MATHEMATICA, Tome 48 (71), N^o 1, 2006, pp. 25–35

HYPERGEOMETRIC STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

MARZIEH GHAEDI and S.R. KULKARNI

Abstract. In this paper we obtain several interesting properties of the hypergeometric function $F(a,b,c,d;e;z)$ where

$$F(a,b,c,d;e;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)}z^n.$$

In the class $H(a,b,c,d,e,z)$ of the hypergeometric functions $F(a,b,c,d;e;z)$ in the open unit disk $U=\{z||z|<1\}$, we consider starlike and convex functions of order α with negative coefficients. These properties include conditions on a,b,c,d,e to guarantee $zF(a,b,c,d;e;z)$ to be in the subclasses of starlike and convex functions. We give also several interesting properties of the class $H(a,b,c,d;e;z)$.

MSC 2000. 30C45.
Key words. Hypergeometric, starlike, convex, gamma function.

1. INTRODUCTION

Let A denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ that are analytic and univalent in the unit disk $U=\{z:|z|<1\}$. A function $f\in A$ is said to be starlike of order α, $0\leq\alpha<1$, if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$ for $z\in U$, and it is said to be convex of order α, $0\leq\alpha<1$, if $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$ for $z\in U$. We denote by $S^*(\alpha)$ the class of functions from A which are starlike of order α and we denote by $K(\alpha)$ the class of functions from A which are convex of order α. The classes $S^*(0)=S$ and $K(0)=K$ are called the classes of starlike and convex functions, respectively.

We define $S^*[\beta],K[\beta]$ for $0<\beta\leq\alpha$, $\beta=1-\alpha$ by

$$S^*[\beta]=\left\{f:f\in A\text{ and }\left|\frac{zf'(z)}{f(z)}-1\right|<\beta\text{ for }z\in U\right\},$$

$$K[\beta]=\left\{f:f\in A\text{ and }\left|\frac{zf''(z)}{f'(z)}\right|<\beta\text{ for }z\in U\right\}.$$

The classes $S^*[\beta]$ and $K[\beta]$ are subclasses of $S^*(\alpha)$ and $K(\alpha)$, respectively [2] (when $\beta=1-\alpha$).

Let consider the hypergeometric function

$$F(a,b,c,d;e;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)}z^n, \tag{1}$$

where $e \neq 0, -1, -2, \cdots$ and (λ, n) is the Pochhammer symbol defined by the relations

$$
\begin{aligned}
(a,n) &= \frac{\Gamma(a+n)}{\Gamma(a)},\\
(a,-n) &= \frac{(-1)^n}{(1-a,n)},\ \frac{(a,n)}{(a,n-1)} = a+n-1\\
(a,n+m) &= (a,m)(a+m,n),
\end{aligned}
\tag{2}
$$

where n is an integer. The series (1) may be written

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!} z^n\\
&= 1 + \frac{abcd}{1.e} z + \frac{a(a+1)b(b+1)c(c+1)d(d+1)}{1.2\ e(e+1)} z^2 + \cdots .
\end{aligned}
$$

This series may be regarded as a representation of the Gauss hypergeometric function which we denote by the symbol ${}_4F_1(a,b,c,d;e;z)$. The elements a, b, c, d and e of the hypergeometric series are called the parameters and the element z is called the variable of the series. Also, the general term of the Gaussian series is given by

$$u_n = \frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!} z^n$$

so that

$$\frac{u_{n+1}}{u_n} = \frac{(a+n)(b+n)(c+n)(d+n)}{(e+n)(1+n)} z$$

from d'Alembert's ratio test, the series (1) converges for all z, real or complex in $|z| < 1$ and diverges in $|z| > 1$. When $z = 1$, it converges absolutely if $\mathrm{Re}(e-d-c-b-a) > 0$ and diverges if $\mathrm{Re}(e-d-c-b-a) \leq 0$.

In addition the function $F(a,b,c,d;e;1)$ is defined with the gamma function in the following form:

$$F(a,b,c,d;e;1) = \frac{\Gamma(e)\Gamma(e-d-c-b-a)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}.$$

A special case of this kind of functions is ${}_2F_1\ (a,b;c;z)$ which was considered by M.K. Aouf, H.M. Hossen and A.Y. Lashin; they obtained several growth and distortion properties of functions in the class of operators of fractional integral and fractional derivative [1]. Also it was considered by H.M. Srivastava and S. Owa in [9] and many other works by S. Ponnusamy, e.g. [5], [6]. A quote from M. Jahangiri [4] reads: "A new criterion for close-to-convexity of partial sums of certain hypergeometric functions. It is well-known that hypergeometric and univalent or multivalent functions play important roles in a large variety of problems encountered in probability, statistics, operations research, applied mathematics and other areas, etc."

