MATHEMATICA, Tome 48 (71), N° 2, 2006, pp. 137–148

A COMPROMISE MODEL FOR SOLVING FUZZY LINEAR PROGRAMMING PROBLEMS

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Abstract. In this paper, we concentrate on two kinds of fuzzy linear programming problems: linear programming problems with only fuzzy technological coefficients and linear programming problems in which both the right-hand side and technological coefficients are fuzzy number. We consider here only the case of fuzzy numbers with linear membership functions. The symmetric method of Bellman and Zadeh [1] is used for a defuzzification of these problems (min operator). Two-phase approach had been proposed to generate an efficient solution for the linear programming problem. In this study, we shall show a revised two-phase approach to the case of the fuzzy linear programming problems [FLP]. This revised model can improve the optimal decision obtained from min operator. Moreover, a compromise model embedded two-phase approach and average operator will be proposed to yield a fuzzy-efficient solution between non-compensatory and full compensatory.

MSC 2000. 90C70.
Key words. Fuzzy linear programming, two-phase approach, fuzzy number.

1. INTRODUCTION

In fuzzy decision making problems, the concept of maximizing decision was proposed by Bellman and Zadeh [1]. This concept was adopted to problems of mathematical programming by Tanaka et al. [15]. Zimmermann [18] presented a fuzzy approach to multiobjective linear programming problems. He also studied the duality relation in fuzzy linear programming. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita [11] and called robust programming. Dubois and Prade [2] investigated linear fuzzy constraints. Tanaka and Asai [14] also proposed a formulation of fuzzy linear programming with fuzzy constraints and gave a method for its solution which bases on inequality relations between fuzzy numbers. To deal with imprecision parameters in mathematical programming problems, fuzzy set theory has been applied to real-word decision making problems. Fuzzy linear programming models and fuzzy multiple objective programming problems are designated for such a purpose [7, 8]. In fuzzy set theory, a corresponding membership function is usually employed to quantify the fuzzy objectives and constraints. Using the linear membership function, Zimmermann proposed the min operator model to the MOLP [19, 20]. Although the min operator method has been proven to have several nice properties [10], the solution generated by min operator does not guarantee compensatory and efficient [3, 9]. Lee and Li [9] proposed two-phase approach to overcome this difficulty. Chen and Chou [10] proposed a fuzzy approach to integrate the min operator, average operator and two-phase methods. Their approach considers simultaneously maximizing the least satisfaction level and total satisfaction. Because the two-phase approach can really yield efficient solution, it is also applied to the multiple objective linear fractional programming problems [4]. While dealing with the fuzzy linear programming problem [FLP], Guu and Wu [5] proposed a similar two-phase model to improve the dominated solution yielded by min operator. This two-phase approach shows that not only should the outcome of a FLP model achieve the highest membership degree in objective, but also pursue a better utilization of each constraint resource.

In this paper, we shall propose a simplified two-phase model to the case of FLP. As the result of experiment made by Zimmermann [21], and most of the decisions taken in the real world are neither non-compensatory (min operator) nor full compensatory (average operator). On the other hand, the decision-maker may prefer to make judgments or evaluations resulting in the fuzzy programming problem between the min and average operator.

2. LINEAR PROGRAMMING PROBLEMS WITH FUZZY TECHNOLOGICAL COEFFICIENTS

We consider a linear programming problem with fuzzy technological coefficients

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j \\ &\text{subject to } \sum_{j=1}^{n} \tilde{a}_{ij} x_j \le b_i, \quad 1 \le i \le m, \\ &x_j \ge 0, \qquad\qquad\qquad\quad 1 \le j \le n, \end{aligned} \tag{2.1}$$

where at least one $x_j > 0$, suppose that the crisp inequality relation between fuzzy number is defined by [12].

We can accept some assumptions.

