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ON THE HALL PRODUCT OF PREINJECTIVE KRONECKER MODULES

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Abstract. Using some results on the Hall algebra of the Kronecker algebra kK over the finite field k, we describe the Hall product of a finite number of isoclasses of preinjective indecomposable Kronecker modules. More precisely, we will characterize those isoclasses which appear with nonzero coefficient in the previously mentioned Hall product. A dual statement can be formulated also for preprojective modules.

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1. PRELIMINARIES

Let $K:\ 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ be the Kronecker quiver and k a finite field with $|k| = q$. We will consider the path algebra kK of K over k (called Kronecker algebra) and the category mod-kK of finite right modules over kK which is equivalent with the category of finite dimensional representations over k of the quiver K. For $M \in$ mod-kK we will denote by $[M]$ the isomorphism class of M.

It is well known (see for example [1] or [2]) that up to isomorphism the indecomposables in mod-kK are of three types: preinjectives, preprojectives and regulars. The preinjective indecomposables (seen as representations) up to isomorphism have the form $I_n:\ k^n \underset{(I0)}{\overset{(0I)}{\leftleftarrows}} k^{n+1}$.

A module with all its indecomposable direct summands preinjective will be called preinjective module and denoted by I.

The Hall algebra $\mathcal{H}(kK, \mathbb{Q})$ associated to the Kronecker algebra kK is the free $\mathbb{Q}$-space having as basis the isomorphism classes in mod-kK together with a multiplication defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M]$, where the structure constants $F^M_{N_1 N_2} = |\{M \supseteq U|\ U \cong N_2,\ M/U \cong N_1\}|$ are called *Hall numbers.*

More generally for $M, N_1, \dots, N_t \in$ mod-kK we can define

$$F^M_{N_1 \dots N_t} = |\{M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0|\ M_{i-1}/M_i \cong N_i, \forall 1 \le i \le t\}|.$$

If $F^M_{N_1 \dots N_t} \neq 0$ then we will use the notation $[M] \in \{[N_1] \dots [N_t]\}$ and call $[M]$ a term in $[N_1] \dots [N_t]$, $\{[N_1] \dots [N_t]\}$ denoting the set of all terms in $[N_1] \dots [N_t]$. So if every term in $[N_1] \dots [N_t]$ is also a term in $[M_1] \dots [M_s]$, then we will denote this by $\{[N_1] \dots [N_t]\} \subseteq \{[M_1] \dots [M_s]\}$.

Our aim is to determine the terms of $[I_{a_1}]\dots[I_{a_n}]$, where $a_i\in\mathbb{N}$. Due to the associativity of the Hall algebra we have

$$\sum_{[N]} F^M_{N_1N}F^N_{N_2N_3}=F^M_{N_1N_2N_3}=\sum_{[N]} F^N_{N_1N_2}F^M_{NN_3}.$$

Using all the definitions and properties above we can easily see

Lemma 1. a) $[M]\in\{[N][N_3]\}$ *and* $[N]\in\{[N_1][N_2]\}$ *implies* $[M]\in\{[N_1][N_2][N_3]\}$.

b) $[M]\in\{[N_1][N]\}$ *and* $[N]\in\{[N_2][N_3]\}$ *implies* $[M]\in\{[N_1][N_2][N_3]\}$.

c) $\{[N_1]\dots[N_t]\}\subseteq\{[M_1]\dots[M_s]\}$ *implies* $\{[N][N_1]\dots[N_t]\}\subseteq\{[N][M_1]\dots[M_s]\}$ *and* $\{[N_1]\dots[N_t][N]\}\subseteq\{[M_1]\dots[M_s][N]\}$

In $\mathbb{Z}^n$ we define the following ordering: for $a=(a_1,\dots a_n), b=(b_1,\dots,b_n)\in\mathbb{Z}^n$ let $a\trianglelefteq b$ iff $a_1\le b_1$, $a_1+a_2\le b_1+b_2$,..., $a_1+\dots+a_{n-1}\le b_1+\dots+b_{n-1}$ and $a_1+\dots+a_n=b_1+\dots+b_n$. The operator w will arrange the components of any element in $\mathbb{Z}^n$ in decreasing order and for $1\le i<j\le n$ and $k\in\mathbb{N}^*$ we define the operator

$$R^k_{i,j}(a_1,\dots,a_n)=(a_1,\dots,a_i-k,\dots,a_j+k,\dots,a_n).$$

Lemma 2. *Let* $a_1\ge\dots\ge a_n\ge 0$ *and* $k\in\mathbb{N}^*$ *such that* $a_i-k\ge a_j+k$. *Then* $wR^k_{i,j}(a_1,\dots,a_n)\trianglelefteq(a_1,\dots,a_n)$.