In this paper we introduce a new approach for studying the relationship between classes of hypergeometric ${}_4F_1$ and analytic univalent functions.

We need the following theorems that Silverman has shown in [8].

THEOREM (I). *Suppose* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$. *Then* $f \in S^*[\beta]$ *if and only if* $\sum_{n=2}^{\infty} (n-\alpha)(a_n) \leq \beta$. *Also* $f \in S^*[\beta]$ *if and only if* $f \in S^*(\alpha)$, *when* $\beta = 1 - \alpha$.

THEOREM (II). *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$. *Then* $f \in K[\beta]$ *if and only if* $\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq \beta$. *Also a necessary and sufficient condition for* $f(z)$ *to be in* $K(\alpha)$ *is* $f(z) \in K[\beta]$ *(when* $\beta = 1 - \alpha$*).*

THEOREM (III). *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$. *Then:*

1) $f \in S^*[\beta]$ *if* $\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq \beta$.
2) $f \in K[\beta]$ *if* $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \beta$
3) $f \in K[\beta]$ *implies* $f \in K(\alpha)$
4) $f \in S^*[\beta]$ *implies* $f \in S^*(\alpha)$.

Also we use the concepts in [8], [7], [3]. Finally, we prove that the class $H(a, b, c, d, e, z)$ is closed under convex linear combinations and we study convolution of members of itself in the open unit disk U.

2. STARLIKENESS PROPERTY

THEOREM 1. *Let* $a, b, c, d > 0$, $e > a + b + c + d + 3, 0 \leq \alpha < 1$, $\beta = 1 - \alpha$. *Then the hypergeometric function* $zF(a, b, c, d; e; z)$ *belongs to* $S^*[\beta]$ *if*

(3)
$$\frac{\Gamma(e)\Gamma(B)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\left[1 + \frac{abcd}{(1-\alpha)(B-3)(B-2)(B-1)}\right] \leq 2,$$

where $B = e - d - c - b - a$.

Proof. We have

(4)
$$\begin{aligned} zF(a, b, c, d; e; z) &= z\left(\sum_{n=0}^{\infty} \frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)} z^n\right) \\ &= z + \sum_{n=2}^{\infty} \frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!} z^n. \end{aligned}$$

On the other hand, in order to use Theorem (III), we get

$$G(a,b,c,d,e,\alpha)=\sum_{n=2}^{\infty}(n-\alpha)\left|\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}\right|.$$

Therefore

$$\begin{aligned}G(a,b,c,d,e,\alpha)&=\sum_{n=1}^{\infty}(n+1-\alpha)\left(\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\right)\\&=\sum_{n=1}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(n-1)!}+\sum_{n=1}^{\infty}(1-\alpha)\left(\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\right).\end{aligned}$$

By (2) we have

$$\begin{aligned}&G(a,b,c,d,e,\alpha)\\&=\frac{abcd}{e}\sum_{n=1}^{\infty}\frac{(a+1,n-1)(b+1,n-1)(c+1,n-1)(d+1,n-1)}{(e+1,n-1)(n-1)!}\\&+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\\&=\frac{abcd}{e}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)(c+1,n)(d+1,n)}{(e+1,n)n!}\\&+(1-\alpha)\sum_{n=0}^{\infty}\left(\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}-1\right)\\&=\frac{abcd}{e}F(a+1,b+1,c+1,d+1;e+1;1)+(1-\alpha)[F(a,b,c,d;e;1)-1]\\&=\frac{abcd}{e}\left(\frac{\Gamma(e+1)\Gamma(e-d-c-b-a-3)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\right)\\&+(1-\alpha)\left(\frac{\Gamma(e)\Gamma(e-d-c-b-a)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}-1\right).\end{aligned}$$