Assumption 1. $\tilde{a}_{ij}$ is a fuzzy number with the following linear membership function

$$\mu_{a_{ij}}(x) = \begin{cases} 1 & \text{if} \quad x < a_{ij} \\ \frac{(a_{ij}+d_{ij}-x)}{d_{ij}} & \text{if} \quad a_{ij} \le x < a_{ij} + d_{ij} \\ 0 & \text{if} \quad x \ge a_{ij} + d_{ij}, \end{cases}$$

where $x \in \mathbb{R}$ and $d_{ij} > 0$ for all $i = 1, \ldots, m, j = 1, \ldots, n$. For defuzzification of this problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values first. The bounds of the optimal values, z_ℓ and z_u are obtained by solving the standard linear programming problems.

$$\begin{aligned} & z_1 = \max \sum_{j=1}^{n} c_j x_j \\ & \text{subject to } \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m, \\ & x_j \ge 0, \qquad\qquad\qquad\qquad j = 1, \ldots, n, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & z_2 = \max \sum_{j=1}^{n} c_j x_j \\ & \text{subject to } \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i, \quad i = 1, \ldots, m, \\ & x_j \ge 0, \qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n. \end{aligned} \tag{2.3}$$

The objective function takes values between z_1 and z_2 while technological coefficients vary between a_{ij} and $a_{ij} + d_{ij}$. The problem (2.2) and (2.3) are in relation. The feasible solutions set of the problem (2.3) is a subset of the feasible solutions set of the problem (2.2). Consequently $z_1 \ge z_2$. Let $z_\ell = \min(z_1, z_2)$ and $z_u = \max(z_1, z_2)$. Then $z_\ell = z_2$ and $z_u = z_1$ are called the lower and upper bounds of the optimal values, respectively.
Assumption 2. The linear crisp problems (2.2) and (2.3) have finite optimal values, otherwise we cannot calculate the lower and upper bounds of the optimal values. In this case let $z_1 \ne z_2$ then, the fuzzy set of optimal values, G, which is a subset of $\mathbb{R}^n$, is defined as (see Klir and Yuan [6]);

$$\mu_G(x) = \begin{cases} 0 & \text{if } \sum_{j=1}^{n} c_j x_j < z_\ell, \\ \frac{\sum_{j=1}^{n} c_j x_j - z_\ell}{z_u - z_\ell} & \text{if } z_\ell \le \sum_{j=1}^{n} c_j x_j < z_u, \\ 1 & \text{if } \sum_{j=1}^{n} c_j x_j \ge z_u. \end{cases} \tag{2.4}$$

The fuzzy set of the i-th constraint, c_i, which is a subset of $\mathbb{R}^m$, is defined by

$$\mu_{c_i}(x) = \begin{cases} 0 & \text{if } b_i < \sum_{j=1}^{n} a_{ij} x_j, \\ \frac{(b_i - \sum_{j=1}^{n} a_{ij} x_j)}{\sum_{j=1}^{n} d_{ij} x_j} & \text{if } \sum_{j=1}^{n} a_{ij} x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j, \\ 1 & \text{if } b_i \ge \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j. \end{cases} \tag{2.5}$$

By using the definition of the fuzzy decision proposed by Bellman and Zadeh [1] (see also Lai and Hweng [8]), we have

$$\mu_D(x) = \min\{\mu_G(x), \mu_{c_1}(x), \mu_{c_2}(x), \ldots, \mu_{c_m}(x)\}. \tag{2.6}$$

In this case the optimal fuzzy decision is a solution of the problem

$$\max(\mu_D(x)) = \max_{x\geq 0} \min\{\mu_G(x), \mu_{c_1}(x), \mu_{c_2}(x), \ldots, \mu_{c_m}(x)\}. \tag{2.7}$$

Consequently, the problem (2.1) becomes to the following optimization problem

$$\begin{aligned}
&\max \lambda,\\
&\mu_G(x) \geq \lambda,\\
&\mu_{c_i}(x) \geq \lambda,\ i = 1, \ldots, m,\\
&x \geq 0.
\end{aligned} \tag{2.8}$$

By using (2.4) and (2.5), the problem (2.8) can be written as

$$\begin{aligned}
&\max \lambda,\\
&\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 \leq 0,\\
&\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i \leq 0,\ 1 \leq i \leq m,\\
&x_j \geq 0,\ j = 1, \ldots, n.
\end{aligned} \tag{2.9}$$

The optimal value of this problem can be yield if the set of feasible solutions is not empty.