Proof. We will use induction on k. For $k=1$ suppose that $a_i=a$ and $a_j=d$, so $(a_1,\dots,a_i,\dots,a_j,\dots,a_n)$ will have the form $(\dots,a,\dots,a,a,b,\dots,b,\dots,c,\dots,c,d,d,\dots,d,\dots)$, where $a-1\ge b\ge c\ge d+1$. This means that under our assumptions $wR^1_{i,j}(a_1,\dots,a_n)$ will have the form $(\dots,a,\dots,a,a-1,b,\dots,b,\dots,c,\dots,c,d+1,d,\dots,d,\dots)$ so in this case we are clearly done. Suppose now, that $k>1$ and the lemma is true for $k-1$. Observe that

$$wR^k_{i,j}(a_1,\dots,a_n)=wR^1_{i_0,j_0}wR^{k-1}_{i,j}(a_1,\dots,a_n),$$

where $wR^{k-1}_{i,j}(a_1,\dots,a_n)=(b_1,\dots,b_n)$, $b_{i_0}=a_i-k+1$ and $b_{j_0}=a_j+k-1$, so we obtain $wR^k_{i,j}(a_1,\dots,a_n)=wR^1_{i_0,j_0}(b_1,\dots,b_n)\trianglelefteq(b_1,\dots,b_n)=wR^{k-1}_{i,j}(a_1,\dots,a_n)\trianglelefteq(a_1,\dots,a_n)$. □

2. THE HALL PRODUCT OF PREINJECTIVES

We start from some formulas obtained by the author in [3] expressing the Hall product of two indecomposable preinjectives:

- $[I_i][I_j]=[I_i\oplus I_j]$ for $i>j$,
- $[I_i][I_j]=q^{j-i+1}[I_j\oplus I_i]+(q^{j-i+1}-q^{j-i-1})\sum_{u=1}^{[\frac{j-i}{2}]}[I_{j-u}\oplus I_{i+u}]$ for $i<j$.

We also know (see [3]):

$$[uI_i][vI_i]=\frac{(q^{u+v}-1)\dots(q^{u+1}-1)}{(q^v-1)\dots(q-1)}[(u+v)I_i].$$

Using the formulas above one easily gets the following lemma

LEMMA 3. a) *For* $i \geq j$ *we have* $[M] \in \{[I_i][I_j]\} \Leftrightarrow [M] = [I_i \oplus I_j]$.

b) *For* $i < j$ *we have* $[M] \in \{[I_i][I_j]\} \Leftrightarrow [M] = [I_j \oplus I_i]$ *or* $[I_{j-1} \oplus I_{i+1}]$ *or* $\ldots$ *or* $[I_{j-[\frac{j-i}{2}]} \oplus I_{i+[\frac{j-i}{2}]}]$.

Now we are ready to determine the terms in a product of the form $[I_{a_n}]\ldots[I_{a_1}]$ denoted here for simplicity also by $[a_1, \ldots, a_n]$.

LEMMA 4. *Suppose* $c_1 \geq \cdots \geq c_n \geq 0$ *and* $\sigma \in S_n$. *Then*

$$\{[I_{c_{\sigma(n)}}]\ldots[I_{c_{\sigma(1)}}]\} \subseteq \{[I_{c_n}]\ldots[I_{c_1}]\}.$$

Proof. Using the notation above, we will need to prove that $\{[c_{\sigma(1)}, \ldots, c_{\sigma(n)}]\} \subseteq \{[c_1, \ldots, c_n]\}$. Since by Lemma 2.1. we have

$$\{[a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n]\} \subseteq \{[a_1, \ldots, a_n]\} \text{ for } a_i \geq a_{i+1},$$

denoting $[a_1, \ldots a_{i-1}, a_j, a_i, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$ by $R_{i,j}[a_1, \ldots, a_n]$ after a finite number of steps we obtain that

$$\{R_{i,j}[a_1, \ldots, a_n]\} \subseteq \{[a_1, \ldots, a_n]\} \text{ for } i < j \text{ and } a_i \geq a_{i+1} \geq \cdots \geq a_{j-1} \geq a_j.$$

Then the inclusion

$$\{R_{\sigma(1),1} \ldots R_{\sigma(n),n}[c_1, \ldots, c_n] = [c_{\sigma(1)}, \ldots, c_{\sigma(n)}]\} \subseteq \{[c_1, \ldots, c_n]\}$$

easily follows. □

THEOREM 1. *Suppose* $c_1 \geq \cdots \geq c_n \geq 0$. *Then* $[M] \in \{[I_{c_n}]\ldots[I_{c_1}]\}$ *if and only if* $[M] = [I_{d_1} \oplus \cdots \oplus I_{d_n}]$ *with* $d_1 \geq \cdots \geq d_n \geq 0$ *and* $(d_1, \ldots, d_n) \trianglelefteq (c_1, \ldots, c_n)$.

