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APPLICATIONS OF THE SOLUTIONS OF TWO ABSTRACT MOMENT PROBLEMS TO THE CLASSICAL MOMENT PROBLEM

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Abstract. We apply Theorems 1 and 4 [12] to some classical moment problems in spaces of analytic or real-differentiable functions, considered as real ordered normed vector spaces. Our solutions are operator-valued and satisfy some natural sandwich-type conditions. The present work is related to the papers [6], [10], [12], [13], [14], [16], [17].

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1. GENERAL-TYPE KNOWN RESULTS ON THE ABSTRACT MOMENT PROBLEM

THEOREM 1. (Theorem 4 [12] or Theorem 2.1 [13]). *Let V be a preordered vector space, let Y be an order-complete vector lattice, $\{v_j; j \in J\} \subset V$, $\{y_j; j \in J\} \subset Y$, $F, G \in L(V, Y)$ two linear operators. Consider the following assertions:*

(a) *there exists $H \in L(V, Y)$ such that*

$$H(v_j) = y_j, \quad j \in J;$$

$$G(\varphi) \le H(\varphi) \le F(\varphi) \quad \forall \varphi \in V_+;$$

(b) *for any finite subset $J_1 \subset J$ and any $\{\alpha_j; j \in J_1\} \subset \mathbf{R}$, the implication*

$$(1) \qquad \sum_{j \in J_1} \alpha_j v_j = \varphi_2 - \varphi_1 \text{ with } \varphi_1, \varphi_2 \in V_+ \Rightarrow \sum_{j \in J_1} \alpha_j y_j \le F(\varphi_2) - G(\varphi_1)$$

holds.

If V is a vector lattice, we also consider the assertion

(b′) *$G(\varphi) \le F(\varphi)$ $\forall \varphi \in V_+$, and for any finite subset $J_1 \subset J$ and any $\{\alpha_j; j \in J_1\} \subset \mathbf{R}$, we have*

$$(1') \qquad \sum_{j \in J_1} \alpha_j v_j \le F\left(\left(\sum_{j \in J_1} \alpha_j v_j\right)^+\right) - G\left(\left(\sum_{j \in J_1} \alpha_j v_j\right)^-\right)$$

(where $v^+ := \sup\{v, 0\}$, $v^- := \sup\{-v, 0\}$, $v = v^+ - v^-$, $|v| = v^+ + v^-$, $\forall v \in V$). Then (a) $\Leftrightarrow$ (b) *holds and, if V is a vector lattice, we have* (b′) $\Leftrightarrow$ (b) $\Leftrightarrow$ (a).

This theorem was published for the first time in [12], without proof. Its proof may be found in [13]. For some of its applications see [10], [13], [14], [17], [18]. Theorem 1 may be considered as a generalization of a result of M.G. Krein [6] (see also [13]).

Now we recall another abstract moment problem, in which the solution H is nonnegative.

THEOREM 2. *Let V, Y, $\{v_j; j \in J\}$, $\{y_j; j \in J\}$ be as in Theorem 1. Let $P : V \to Y$ be a convex operator. The following assertions are equivalent:*

(a) *there exists $H \in L(V, Y)$ such that*

$$\begin{aligned} &H(v_j) = y_j, && j \in J,\\ &H(\varphi) \geq 0, && \forall \varphi \in V_+,\\ &H(v) \leq P(v), && \forall v \in V; \end{aligned}$$

(b) *for any finite subset $J_1 \subset J$ and any $\{\alpha_j; j \in J_1\} \subset \mathbf{R}$ we have*

$$\sum_{j\in J_1} \alpha_j v_j \leq v \in V \Rightarrow \sum_{j\in J_1} \alpha_j y_j \leq P(v) \text{ in } Y.$$

Theorem 2 was published for the first time in [12], without proof. Its proof can be found in [14]. For some of its applications see [10], [12], [14], [15], [16], [17].