Solving the model (2.8) or (2.9), one optimal value λ^* can be yielded. In fact, this λ^* dentoes that the satisfaction level for all membership functions can simultaneously obtain. Further, let us assume that the membership function of objective and all constraint are equally important.

3. LINEAR PROGRAMMING PROBLEMS WITH FUZZY TECHNOLOGICAL COEFFICIENTS AND FUZZY RIGHT-HAND-SIDE NUMBERS

In this section we consider a linear programming problem with fuzzy technological coefficients and fuzzy right-hand-side numbers

$$\begin{aligned}
&\max \sum_{j=1}^{n} c_j x_j\\
&\text{subject to } \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i,\ 1 \leq i \leq m,\\
&x_j \geq 0,\ 1 \leq j \leq n,
\end{aligned} \tag{3.1}$$

where at least one $x_j > 0$.

Assumption 3. $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers with the following linear membership functions:

$$\mu_{a_{ij}}(x) = \begin{cases} 1 & \text{if} \quad x < a_{ij},\\ \frac{(a_{ij}+d_{ij}-x)}{d_{ij}} & \text{if} \quad a_{ij} \leq x < a_{ij} + d_{ij},\\ 0 & \text{if} \quad x \geq a_{ij} + d_{ij}, \end{cases}$$

and

$$\mu_{b_i}(x) = \begin{cases} 1 & \text{if} \quad x < b_i, \\ \frac{(b_i+p_i-x)}{p_i} & \text{if} \quad b_i \leq x < b_i + p_i, \\ 0 & \text{if} \quad x \geq b_i + p_i, \end{cases}$$

where $x \in \mathbb{R}$. For defuzzification of the problem (3.1), we first calculate the lower and upper bounds of the optimal values. The optimal values z_ℓ and z_u can be defined by solving the following standard linear programming problems, for which we assume that all they have the finite optimal values.

$$\begin{aligned} & z_1 = \max \sum_{j=1}^{n} c_j x_j, \\ & \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i, \ 1 \leq i \leq m, \\ & x_j \geq 0, \ 1 \leq j \leq n; \end{aligned} \tag{3.2}$$

$$\begin{gathered} z_2 = \max \sum_{j=1}^{n} c_j x_j, \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i + p_i, \ 1 \leq i \leq m, \\ x_j \geq 0, \ 1 \leq j \leq n; \end{gathered} \tag{3.3}$$

$$\begin{gathered} z_3 = \max \sum_{j=1}^{n} c_j x_j, \\ \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i + p_i, \ 1 \leq i \leq m, \\ x_j \geq 0, \ 1 \leq j \leq n \end{gathered} \tag{3.4}$$

and

$$\begin{aligned} & z_4 = \max \sum_{j=1}^{n} c_j x_j, \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ 1 \leq i \leq m, \\ & x_j \geq 0, \ 1 \leq j \leq n. \end{aligned} \tag{3.5}$$

The problems (3.2), (3.3), (3.4) and (3.5) are in relation. If we denote S_1, S_2, S_3 and S_4 the feasible solutions set of the problems (3.2), (3.3), (3.4) and (3.5), respectively, then $S_1 \subseteq S_3 \subseteq S_2$ and $S_1 \subseteq S_4 \subseteq S_2$. Consequently $z_1 \leq z_3 \leq z_2$ and $z_1 \leq z_4 \leq z_2$.

Let $z_\ell = \min(z_1, z_2, z_3, z_4)$ and $z_u = \max(z_1, z_2, z_3, z_4)$. The objective function takes values between $z_\ell = z_1$ and $z_u = z_2$ while technological coefficients take values between a_{ij} and $a_{ij} + d_{ij}$ and the right-hand side numbers takes values between b_i and $b_i + p_i$.