By using the two relations of $\Gamma(e+1)=e\Gamma(e)$ and

$$\Gamma(e-d-c-b-a-3)=\frac{\Gamma(B)}{(B-3)(B-2)(B-1)}\text{ for }B=e-d-c-b-a$$

we obtain

$$\begin{aligned}G(a,b,c,d,e,\alpha)&=\frac{\Gamma(e)\Gamma(e-d-c-b-a)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\\&\left[\frac{abcd}{(B-3)(B-2)(B-1)}+(1-\alpha\right]-(1-\alpha).\end{aligned}\tag{5}$$

But $G(a,b,c,d,e,\alpha)$ is bounded by $(1-\alpha)$ if and only if

$$\frac{\Gamma(e)\Gamma(B)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\left[1+\frac{abcd}{(1-\alpha)(B-3)(B-2)(B-1)}\right]\leq 2$$

for $\beta=1-\alpha$. Then, according to Theorem (III), $F(a,b,c,d;e;z)\in S^*[\beta]$. $\square$

THEOREM 2. *Let $a,b,c,d>-1$, $e>0$, $abcd<0$. Then $zF(a,b,c,d;e;z)\in S^*(\alpha)$ if $e\geq a+b+c+d+3-\frac{abcd}{1-\alpha}$. Also, the condition $e\geq a+b+c+d+3-abcd$ is necessary and sufficient for $zF(a,b,c,d;e;z)$ to be in S.*

Proof. We have

$$\begin{aligned} zF(a,b,c,d;e;z) &= z\left(\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)}z^n\right)\\ &= z+\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}z^n\\ &= z-\left|\frac{abcd}{e}\right|\sum_{n=2}^{\infty}\frac{(a+1,n-2)(b+1,n-2)(c+1,n-2)(d+1,n-2)}{(e+1,n-2)(n-1)!}z^n. \end{aligned}$$

By using Theorem (I) we must show that

$$\begin{aligned} &\sum_{n=2}^{\infty}(n-\alpha)\frac{(a+1,n-2)(b+1,n-2)(c+1,n-2)(d+1,n-2)}{(e+1,n-2)(n-1)!}\\ &\leq\frac{e}{abcd}(1-\alpha)\leq\left|\frac{e}{abcd}\right|(1-\alpha). \end{aligned}\tag{6}$$

Let the left side of (6) be denoted by $L(a,b,c,d,e,\alpha)$. Thus

$$\begin{aligned} L(a,b,c,d,e,\alpha) &= \sum_{n=0}^{\infty}(n+2-\alpha)\frac{(a+1,n)(b+1,n)(c+1,n)(d+1,n)}{(e+1,n)(n+1)!}\\ &= \sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)(c+1,n)(d+1,n)}{(e+1,n)n!}\\ &\quad+\sum_{n=1}^{\infty}(1-\alpha)\frac{e}{abcd}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\\ &= \frac{\Gamma(e+1)\Gamma(e-a-b-c-d-3)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\\ &\quad+(1-\alpha)\frac{e}{abcd}\left[\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}-1\right]. \end{aligned}$$

Therefore

$$
\begin{aligned}
L(a,b,c,d,e,\alpha) &= \frac{\Gamma(e+1)\Gamma(e-a-b-c-d-3)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} \\
&\quad \left[1+(1-\alpha)\frac{e-a-b-c-d-3}{abcd}\right] \\
&\leq (1-\alpha)\left[\frac{e}{|abcd|}+\frac{e}{abcd}\right] = 0.
\end{aligned}
$$

Hence $1+(1-\alpha)\frac{e-a-b-c-d-3}{abcd} \leq 0$, where $cabd < 0$ or $e \geq a+b+c+d+3$. □

Corollary 1. *Let $F_1(a,b,c,d;e;z) = z(2-F(a,b,c,d;e;z)$. Then we have $F_1(a,b,c,d;e;z) \in S^*(\alpha), S^*[\beta]$ if and only if the parameters a,b,c,d,e satisfy in the condition* (3).