Proof. "⇒" We will use lexicographical induction on $(c_1, \ldots, c_n) \in \mathbb{N}^n$ with $c_1 \geq \cdots \geq c_n \geq 0$ and $c_1 + \cdots + c_n = m$ fixed. One can easily see that the smallest such sequence will have the form $(a, \ldots, a, b, \ldots, b)$, where $a = b$ or $a = b + 1$, so in this case the only term in $[I_b]\ldots[I_b][I_a]\ldots[I_a]$ will be $[I_a \oplus \cdots \oplus I_a \oplus I_b \oplus \cdots \oplus I_b]$. In general observe that

$$\{[c_1, \ldots, c_n]\} = \{[I_{c_1} \oplus \cdots \oplus I_{c_n}]\}$$
$$\cup \Big(\bigcup_{i<j, k=1,[\frac{c_i-c_j}{2}]} \{[c_{i+1}, \ldots, c_j + k, c_i - k, c_{j+1}, \ldots, c_n, c_{i-1}, \ldots, c_1]\} \Big),$$

so using the previous lemma we obtain that

$$\{[c_1, \ldots, c_n]\} \subseteq \{[I_{c_1} \oplus \cdots \oplus I_{c_n}]\} \cup \Big(\bigcup_{i<j, k=1,[\frac{c_i-c_j}{2}]} \{[wR^k_{i,j}(c_1, \ldots, c_n)]\} \Big).$$

Now we use lemma 1.2. and the induction hypothesis.

"⇐" Let $c_1 \geq \cdots \geq c_n \geq 0$ and $d_1 \geq \cdots \geq d_n \geq 0$ such that $(d_1, \ldots, d_n) \triangleleft (c_1, \ldots, c_n)$ adjacently. We will show that in this case $\{[d_1, \ldots, d_n]\} \subseteq \{[c_1, \ldots, c_n]\}$. Let i be the smallest in $\{1, \ldots, n-1\}$ such that $c_i > d_i$

and k the smallest in $\{i+1,\dots,n\}$ such that $c_i+\dots+c_k = d_i+\dots+d_k$. Such a k exists since we have $c_i+\dots+c_n = d_i+\dots+d_n$ and $i+1 \le n$. Then the conditions on i,k and also $(d_1,\dots,d_n) \lhd (c_1,\dots,c_n)$ will imply that $c_i+\dots+c_{k-1} > d_i+\dots+d_{k-1}$, so $d_k > c_k \ge c_{k+1} \ge d_{k+1}$, where $c_{n+1} := d_{n+1} := 0$. Finally let j the smallest in $\{i+1,\dots,k\}$ such that $d_j > d_{j+1}$, so $d_{i+1} = \dots = d_j > d_{j+1}$. Using all the conditions above we obtain

$$\begin{aligned}(d_1,\dots,d_n) &\lhd (d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_{j-1},d_j-1,d_{j+1},\dots,d_n)\\ &\unlhd (c_1,\dots,c_n),\end{aligned}$$

so by the adjacency we get

$$(d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_{j-1},d_j-1,d_{j+1},\dots,d_n) = (c_1,\dots,c_n).$$

Since $d_{i+1} = \dots = d_j$, by Lemma 1.1. and 2.1. we get

$$\begin{aligned}\{[c_1,\dots,c_n]\} &= \{[d_1,\dots,d_{i-1},d_i+1,d_j-1,d_{i+1},\dots,d_{j-1},d_{j+1},\dots,d_n]\}\\ &\subseteq \{[d_1,\dots,d_{i-1},d_i,d_j,d_{i+1},\dots,d_{j-1},d_{j+1},\dots,d_n]\}\\ &= \{[d_1,\dots,d_n]\},\end{aligned}$$

which finishes the proof. □

COROLLARY 1. *Let $c_1 \ge \dots \ge c_n \ge 0$ and $d_1 \ge \dots \ge d_n \ge 0$ such that $(d_1,\dots,d_n) \unlhd (c_1,\dots,c_n)$. Then $\{[d_1,\dots,d_n]\} \subseteq \{[c_1,\dots,c_n]\}$.*

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