Let $z_1 \neq z_2$ then, the fuzzy set of optimal values, G, which is a subset of $\mathbb{R}^n$, is defined by (see Klir and Yuan [6]);

$$\mu_G(x) = \begin{cases} 0 & \text{if} \quad \sum_{j=1}^{n} c_j x_j < z_\ell, \\ \frac{(\sum_{j=1}^{n} c_j x_j - z_\ell)}{(z_u - z_\ell)} & \text{if} \quad z_\ell \leq \sum_{j=1}^{n} c_j x_j < z_u, \\ 1 & \text{if} \quad \sum_{j=1}^{n} c_j x_j \geq z_u. \end{cases} \tag{3.6}$$

The fuzzy set of the i-th constraint, c_i, which is a subset of $\mathbb{R}^m$ is defined by

$$\mu_{c_i}(x) = \begin{cases} 0 & \text{if} \quad b_i < \sum_{j=1}^{n} a_{ij} x_j, \\ \frac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{\sum_{j=1}^{n} d_{ij} x_j + p_i} & \text{if} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i, \\ 1 & \text{if} \quad b_i \geq \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i. \end{cases} \tag{3.7}$$

Then, by using the method of defuzzification as for the problem (2.8), the problem (3.1) is reduced to the following crisp problem:

$$\begin{gathered} \max \lambda, \\ \lambda(z_2 - z_1) - \sum_{j=1}^{n} c_j x_j + z_1 \leq 0, \\ \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i - b_i \leq 0, \ 1 \leq i \leq m, \\ x \geq 0. \end{gathered} \tag{3.8}$$

The optimal value of problem (3.8) can be yield if the set of feasible solutions is not empty.

Solving the model (3.8), one optimal value λ^* can be yielded. In fact, this λ^* denotes that the satisfaction level for all membership functions can simultaneously obtain. Further, let us assume that the membership functions of objective and all constraint are equally important.

4. AVERAGE OPERATOR MODEL AND TWO-PHASE APPROACH MODEL

The problem (2.1) and (3.1) can be solved to the following average operator model.

$$\begin{aligned} & \max \lambda^{\#} = \tfrac{1}{m+1} \sum_{k=1}^{m+1} \lambda_k, \\ & \text{subject to } 1 \geq \mu_G(x) \geq \lambda_1 \geq 0, \\ & 1 \geq \mu_{c_i}(x) \geq \lambda_i \geq 0, \ \forall \ i = 2, \ldots, m+1, \\ & x \geq 0. \end{aligned} \tag{4.1}$$

It is easy to understand that the optimal value $\lambda^{\#}$ represent the total amount of all membership functions.

Definition 4.1. x^* is a fuzzy-efficient solution to the fuzzy linear programming (FLP) if there does not exist any $x \in S$ such that $\mu_k(x^*) \leq \mu_k(x)$ for all k and $\mu_\rho(x^*) < \mu_\rho(x)$ for at least one ρ.

Guu and Wu [16] have shown that the optimal solution yielded by the two-phase approach is afuzzy-efficient solution [16, 17]. The two-phase approach employs the min operator (model (2.8)) as phase I; and it associates the average operator (model (4.1)) as phase II. The second phase is defined as following:

$$\begin{array}{l} \max \overline{\lambda} = \frac{1}{m+1} \sum\limits_{k=1}^{m+1} \lambda_k \\ \text{subject to } 1 \geq \mu_G(x) \geq \lambda_1 \geq \lambda^* \geq 0, \\ 1 \geq \mu_{c_i}(x) \geq \lambda_i \geq \lambda^* \geq 0, \ \forall\, i = 2, \ldots, m+1, \\ x \geq 0. \end{array} \tag{4.2}$$

where λ^* denotes the solution yielded by the min operator. For model (4.2), we know that the two-phase approach is a combination of the min operator and the average operator model. The optimal value $\overline{\lambda}$ is absolutely not less than the value λ^*.

5. THE COMPROMISE MODEL

Let us note the framework of average operator model (4.1) that the decision variables in objective function are separable and with positive coefficients. Therefore, at any optimal solutions certain inequality must hold as equalities. The following lemma can support this assertion.

LEMMA 1. *The inequality constraints $\lambda_1 \leq \mu_G(x)$ and $\lambda_i \leq \mu_{c_i}(x)$, $\forall\, i$ must hold as equalities at any optimal solution in model* (4.1).

