k$ and indices ρ_l, $l = 1,2,\ldots,m$ taking values in the set $M = \{0,1,2,\ldots,m\}$ such that $\rho_l+\rho_2+\cdots+\rho_m = m$ (see page 10 of [2]). From (12) we write

$$f(z) = z+\sum_{k=2}^{\infty}\widetilde{A}_k^{(m)} z^k, \tag{14}$$

where

$$\widetilde{A}_k^{(m)} = \sum_{m=1}^{k-1}\widetilde{B}_m C_{k-1}^{(m)}, \quad k = 2,\ 3,\ \ldots .$$

Comparing coefficients in (1) and (14), we see that $a_k = \widetilde{A}_k^{(m)}$ which gives

$$a_k = \sum_{m=1}^{k-1}\frac{(1-\beta)^m\alpha^{m(n-1)}\prod_{j=0}^{m-1}(1-j\alpha)}{m!}\left(\sum_{j=1}^{k} C_j\prod_{l=1}^{m}\frac{b_l^{\rho_l}}{(\alpha+l)^{n\rho_l}}\right). \tag{15}$$

Now we compute the leading coefficients, A_v, in the expression (8). From (8) we have

$$\sum_{m=1}^{\infty}(-1)^{m+1}\eta_{m-1}G_1^m(z) = G_1(z) - \eta_1 G_1^2(z) + \cdots = \sum_{v=1}^{\infty} A_v z^v \tag{16}$$

with $G_1(z) = \sum_{v=1}^{\infty} b'_v z^v$. Using Lemma 3 again we have

$$G_1^m(z) = \sum_{v=m}^{\infty} C_v^{(m)} z^v, \quad m = 1,\ 2,\ \ldots, \tag{17}$$

where

$$C_v^{(m)} = \sum_{j=1}^{v} C_j \prod_{l=1}^{m} b'^{\rho_l}_l \tag{18}$$

with C_j and indices ρ_l as already defined for (13). Using (17) and (18) in (16) we get

$$\begin{aligned}\sum_{m=1}^{\infty}(-1)^{m+1}\eta_{m-1}G_1^m(z) &= \sum_{v=1}^{\infty}\left(\sum_{m=1}^{v}(-1)^{m+1}\eta_{m-1}C_v^{(m)}\right) z^v \\ &= \sum_{v=1}^{\infty}\left(\sum_{m=1}^{v}(-1)^{m+1}\eta_{m-1}\left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} b'^{\rho_l}_l\right)\right) z^v \\ &= \sum_{v=1}^{\infty} A_v z^v,\end{aligned}$$

so that

$$A_v = \sum_{m=1}^{v}(-1)^{m+1}\eta_{m-1}\left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} b'^{\rho_l}_l\right) z^v.$$

By corollary 1, these coefficients, A_v, satisfy the inequality (9) if $1 + G(z) = 1 + b'_1 z + b'_2 z^2 + \cdots$ is a function of the class P, and by Lemma 1 we may set $b'_l = \frac{1}{2} b_l c_l$ where $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is the function (4) and $H(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an arbitrary function in P. Then

$$|A_v| = \left|\sum_{m=1}^{v}(-1)^{m+1}\frac{\eta_{m-1}}{2^m}\left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l}\right)\right| \leq \frac{2(1-\beta)\alpha^n}{(\alpha+v)^n}. \tag{19}$$

Using (7) together with the fact that $\alpha > 0$, we can write (19) equivalently as (20)

$$|A_v| = \left|\sum_{m=1}^{v}(-1)^{m+1}\frac{(1-\beta)\alpha^n}{(\alpha+m-1)^n}\frac{\gamma_{m-1}}{2^m\alpha}\left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l}\right)\right| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+v)^n}.$$