Proof. By using (4) and (5), we have

$$
\begin{aligned}
F_1(a,b,c,d;e;z) &= z\left(2-\left(\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)}z^n\right)\right) \\
&= z\left(2-\left(1+\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(1,n-1)}z^{n-1}\right)\right) \\
&= z-\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(1,n-1)}z^n.
\end{aligned}
$$

Similarly, by making use of Theorem 1, we obtain that the relation (3) is a necessary condition for $F_1(a,b,c,d;e;z) \in S^*(\alpha), S^*[\beta]$. □

3. CONVEXITY PROPERTY

Theorem 3. *Let $a,b,c,d > 0$, $e > a+b+c+d+1$. Then $zF(a,b,c,d;e;z) \in K[\beta]$, $0 \leq \alpha < 1$, $\beta = 1-\alpha$ if*

$$
\begin{aligned}
&\frac{\Gamma(e-1)\Gamma(e-1-a-b-c-d+1)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} \\
&\times\left[\frac{abcd}{(1-\alpha)(e-a-b-c-d-1,2)}+\left(\frac{3-\alpha}{1-\alpha}\right)\frac{1}{e-a-b-c-d}+1\right] \leq 2. \qquad (7)
\end{aligned}
$$

Proof. By using (4), we have

$$
zF(a,b,c,d;e;z) = z+\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(1,n-1)}z^n.
$$

According to Theorem (III), we must show that

$$
\sum_{n=2}^{\infty} n(n-\alpha)\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(c,n-1)(1,n-1)} \leq \beta. \qquad (8)
$$

Let the left side of (8) be denoted by $N(a,b,c,d,e,\alpha)$. Thus

$$\begin{aligned} N(a,b,c,d,e,\alpha) &= \sum_{n=1}^{\infty}(n+1-\alpha)\frac{(a,n)(b,n)(c,n)(d,n)}{(c,n)(1,n)} \\ &= \sum_{n=1}^{\infty}(n+1)^2\frac{(a,n)(b,n)(c,n)(d,n)}{(c,n)(1,n)} \\ &\quad -\alpha\sum_{n=1}^{\infty}(n+1)\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)}. \end{aligned}$$

Writing $(A,n)=\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}(n+1)(n+1-\alpha)\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)} \\ &= \sum_{n=1}^{\infty}(n+1)^2\frac{(A,n)}{(1,n)} - \alpha\sum_{n=1}^{\infty}\frac{(n+1)(A,n)}{(1,n)}, \end{aligned} \tag{9}$$

$$\begin{aligned} \sum_{n=1}^{\infty}(n+1)^2\frac{(A,n)}{(1,n)} &= \sum_{n=1}^{\infty}(n^2+2n+1)\frac{(A,n)}{(1,n)} \\ &= \sum_{n=1}^{\infty}(A,n)\left(\frac{n}{(1,n-1)}+\frac{2}{(1,n-1)}+\frac{1}{(1,n)}\right) \\ &= \sum_{n=1}^{\infty}\frac{(A,n)}{(1,n-2)} + 3\sum_{n=1}^{\infty}\frac{(A,n)}{(1,n-1)} + \sum_{n=1}^{\infty}\frac{(A,n)}{(1,n)}, \end{aligned} \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{(n+1)(A,n)}{(1,n)} = \sum_{n=1}^{\infty}\frac{(A,n)}{(1,n-1)} + \sum_{n=1}^{\infty}\frac{(A,n)}{(1,n)}. \tag{11}$$

We put (10), (11) in (9), yielding

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[\frac{(A,n)}{(1,n-2)} + (3-\alpha)\frac{(A,n)}{(1,n-1)} + \frac{(1-\alpha)(A,n)}{(1,n)}\right] \\ &= \sum_{n=0}^{\infty}\frac{(A,n+1)}{(1,n-1)} + \sum_{n=1}^{\infty}(3-\alpha)\frac{(A,n)}{(1,n-1)} + \sum_{n=0}^{\infty}\frac{(1-\alpha)(A,n-1)}{(1,n-1)} \\ &= \frac{(a,1)(b,1)(c,1)(d,1)\Gamma(e+1)\Gamma(e-a-b-c-d-1)}{(e,1)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} \\ &\quad + (3-\alpha)\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} \\ &\quad + (1-\alpha)\left[\frac{\Gamma(e-1)\Gamma(e-a-b-c-d+1)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} - 1\right]. \end{aligned} \tag{12}$$

The relation (12) is bounded by $(1-\alpha)$ if and only if (7) holds. This completes the proof. □

Corollary 2. *Let* $F_1(a,b,c,d;e;z) = z(2-F(a,b,c,d;e;z)$. *Then we have* $F_1(a,b,c,d;e;z) \in K[\beta], K(\alpha)$ *if and only if the parameters* a,b,c,d,e *satisfy the condition* (7).