Proof. : See (Y. K. Wu and S. M. Guu [22]). □

This lemma implies that the average operator model (4.1) can be modified into following problem:

$$\begin{array}{l} \max \lambda^{\#} = \frac{1}{m+1} \left[\mu_G(x) + \sum\limits_{i=2}^{m+1} \mu_{c_i}(x) \right] \\ \text{subject to } x \geq 0. \end{array} \tag{5.1}$$

Let us look at the precious models closely, then we can understand that solutions obtained by min and average operators represent two extreme situations. The optimal value λ^* generated by min operator model denotes to maximize the least satisfaction level among all membership functions simultaneously. However, it is a non-compensatory model and can't guarantee to get fuzzy-efficient solutions. The optimal value $\lambda^{\#}$ comes from average operator stands for maximizing the total amount of membership functions, but it is classified into the fully compensatory model. In order to offer any desirable compromise solutions between non-compensatory and fully compensatory to the decision-maker, we associate preceding two-phase approach with the results obtained by min operator and propose following compromise model to solve the FLP.

$$\begin{aligned}
&\max \overline{\lambda} = \tfrac{1}{m+1} \sum_{k=1}^{m+1} \lambda_k \\
&\text{subject to } 1 \geq \mu_G(x) \geq \lambda_1 \geq \lambda' \geq 0, \\
&1 \geq \mu_{c_i}(x) \geq \lambda_i \geq \lambda' \geq 0,\ i = 2, \ldots, m+1.
\end{aligned} \tag{5.2}$$

where the parameter $\lambda' \in [0, \lambda^*]$ is given by the decision-maker. Furthermore, the compromise model (5.2) has been proven and can guarante to obtain the fuzzy-efficient solution by Guu and Wu [5]. Note that if the decision-maker assigns λ^* to λ', model (5.2) reduces to be the two-phase approach model (4.2). On the other hand, let λ' be equal to 0, then it becomes to the average operator as model (4.1). Therefore, the two-phase approach and average method are special cases to our compromise model. However, according to the Lemma 5.1 the model (5.2) can be equivalent to the following reduced-form problem:

$$\begin{aligned}
&\max \tilde{\lambda} = \tfrac{1}{m+1} \left[\mu_G(x) + \sum_{i=2}^{m+1} \mu_{c_i}(x) \right] \\
&\text{subject to } 1 \geq \mu_G(x) \geq \lambda', \\
&1 \geq \mu_{c_i}(x) \geq \lambda',\ \forall\, i = 2, \ldots, m+1, \\
&x \geq 0.
\end{aligned} \tag{5.3}$$

Note that the model (5.3) has less number of decision variables and constraints then that the model (5.2), yet yields the same fuzzy-efficient solutions. In fact, the parameter λ' may be considered as a compromise index for all membership functions. As long as the decision-maker determines the compromise degree among 0 and λ^* to index λ', the model (5.3) can be solved and obtained a fuzzy-efficient solution between non-compensatory and fully compensatory.

6. NUMERICAL EXAMPLE

In this section, an example will be used to show that every membership function yielded by the two-phase approach guarantees at least as large a then the min operator can offer. Consider the following FLP.

$$\begin{array}{rl}
\max & z = x_1 + x_2 \\
\text{subject to} & \tilde{1}x_1 + \tilde{2}x_2 \leq \tilde{3}, \\
& \tilde{2}x_1 + \tilde{3}x_2 \leq \tilde{4}, \\
& x_1, x_2 \geq 0.
\end{array} \tag{6.1}$$

which take fuzzy parameters as; $\tilde{1} = L(1,1), \tilde{2} = L(2,1), \tilde{3} = L(3,2), b_1 = \tilde{3} = L(3,2)$ and $b_2 = \tilde{4} = L(4,3)$ as used by Shaocheng [13]. That is

$$(a_{ij}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},\ (d_{ij}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},$$

$$(b_i) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ (p_i) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow (b_i + p_i) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

To solve this problem, first, we must solve the following two subproblems.