Since for each $m = 1, 2, \dots$ and any $\alpha > 0$,

$$\prod_{l=1}^{m} \frac{\alpha^{n\rho_l}}{(\alpha+l)^{n\rho_l}} \le \prod_{l=1}^{m} \frac{\alpha^{n\rho_l}}{(\alpha+1)^{n\rho_l}} = \frac{\alpha^{mn}}{(\alpha+1)^{mn}} \le \frac{\alpha^n}{(\alpha+m-1)^n},$$

it is evident that for each $v = 1, 2, \dots$

$$\sum_{m=1}^{v} (-1)^{m+1} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} \frac{(1-\beta)^{\rho_l} \alpha^{n\rho_l}}{(\alpha+l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \le$$
$$\sum_{m=1}^{v} (-1)^{m+1} \frac{(1-\beta)\alpha^n}{(\alpha+m-1)^n} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l} \right).$$

Therefore by (20), we have

$$\left| \sum_{m=1}^{v} (-1)^{m+1} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} \frac{(1-\beta)^{\rho_l} \alpha^{n\rho_l}}{(\alpha+l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \le \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+v)^n}$$

that is,

(21)
$$\left| \sum_{m=1}^{v} (-1)^{m+1} \frac{(1-\beta)^m \alpha^{mn-1} \gamma_{m-1}}{2^m} \left(\sum_{j=1}^{v} C_j \prod_{l=1}^{m} \frac{b_l^{\rho_l} c_l^{\rho_l}}{(\alpha+l)^{n\rho_l}} \right) \right| \le \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+v)^n}.$$

Now comparing (15) and the term in the absolute value in (21) (with $v = k-1$), we would conclude that the inequalities (3) hold if we are able to find two members $h(z) = 1 + d_1 z + d_2 z^2 + \cdots$ and $H(z) = 1 + c_1 z + c_2 z^2 + \cdots$ of P which give rise to the constants γ_m (as required by (6)) and c_l. For $H \in P$, a natural choice of the Moebius function is suitable. That is, $H(z) = (1+z)/(1-z) = 1 + 2z + 2z^2 + \cdots$. Thus we have $c_l = 2, \quad l = 1,\ 2,\ \dots$. Using this in (21) we get

(22)
$$\left| \sum_{m=1}^{k-1} (-1)^{m+1} (1-\beta)^m \alpha^{mn-1} \gamma_{m-1} \left(\sum_{j=1}^{k-1} C_j \prod_{l=1}^{m} \frac{b_l^{\rho_l}}{(\alpha+l)^{n\rho_l}} \right) \right| \le \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+k-1)^n}.$$

Comparing (15) and the term in the absolute value in (22) we find that

$$(-1)^{m+1} \frac{\gamma_{m-1}}{\alpha} = \frac{\prod_{j=0}^{m-1} (1 - j\alpha)}{m! \alpha^m}$$

that is

$$\gamma_{m-1} = \frac{\prod_{j=1}^{m-1} (j\alpha - 1)}{m! \alpha^{m-1}}, \quad \gamma_0 = 1. \tag{23}$$

Now with d_μ, $\mu = 1, 2, \ldots, m-1$ defined by

$$\frac{1}{2^{m-1}}\left[1 + \frac{1}{2}\sum_{\mu=1}^{m-1}\binom{m-1}{\mu}d_\mu\right] = \frac{\prod_{j=1}^{m-1}(j\alpha - 1)}{m!\alpha^{m-1}}, \tag{24}$$

we need to find $h(z)_k$, corresponding to each a_k, $k = 2, 3, 4, \ldots$, such that the coefficients d_μ, $\mu = 1, 2, \ldots, m-1$ of each $h(z)_k$ satisfy (24) and the proof would be complete. Observe from (22) that $m = 1, 2, \ldots, k-1$ as we begin to implement our scheme for each $k = 2, 3, 4, \ldots$.

$k = 2$: In this case $m = 1$. Hence by (23) $\gamma_0 = 1$ and we can therefore define $d_\mu = 0$ for all μ so that $h(z)_2 = 1$.