2. APPLICATIONS TO THE CLASSICAL MOMENT PROBLEM

For the first applications stated below, V will be the space of all functions v which can be represented as an absolutely convergent power series

$$v(z) = \sum_{j=0}^{\infty} a_j z^j, \quad a_j \in \mathbf{R}$$

in the open disc $|z| < b$, v being assumed to be continuous in the closed disc $|z| \leq b$. Endowed with the order relation defined by the convex cone

$$V_+ := \left\{ v \in V;\ v(z) = \sum_{j\in\mathbf{N}} a_j z^j,\ |z| < b,\ a_j \geq 0\ \forall j \in \mathbf{N} \right\},$$

V is a real ordered vector space. On V we consider the norm

$$||v|| := \sup_{|z|\leq b} |v(z)|, \quad v \in V.$$

On the other hand, let E be an arbitrary Hilbert space, and $U_0 \in \mathcal{A}(E)$, where $\mathcal{A}(E)$ is the real vector space of all self-adjoint (linear bounded) operators acting on E. Denote

$$\begin{gathered} \mathcal{A}_1 := \{U \in \mathcal{A}(E);\ U_0 U = U U_0\},\\ Y := \{U \in \mathcal{A}_1;\ UV = VU \quad \forall V \in \mathcal{A}_1\}, \end{gathered}$$

and

$$Y_+ := \{U \in Y;\ \langle U(h), h\rangle \geq 0 \quad \forall h \in E\}.$$

It is well known that Y is an order-complete vector lattice, and a commutative algebra of operators (see [4], pp. 303–305).

Now we can state the first application of Theorem 1.

THEOREM 3. *Let V, Y, U_0 be as above. Denote $v_j(z) := z^j$, $|z| \leq b$, $j \in \mathbf{N}$. Assume that $b > 1$ and let $A \in Y$ be such that $||A|| < b$. Let $\varepsilon > 0$ and $\{B_j\}_{j\in\mathbf{N}} \subset Y$.*

The following assertions are equivalent:

(a) *there exists a linear operator $H \in L(V, Y)$ such that*

$$H(v_j) = B_j, \quad \forall j \in \mathbf{N};$$

$$\varphi(A) - \varepsilon\varphi(I) \leq H(\varphi) \leq \varphi(A) + \varepsilon\varphi(I) \quad \forall \varphi \in V_+ , \tag{2}$$

$$||H(\varphi)|| \leq 2[||\varphi|| + \varepsilon\varphi(1)] \quad \forall \varphi \in V_+ \tag{3}$$

(b) *we have*

$$A^j - \varepsilon I \leq B_j \leq A^j + \varepsilon I, \quad j \in \mathbf{N}.$$

Proof. (a)⇒(b) is almost obvious, since (a) implies

$$B_j = H(v_j) \overset{(2)}{\in} [v_j(A) - \varepsilon v_j(I),\ v_j(A) + \varepsilon v_j(I)] =$$
$$= [A^j - \varepsilon I,\ A^j + \varepsilon I], \quad j \in \mathbf{N}.$$

(b)⇒(a) We use Theorem 1, (b)⇒(a), for $J = \mathbf{N}$. Let $J_1 \subset \mathbf{N}$ be a finite subset, such that

$$\sum_{j\in J_1} \alpha_j v_j = \varphi_2 - \varphi_1 = \sum_{j\in\mathbf{N}} a_j v_j - \sum_{j\in\mathbf{N}} b_j v_j, \tag{4}$$

where $\varphi_1, \varphi_2 \in V_+$, i.e. $a_j, b_j \geq 0$, $j \in \mathbf{N}$ ($\varphi_2 = \sum_{j\in\mathbf{N}} a_j v_j$, $\varphi_1 = \sum_{j\in\mathbf{N}} b_j v_j$).

Then we have

$$\alpha_j = a_j - b_j \quad \forall j \in J_1, \quad a_j = b_j \quad \forall j \in \mathbf{N} \setminus J_1.$$

Since a_j, b_j are nonnegative, we have

$$-(a_j + b_j) \leq -b_j \leq \alpha_j = a_j - b_j \leq a_j \leq a_j + b_j, \quad j \in J_1,$$

which lead to

$$|\alpha_j| \leq a_j + b_j, \quad j \in J_1. \tag{5}$$

On the other hand, (b) leads to

$$\alpha_j B_j \leq \alpha_j A^j + \varepsilon\alpha_j I \quad \forall j \in J_1^+,\ \alpha_j B_j \leq \alpha_j A^j - \varepsilon\alpha_j I \quad \forall j \in J_1^- , \tag{6}$$

where

$$J_1^+ := \{j \in J_1; \alpha_j \geq 0\}, \quad J_1^- := \{j \in J_1; \alpha_j < 0\}.$$