$$\begin{array}{ll} \max & z_1 = x_1 + x_2 \\ \text{subject to} & 2x_1 + 3x_2 \leq 3, \\ & 4x_1 + 5x_2 \leq 4, \\ & x_1, x_2 \geq 0, \end{array}$$

and

$$\begin{array}{ll} \max & z_2 = x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 5, \\ & 2x_1 + 3x_2 \leq 7, \\ & x_1, x_2 \geq 0. \end{array}$$

Optimal solutions of these subproblems are

$$\begin{array}{ccc} x_1 = 1 & & x_1 = 3.5 \\ x_2 = 0 & \text{and} & x_2 = 0 \\ z_1 = 1 & & z_2 = 3.5 \end{array}$$

$$z_\ell = \min(z_1, z_2) = \min(1, 3.5) = 1 = z_1$$

$$z_u = \max(z_1, z_2) = \max(1, 3.5) = 3.5 = z_2$$

Then the membership function of the objective function can be defined as follows:

$$\mu_G(x) = \begin{cases} 0 & \text{if} \quad x_1 + x_2 < 1, \\ \frac{x_1+x_2-1}{3.5-1} & \text{if} \quad 1 \leq x_1 + x_2 < 3.5, \\ 1 & \text{if} \quad x_1 + x_2 \geq 3.5. \end{cases}$$

For each of fuzzy constraints, the non-increasing linear membership functions are designed to as follows:

$$\mu_{c_1}(x) = \begin{cases} 0 & \text{if} \quad 3 < x_1 + 2x_2, \\ \frac{3-x_1-2x_2}{x_1+x_2+2} & \text{if} \quad x_1 + 2x_2 \leq 3 < 2x_1 + 3x_2 + 2, \\ 1 & \text{if} \quad 3 \geq 2x_1 + 3x_2 + 2; \end{cases}$$

$$\mu_{c_2}(x) = \begin{cases} 0 & \text{if} \quad 4 < x_1 + 2x_2, \\ \frac{4-2x_1-3x_2}{2x_1+2x_2+3} & \text{if} \quad x_1 + 2x_2 \leq 4 < 4x_1 + 5x_2 + 3, \\ 1 & \text{if} \quad 4 \geq 4x_1 + 5x_2 + 3. \end{cases}$$

When the membership function of the objective function and fuzzy constraint are determined, the phase I of two-phase approach as the same with min operator will be ready to solve the following problem.

$$\begin{array}{ll} & \max \lambda \\ \text{subject to} & \frac{x_1+x_2-1}{2.5} \geq \lambda, \\ & \frac{3-x_1-2x_2}{x_1+x_2+2} \geq \lambda, \\ & \frac{4-2x_1-3x_2}{2x_1+2x_2+3} \geq \lambda, \\ & x_1, x_2 \geq 0, \end{array}$$

that is

$$\begin{array}{rc}
 & \max \lambda \\
\text{subject to} & x_1 + x_2 \geq 1 + 2.5\lambda, \\
 & (1+\lambda)x_1 + (2+\lambda)x_2 \leq 3 - 2\lambda, \\
 & (2+2\lambda)x_1 + (2\lambda+3)x_2 \leq 4 - 3\lambda, \\
 & x_1, x_2 \geq 0.
\end{array} \tag{6.2}$$

Solving above problem, the optimal solution is $x^* = (1.45804, 78 \times 10^{-8} \simeq 0)$. According to x^*, we can obtain the following optimal value and membership function for objective and constraints:

$$\lambda^* = 0.1832159, \ z(x^*) = 1.45804,$$

$$\mu_G(x^*) = 0.183216, \ \mu_{c_1}(x^*) = 0.44590, \ \mu_{c_2}(x^*) = 0.183216. \tag{6.3}$$

Besides, using the average operator as model (4.1) or (5.1) to solve the numerical example can yield the optimal solution $x^{\#} = (1, 0)$ and the following optimal value: $\lambda^{\#} = 0.355533, \mu_G(x^{\#}) = 0, \mu_{c_1}(x^{\#}) = 0.6666, \mu_{c_2}(x^{\#}) = 0.4$. Applying the results x^* generated from phase I, the second phase of two-phase approach as model (4.2) is to solve the following problem.