$k = 3$: Here we have $m = 1, 2$. Using (24) we get $d_1 = -\frac{2}{\alpha}$, so that

$$h(z)_3 = \frac{\alpha - 1}{\alpha} + \frac{1}{\alpha}\left(\frac{1-z}{1+z}\right) = 1 - \frac{2}{\alpha}z + \cdots.$$

$k = 4$: Now $m = 1, 2, 3$. From (24) we have

$$\frac{1}{4}\left(1 + \frac{1}{2}(2d_1 + d_2)\right) = \frac{(\alpha-1)(2\alpha-1)}{3!\alpha^2}.$$

Taking $d_1 = 0$, we get $\frac{d_2}{2} = \frac{\alpha^2 - 6\alpha + 2}{3\alpha^2}$ (note that $|d_2| \le 2$), and we have

$$h(z)_4 = \begin{cases} \frac{2(\alpha^2+3\alpha-1)}{3\alpha^2} + \frac{\alpha^2-6\alpha+2}{3\alpha^2}\left(\frac{1+z^2}{1-z^2}\right) & \text{if } \alpha \ge \alpha_0, \\ \frac{2(2\alpha^2-3\alpha+1)}{3\alpha^2} + \frac{6\alpha-\alpha^2-2}{3\alpha^2}\left(\frac{1-z^2}{1+z^2}\right) & \text{if } 1 < \alpha \le \alpha_0 \end{cases}$$

$$= 1 + \frac{2(\alpha^2 - 6\alpha + 2)}{3\alpha^2}z^2 + \cdots,$$

where $\alpha_0 > 1$ is the solution of $\alpha^2 - 6\alpha + 2 = 0$.

$k = 5$: In this case $m = 1, 2, 3, 4.$, and from (24) we get

$$\frac{1}{8}\left(1 + \frac{1}{2}(3d_1 + 3d_2 + d_3)\right) = \frac{(\alpha-1)(2\alpha-1)(3\alpha-1)}{4!\alpha^3}.$$

Taking $d_1 = d_2 = 0$, we get $\frac{d_3}{2} = \frac{3\alpha^3 - 11\alpha^2 + 6\alpha - 1}{3\alpha^3}$ (note also that $|d_3| \le 2$) and

$$h(z)_5 = \begin{cases} \frac{11\alpha^2-6\alpha+1}{3\alpha^3} + \frac{3\alpha^3-11\alpha^2+6\alpha-1}{3\alpha^3}\left(\frac{1+z^3}{1-z^3}\right) & \text{if } \alpha \ge \alpha_0, \\ \frac{6\alpha^3-11\alpha^2+6\alpha-1}{3\alpha^3} + \frac{1-6\alpha+11\alpha^2-3\alpha^3}{3\alpha^3}\left(\frac{1-z^3}{1+z^3}\right) & \text{if } 1 < \alpha \le \alpha_0 \end{cases}$$

$$= 1 + \frac{2(3\alpha^3 - 11\alpha^2 + 6\alpha - 1)}{3\alpha^3}z^3 + \cdots.$$

In this case $\alpha_0 > 1$ is the solution of $3\alpha^3 - 11\alpha^2 + 6\alpha - 1 = 0$.

$k \ge 6$: In general, $m = 1, 2, 3, \ldots, k-1$. In (24) we set $d_1 = \frac{-2}{m-1}$, $d_2 = d_4 = \cdots = d_\xi = \sigma$ where ξ equals $m-1$ if $m-1$ is *even* and $m-2$ otherwise, and $d_3 = d_5 = \cdots = d_\omega = 0$ where ω equals $m-1$ if $m-1$ is *odd* and $m-2$ otherwise. With this we get $\frac{\sigma}{2} = \frac{2^{m-1}\prod_{j=1}^{m-1}\left(\frac{j\alpha-1}{j\alpha}\right)}{m\left(\binom{m-1}{2}+\binom{m-1}{4}+\cdots+\binom{m-1}{\xi}\right)}$. In this sense $|d_\mu| \le 2$ for all $\mu = 1, 2, \ldots, m-1$. Setting $m = k-1$, we now define $h(z)_k$, $k \ge 6$ as follows