The preceding relations yield

$$\begin{aligned}
\sum_{j\in J_1}\alpha_j B_j &= \sum_{j\in J_1^+}\alpha_j B_j+\sum_{j\in J_1^-}\alpha_j B_j\\
&\overset{(6)}{\le} \sum_{j\in J_1^+}\alpha_j A^j+\varepsilon\left(\sum_{j\in J_1^+}\alpha_j\right)I+\sum_{j\in J_1^-}\alpha_j A^j-\varepsilon\left(\sum_{j\in J_1^-}\alpha_j\right)I\\
&= \sum_{j\in J_1}\alpha_j A^j+\varepsilon\left(\sum_{j\in J_1}|\alpha_j|\right)I \overset{(4)}{=} \varphi_2(A)-\varphi_1(A)\\
&+ \varepsilon\left(\sum_{j\in J_1}|\alpha_j|\right)I \overset{(5)}{\le} \varphi_2(A)-\varphi_1(A)+\varepsilon\left(\sum_{j\in J_1}(a_j+b_j)\right)I\\
&\le \varphi_2(A)-\varphi_1(A)+\varepsilon\left(\sum_{j\in \mathbf{N}}a_j\right)I+\varepsilon\left(\sum_{j\in \mathbf{N}}b_j\right)I\\
&= \varphi_2(A)+\varepsilon\varphi_2(1)I-[\varphi_1(A)-\varepsilon\varphi_1(1)I]=F(\varphi_2)-G(\varphi_1),
\end{aligned}$$

where

$$F(\varphi):=\varphi(A)+\varepsilon\varphi(1)I,\ G(\varphi):=\varphi(A)-\varepsilon\varphi(1)I,\quad \varphi\in V.$$

Thus all conditions of the hypothesis of Theorem 1 are accomplished, and by this Theorem, there exists a linear operator $H\in L(V,Y)$ such that $H(v_j)=B_j=:y_j$, $j\in\mathbf{N}$ and (2) hold.

To prove (3), we use relations (2), which lead to

$$(7)\qquad |H(\varphi)|=\sup\{H(\varphi),-H(\varphi)\}\overset{(2)}{\le}|\varphi(A)|+\varepsilon\varphi(I),\quad \varphi\in V_+.$$

On the other hand, using the spectral measure E_A associated to the self-adjoint operator $A(||A||<b$ by hypothesis), one obtains

$$(8)\qquad |\varphi(A)|=\left|\int_{\sigma(A)}\varphi(t)dE_A(t)\right|\le||\varphi||\int_{\sigma(A)}dE_A(t)=||\varphi||I\quad \forall\varphi\in V\,;$$

$$(||A||<b\Rightarrow\sigma(A)\subset]-b,b[\Rightarrow|\varphi(t)|<||\varphi||:=\sup_{|z|\le b}|\varphi(z)|\quad\forall t\in\sigma(A),$$

where $\sigma(A)$ is the spectrum of A).

Relations (7) and (8) lead to

$$(9)\quad |H(\varphi)|\le||\varphi||I+\varepsilon\varphi(I)=||\varphi||I+\varepsilon\varphi(1)I=[||\varphi||+\varepsilon\varphi(1)]I\quad\forall\varphi\in V_+.$$

On the other hand, for any $\varphi\in V$ we obviously have $|H(\varphi)|=(H(\varphi))^++(H(\varphi))^-$. Using this, from (9) one gets

$$(10)\qquad (H(\varphi))^+\le[||\varphi||+\varepsilon\varphi(1)]I\quad\forall\varphi\in V_+.$$

But it is easy to see that in the vector lattice Y we have

$$0 \leq U \leq V \Rightarrow ||U|| = \sup_{||h||=1} \langle U(h), h\rangle \leq \sup_{||h||=1} \langle V(h), h\rangle = ||V||.$$

Whence (10) implies

$$||(H(\varphi))^+|| \leq ||[||\varphi|| + \varepsilon\varphi(1)]I|| = ||\varphi|| + \varepsilon\varphi(1), \quad \varphi \in V_+.$$

Similarly,

$$||(H(\varphi))^-|| \leq ||\varphi|| + \varepsilon\varphi(1), \quad \varphi \in V_+.$$

The conclusion is

$$||H(\varphi)|| = ||(H(\varphi))^+ - (H(\varphi))^-|| \leq$$
$$\leq ||(H(\varphi))^+|| + ||(H(\varphi))^-|| \leq 2[||\varphi|| + \varepsilon\varphi(1)], \quad \varphi \in V_+,$$

i.e. (3) holds. The proof is complete. □

The scalar version of Theorem 3 is:

COROLLARY 1. *Let V, v_j, $j \in \mathbf{N}$ be as above, and assume that $b > 1$. Let $\varepsilon > 0$, $\{y_j\}_{j\in\mathbf{N}} \subset \mathbf{R}$, $a \in]-b, b[$.*

The following assertions are equivalent:

(a) *there exists a linear functional $H \in V^*$ such that*

$$H(v_j) = y_j \quad \forall j \in \mathbf{N},$$
$$\varphi(a) - \varepsilon\varphi(1) \leq H(\varphi) \leq \varphi(a) + \varepsilon\varphi(1) \quad \forall \varphi \in V_+,$$
$$|H(\varphi)| \leq 2[||\varphi|| + \varepsilon\varphi(1)] \quad \forall \varphi \in V_+;$$

(b) *we have*

$$a^j - \varepsilon \leq y_j \leq a^j + \varepsilon \quad \forall j \in \mathbf{N}.$$

Problem. What can we say about the continuity of the linear operator H, which is the solution of the moment problem stated in Theorem 3? If H is continuous, find an estimation of $||H||$.

We go on by an application of Theorem 2.

THEOREM 4. *Let $b > 1$, V, Y, $\{v_j\}_{j\in\mathbf{N}}$ be as above. Let $A \in Y$, with $\sigma(A) \subset]0, b[$, $\{B_j\}_{j\in\mathbf{N}} \subset Y$, $\varepsilon > 0$. Assume that*

$$0 \leq B_j \leq A^j + \varepsilon I \quad \forall j \in \mathbf{N}.$$

Then there exists a positive continuous linear operator $H \in L_+(V, Y)$, such that

$$H(v_j) = B_j \quad \forall j \in \mathbf{N},$$

$$|H(v)| \leq ||v|| \left[(I - b^{-1}A)^{-1} + \varepsilon\frac{b}{b-1}I\right] \quad \forall v \in V, \tag{11}$$

$$||H|| \leq 2 \cdot \frac{b[(1+\varepsilon)b - (1 + \varepsilon||A||)]}{(b-1)(b-||A||)}. \tag{12}$$

Proof. We shall apply Theorem 2, (b)⇒(a) to $y_j := B_j$, $j \in \mathbf{N}$. We have to check the implication mentioned at (b), Theorem 2. Let $J_1 \subset \mathbf{N}$ be a finite subset, $\{\alpha_j; j \in J_1\} \subset \mathbf{R}$ such that

$$\sum_{j\in J_1} \alpha_j v_j \leq v = \sum_{j\in\mathbf{N}} a_j v_j \in V \quad (a_j \in \mathbf{R},\ j \in \mathbf{N}).$$

By the definition of the order relation on V, and using also the Cauchy inequalities for the analytic function $v = \sum_{j\in\mathbf{N}} a_j v_j$, one obtains

$$\alpha_j \leq a_j \leq |a_j| \leq \frac{||v||}{b^j}, \quad j \in J_1. \tag{13}$$

Put $J_1^+ := \{j \in J_1; \alpha_j \geq 0\}$, $J_1^- := \{j \in J_1; \alpha_j < 0\}$. From (13) and using the relations

$$0 \leq B_j \leq A^j + \varepsilon\, I \quad \forall j \in \mathbf{N}$$

from the hypothesis of the present Theorem, one gets:

$$\begin{aligned}
\sum_{j\in J_1} \alpha_j B_j &\leq \sum_{j\in J_1^+} \alpha_j B_j \leq \sum_{j\in J_1^+} \alpha_j (A^j + \varepsilon\, I) \\
&\overset{(13)}{\leq} ||v|| \left[\sum_{j\in J_1^+} b^{-j} A^j + \varepsilon \left(\sum_{j\in J_1^+} b^{-j}\right) I\right] \\
&\leq ||v|| \left[\sum_{j\in \mathbf{N}} b^{-j} A^j + \varepsilon \left(\sum_{j\in \mathbf{N}} b^{-j}\right) I\right] \\
&= ||v|| \left[(I - b^{-1}A)^{-1} + \varepsilon \frac{b}{b-1} I\right] =: P(v)
\end{aligned}$$