$$\begin{array}{rc}
 & \max \overline{\lambda} = \frac{1}{3} \sum_{k=1}^{3} \lambda_k \\
\text{subject to} & \lambda_k \geq 0.1832159, \ k = 1, 2, 3; \quad x_1 + x_2 \leq 3.5, \\
 & x_1 + x_2 \geq 1 + 2.5\lambda_1, \\
 & 2x_1 + 3x_2 \geq 1 \\
 & (\lambda_2 + 1)x_1 + (\lambda_2 + 2)x_2 \leq 3 - 2\lambda_2, \\
 & 4x_1 + 5x_2 \geq 1, \\
 & (2\lambda_3 + 2)x_1 + (2\lambda_3 + 3)x_2 \leq 4 - 3\lambda_3, \\
 & x_1, x_2 \geq 0.
\end{array}$$

The optimal solution is $x^{**} = (0.5500, 0.9081)$. Value of objective function and membership functions are as follow:

$$\overline{\lambda} = 0.352933, \ z(x^{**}) = 1.45804, \ \mu_G(x^{**}) = 0.58326,$$

$$\mu_{c_1}(x^{**}) = 0.4458598, \ \mu_{c_2}(x^{**}) = 0.029679. \tag{6.4}$$

Let us compare the two-phase approach results (6.4) with the solution (6.3) obtained by min operator. It is obvious that the membership function $\mu_G(x^{**}) = 0.58326$ is larger than $\mu_G(x^*) = 0.183216$, which means that the two-phase approach really obtains fuzzy-efficient solution and improves the min operator's solution. Applying the results $\lambda^* = 0.1832159$ obtained in phase I, we are ready to employ the compromise model (5.3) to solve this example.

$$\begin{array}{rl}
 & \max \tilde{\lambda} = \frac{1}{3}(\mu_G(x) + \mu_{c_1}(x) + \mu_{c_2}(x)) \\
\text{subject to} & x_1 + x_2 \leq 3.5, \\
 & x_1 + x_2 \geq 1 + 2.5\lambda'; \quad 2x_1 + 3x_2 \geq 1, \\
 & (\lambda' + 1)x_1 + (\lambda' + 2)x_2 \leq 3 - 2\lambda'; \quad 4x_1 + 5x_2 \geq 1, \\
 & (2\lambda' + 2)x_1 + (2\lambda' + 3)x_2 \leq 4 - 3\lambda'; \quad x_1, x_2 \geq 0,
\end{array}$$

where the parameter λ' is between 0 and 0.1832159. To illustrate the results of compromise model, we give different value to λ' and solve it by general linear programming method. We obtain the various solutions in Table 1.

The Table 1 shows that the solutions obtained by two-phase approach and average operator represent two extreme situations of the compromise model. For instance, if λ' is equal to 0, then the result is the same as the solutions of average operator model. When the decision-maker gives $\lambda' = 0.1832159$, the compromise model provides the same results with the two-phase approach. Giving λ' among 0 and 0.1832159, these fuzzy-efficient solutions between the non-compensatory and fully compensatory can be obtained by the compromise model. This model also provides other alternatives as long as the decision-maker depends upon his/her preference to determine the compromise index λ'.

7. CONCLUSIONS

In this paper, we make a study of the linear programming problems with imprecision parameter and propose a simplified two-phase model to improve the solution yielded by min operator. Moreover, to generate fuzzy-efficient solutions between non-compensatory and full compensatory we provide one compromise model and discover that the two-phase approach and average operator method are special cases to our method. This compromise model also employs an adjustable parameter index that the decision-maker can depend upon his/her preference to determine this index as well as obtain different alternatives.

Table 1 : Some results by compromise model

λ'	objective value $\tilde{\lambda}$	$\mu_G(x)$	$\mu_{c_1}(x)$	$\mu_{c_2}(x)$
0.00	0.355533	0.00	0.6666	0.400
0.05	0.103846	0.100	0.161538	0.0500
0.1	0.1487176	0.100	0.246153	0.1000
0.15	0.240942	0.5500	0.1500	0.022826
0.1832159	0.270777	0.183216	0.44590	0.183216

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

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August 30, 2005

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