$$h(z)_k = 1 - \frac{2}{k-2} - \frac{2^{k-2}\prod_{j=1}^{k-2}\left(\frac{j\alpha-1}{j\alpha}\right)}{(k-1)\left(\binom{k-2}{2}+\binom{k-2}{4}+\cdots+\binom{k-2}{\xi}\right)} + \frac{2}{k-2}(1-z)$$
$$+ \frac{2^{k-2}\prod_{j=1}^{k-2}\left(\frac{j\alpha-1}{j\alpha}\right)}{(k-1)\left(\binom{k-2}{2}+\binom{k-2}{4}+\cdots+\binom{k-2}{\xi}\right)}\left(\frac{1+z^2}{1-z^2}\right).$$

In other words,

$$h(z)_k = 1 - \frac{2}{k-2}z + \frac{2^{k-2}\prod_{j=1}^{k-2}\left(\frac{j\alpha-1}{j\alpha}\right)}{(k-1)\left(\binom{k-2}{2}+\binom{k-2}{4}+\cdots+\binom{k-2}{\xi}\right)}z^2$$
$$+ \frac{2^{k-2}\prod_{j=1}^{k-2}\left(\frac{j\alpha-1}{j\alpha}\right)}{(k-1)\left(\binom{k-2}{2}+\binom{k-2}{4}+\cdots+\binom{k-2}{\xi}\right)}z^4 + \cdots.$$

That these functions $h(z)_k$ belong to P follows from the fact that the function $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m$ belongs to P if $f_1, f_2, \ldots, f_m$ belong, $\lambda_1, \lambda_2, \ldots, \lambda_m \ge 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$. This completes the proof of the theorem.

Remark 1. We compute $h(z)_6, \ldots, h(z)_{10}$ for the purpose of illustration.

$$h(z)_6 = 1 - \frac{1}{2}z + \frac{4(\alpha-1)(2\alpha-1)\ldots(4\alpha-1)}{105\alpha^4}z^2$$
$$\frac{4(\alpha-1)(2\alpha-1)\ldots(4\alpha-1)}{105\alpha^4}z^4 + \cdots,$$

$$h(z)_7 = 1 - \frac{2}{5}z + \frac{4(\alpha-1)(2\alpha-1)\ldots(5\alpha-1)}{675\alpha^5}z^2$$
$$\frac{4(\alpha-1)(2\alpha-1)\ldots(5\alpha-1)}{675\alpha^5}z^4 + \cdots,$$

$$h(z)_8 = 1 - \frac{1}{3}z + \frac{8(\alpha-1)(2\alpha-1)\ldots(6\alpha-1)}{10765\alpha^6}z^2$$
$$\frac{8(\alpha-1)(2\alpha-1)\ldots(6\alpha-1)}{10765\alpha^6}z^4 + \cdots,$$

$$h(z)_9 = 1 - \frac{2}{7}z + \frac{2(\alpha-1)(2\alpha-1)\ldots(7\alpha-1)}{19845\alpha^7}z^2$$
$$\frac{2(\alpha-1)(2\alpha-1)\ldots(7\alpha-1)}{19845\alpha^7}z^4 + \cdots,$$

$$h(z)_{10} = 1 - \frac{1}{4}z + \frac{4(\alpha-1)(2\alpha-1)\ldots(8\alpha-1)}{360045\alpha^8}z^2$$
$$\frac{4(\alpha-1)(2\alpha-1)\ldots(8\alpha-1)}{360045\alpha^8}z^4 + \cdots .$$

With this work the coefficient problem of functions in the class $T_n^\alpha(\beta)$ is settled for any $\alpha > 1$. Of course the case $\alpha = 1$ is trivial as this simply gives $|a_k| \leq \frac{2(1-\beta)}{(k+1)^n}$, $k \geq 2$, as can be seen easily from (12). Thus the problem only remains open for $(k-3)^{-1} \leq \alpha < 1$, $k \geq 5$. Finally, we note a humble attempt at this problem made by the authors in [5]. Their results depended wholly on the triangle inequality, and were not sharp.

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