Thus the implication

$$\sum_{j\in J_1} \alpha_j v_j \leq v \Rightarrow \sum_{j\in J_1} \alpha_j B_j \leq P(v)$$

is proved, where $P(v) := ||v|| \left[(I - b^{-1}A)^{-1} + \varepsilon \frac{b}{b-1} I\right]$, $v \in V$. Applying (b)⇒(a) of Theorem 2, we infer that there exists a linear positive operator $H \in L_+(V, Y)$, such that $H(v_j) = B_j$, $j \in \mathbf{N}$ and

$$H(v) \leq P(v) = ||v|| \left[(I - b^{-1}A)^{-1} + \varepsilon \frac{b}{b-1} I\right], \quad v \in V.$$

Since $P(-v) = P(v)$, it follows that

$$|H(v)| \leq P(v), \quad \forall v \in V, \tag{14}$$

so that (11) is proved. Next we observe that (12) can be deduced from (11). In fact, because of $A \in \mathcal{A}(E)$, $\sigma(A) \subset]0, b[$, we have

$$||A|| = \sup_{||h||=1} \langle A(h), h \rangle = \Omega_A < b, \quad \text{i.e. } ||b^{-1}A|| < 1$$

(and $A > 0$). On the other hand, (14) implies

$$\sup\{(H(v))^+, (H(v))^-\} \leq P(v) \Rightarrow (H(v))^+ \leq P(v) \Rightarrow ||(H(v))^+|| \leq ||P(v)||$$

and also

$$||(H(v))^-|| \leq ||P(v)||.$$

It follows that

$$||H(v)|| = ||(H(v))^+ - (H(v))^-|| \leq 2||P(v)|| \tag{15}$$

On the other hand, we have by the definition of

$$P(v) := ||v|| \left[(I - b^{-1}A)^{-1} + \varepsilon \frac{b}{b-1} I \right];$$

$$\begin{aligned}
||P(v)|| &\leq ||v|| \left[||(I - b^{-1}A)^{-1}|| + \varepsilon \frac{b}{b-1} \right] \\
&= ||v|| \left[||I + (b^{-1}A) + (b^{-1}A)^2 + \ldots || + \varepsilon \frac{b}{b-1} \right] \\
&\leq ||v|| \left[1 + \frac{||A||}{b} + \frac{||A||^2}{b^2} + \ldots + \varepsilon \frac{b}{b-1} \right] \\
&= ||v|| \left[\frac{1}{1 - \frac{||A||}{b}} + \varepsilon \frac{b}{b-1} \right] \\
&= ||v|| \cdot \frac{b[(1+\varepsilon)b - (1 + \varepsilon||A||)]}{(b-1)(b - ||A||)} \quad \forall v \in V.
\end{aligned}$$

From this and using also (15), we get

$$||H(v)|| \overset{(15)}{\leq} 2||P(v)|| \leq 2||v|| \cdot \frac{b[(1+\varepsilon)b - (1 + \varepsilon||A||)}{(b-1)(b - ||A||)}.$$

Thus (12) is proved and the proof is complete. □

Obviously, a "scalar version" of Theorem 4 can be deduced, taking $Y = \mathbf{R}$, $I = 1$, $A = a \in]0, b[$, $B_j = y_j \in [0, a^j + \varepsilon]$.

The last result is an application of Theorem 1, this time to a space of $C^\infty_{\mathbf{R}}$ functions, which are not necessarily analytic. Denote $V := C^\infty_{\mathbf{R}}([0, b])$, where $b \geq 1$, and let $\varepsilon > 0$. Put $v_j(t) = t^j$, $t \in [0, b]$, $j \in \mathbf{N}$. We endow V with the convex cone

$$V_+ := \{v \in V; v^{(k)}(t) \geq 0 \quad \forall t \in [0, b], \ \forall k \in \mathbf{N}\}.$$

Let Y be as above and $\{B_j\}_{j \in \mathbf{N}} \subset Y$.

Let $A \in Y$ be such that $\sigma(A) \subset [0, b]$. Under these assumptions, we have.

THEOREM 5. *The following assertions are equivalent:*
(a) *there exists a linear operator* $H \in L(V, Y)$ *such that* $H(v_j) = B_j$, $j \in \mathbf{N}$,

$$\varphi(A) - \varepsilon\varphi(1)I \le H(\varphi) \le \varphi(A) + \varepsilon\varphi(1)I \quad \forall \varphi \in V_+ \;;$$

(b) $A^j - \varepsilon I \le B_j \le A^j + \varepsilon I \quad \forall j \in \mathbf{N}$.

Proof. The implication (a)⇒(b) is almost obvious, because of the implication: $v_j \in V_+$ and $B_j \overset{(a)}{=} H(v_j)$

$$\Rightarrow B_j = H(v_j) \overset{(a)}{\in} [v_j(A) - \varepsilon v_j(1)I, v_j(A) + \varepsilon v_j(1)I] =$$

$$= [A^j - \varepsilon I,\ A^j + \varepsilon I].$$

To prove (b)⇒(a), we apply Theorem 1, (b)⇒(a). We have to prove the implication at (b) of Theorem 1. Let $J_1 \subset \mathbf{N}$ be a finite subset, $\{\alpha_j; j \in J_1\} \subset \mathbf{R}$, $\varphi_1, \varphi_2 \in V_+$ such that

$$\sum_{j \in J_1} \alpha_j v_j(t) \left(= \sum_{j \in J_1} \alpha_j t^j \right) = \varphi_2(t) - \varphi_1(t) \quad \forall t \in [0, b].$$

This implies $\alpha_j = \dfrac{\varphi_2^{(j)}(0)}{j!} - \dfrac{\varphi_1^{(j)}(0)}{j!}$, $j \in J_1$. Since $\varphi_1, \varphi_2 \in V_+$, we have in particular

$$\varphi_k^{(j)}(0) \ge 0 \quad \forall j \in \mathbf{N}, \quad k \in \{1, 2\}.$$

It follows that

$$-\frac{\varphi_1^{(j)}(0)}{j!} \le \alpha_j \le \frac{\varphi_2^{(j)}(0)}{j!},$$

which leads to

$$|\alpha_j| \le \frac{\varphi_2^{(j)}(0)}{j!} + \frac{\varphi_1^{(j)}(0)}{j!}, \quad j \in J_1. \tag{16}$$

If we denote $J_1^+ := \{j \in J_1; \alpha_j \ge 0\}$, $J_1^- := \{j \in J_1; \alpha_j < 0\}$, the preceding relations lead to

$$
\begin{aligned}
\sum_{j\in J_1}\alpha_j B_j &= \sum_{j\in J_1^+}\alpha_j B_j + \sum_{j\in J_1^-}\alpha_j B_j \\
&\overset{(b)}{\leq} \sum_{j\in J_1^+}\alpha_j (A^j+\varepsilon I) + \sum_{j\in J_1^-}\alpha_j (A^j-\varepsilon I) \\
&= \sum_{j\in J_1}\alpha_j A^j + \varepsilon\left(\sum_{j\in J_1}|\alpha_j|\right) I \\
&= \varphi_2(A)-\varphi_1(A)+\varepsilon\left(\sum_{j\in J_1}|\alpha_j|\right) I \\
&\overset{(16)}{\leq} \varphi_2(A)-\varphi_1(A)+\varepsilon\left(\sum_{j\in J_1}\frac{\varphi_2^{(j)}(0)}{j!}\right) I + \varepsilon\left(\sum_{j\in J_1}\frac{\varphi_1^{(j)}(0)}{j!}\right) I \\
&\leq \varphi_2(A)-\varphi_1(A)+\varepsilon(\varphi_2(1))I+\varepsilon(\varphi_1(1))I = F(\varphi_2)-G(\varphi_1),
\end{aligned} \tag{17}
$$

where

$$F(\varphi) := \varphi(A)+\varepsilon\varphi(1)I, \quad G(\varphi)=\varphi(A)-\varepsilon\varphi(1)I.$$

Note that we have used Taylor's formula and the definition of the order relation on V when we write

$$\sum_{j\in J_1}\frac{\varphi_k^{(j)}(0)}{j!} \leq \varphi_k(1), \quad k\in\{1,2\}.$$

In fact, let $n\in\mathbf{N}$ be such that $J_1\subset\{0,1,\ldots,n\}$. Then we have

$$\varphi_k(1)=\sum_{j=0}^{n}\frac{\varphi_k^{(j)}(0)}{j!}+\frac{\varphi_k^{(n+1)}(t)}{(n+1)!} \geq \sum_{j=0}^{n}\frac{\varphi_k^{(j)}(0)}{j!} \geq \sum_{j\in J_1}\frac{\varphi_k^{(j)}(0)}{j!},$$

$k\in\{1,2\}$, becauseof $\varphi_k^{(n+1)}(t)\geq 0$, $\varphi_k^{(j)}(0)\geq 0$, $\forall j\in\mathbf{N}$, $k\in\{1,2\}$, $\forall t\in[0,b]$.

Now, from (17) and using Theorem 1 (b)⇒(a), the conclusion follows. □

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