Proof. By using Theorem (II) and a similar way with the proof of Corollary 1, this proof is sharp. □

Theorem 4. *Let* $zF(a,b,c,d;e;z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ *and* $F(a,b,c,d;e;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(1,n)} z^n$. *Then* $|b_n| \leq \frac{2}{n}$, $n \in N$.

Proof. We proved in (4) that $b_n = \frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}$. Then [3, Lemma] implies that

$$\left|\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1).(n-1)!}\right| \leq \frac{2}{n},\ n \in N.$$

□

4. CONVOLUTION OPERATOR

Definition. Let $g(z)$ be an analytic, univalent function in the unit disk $U = \{z : |z| < 1\}$ defined in the following form: $g(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and denote by $F(a,b,c,d;e;z)$ the Gaussian hypergeometric function as defined in (1) for $|z| < 1$. Also let A denote the class of every $g(z)$, $z \in U$. Then we define the Hadamard or convolution operator $HO_{a,b,c,d,e}(g)(z)$ as

$$\begin{aligned} HO_{a,b,c,d,e}(g)(z) &= zF(a,b,c,d;e;z) * g(z) \\ &= z - \sum_{n=1}^{\infty} \frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!} a_{n+1} z^{n+1}. \end{aligned} \tag{13}$$

Theorem 5. *Let* $g(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n$ *and* $F(a,b,c,d;e;z)$ *be the Gaussian hypergeometric function defined in* (1). *Then* $HO_{a,b,c,d,e}(g)(z)$ *belongs to* $S^*[\beta](S^*(\alpha))$ *if and only if*

(14)
$$\frac{\Gamma(e)\Gamma(B)\max\{a_n\}_{n\in N}}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\left[1 + \frac{abcd}{(1-\alpha)(B-3)(B-2)(B-1)}\right] \leq 2,$$

where $B = e-a-b-c-d$, $a,b,c,d > 0$ *and* $e > a+b+c+d+3$.

Proof. According to the definition of the operator $HO_{a,b,c,d,e}(g)(z)$ in (13) we have

$$HO_{a,b,c,d,e}(g)(z) = z - \sum_{n=2}^{\infty} \frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!} a_n z^n. \tag{15}$$

In view of Theorem (I), we need only to show that

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}a_n \leq 1-\alpha.$$

We have (see (15) and Theorem (I))

$$P(a,b,c,d,e,\alpha) = \sum_{n=1}^{\infty}(n+1-\alpha)\left(\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\right)a_{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)(n-1)!}a_{n+1} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}a_{n+1}$$

$$= \frac{abcd}{e}\sum_{n=1}^{\infty}\frac{(a+1,n-1)(b+1,n-1)(c+1,n-1)(d+1,n-1)}{(e+1,n-1)(n-1)!}a_{n+1}$$

$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}a_{n+1}$$

$$= \frac{abcd}{e}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)(c+1,n)(d+1,n)}{(e+1,n)n!}a_{n+2}$$

$$+(1-\alpha)\left[\left(\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}a_{n+1}\right) - a_1\right].$$

Suppose $A = \max\{a_n\}_{n\in N}$. By using (5) $a_1 = 1$ we have

$$P(a,b,c,d,e,\alpha) \leq \frac{abcdA}{e}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)(c+1,n)(d+1,n)}{(e+1,n)n!}$$

$$+(1-\alpha)\left[\left(A\sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}\right) - 1\right].$$

On the other hand, (5) and $B = e-d-c-b-a$ imply that

$$P(a,b,c,d,e,\alpha) \leq \left[\frac{abcdA}{e}\left(\frac{\Gamma(e+1)\Gamma(e-d-c-b-a-3)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\right)\right.$$

$$\left.+(1-\alpha)\left(A\frac{\Gamma(e)\Gamma(e-d-c-b-a)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} - 1\right)\right]$$

$$= \frac{\Gamma(e)\Gamma(e-d-c-b-a)A(1-\alpha)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}$$

$$\left[1 + \frac{abcd}{(1-\alpha)(B-3)(B-2)(B-1)}\right] - (1-\alpha).$$

Therefore $P(a,b,c,d,e,\alpha)$ is bounded by $(1-\alpha)$ if and only if (14) holds. □

THEOREM 6. *Let $g(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $F(a,b,c,d;e;z)$ be the Gaussian hypergeometric function defined in (1) for $a,b,c,d > 0$ and $e > a+b+c+d+3$. Then $HO_{a,b,c,d,e}(g)(z)$ belongs to $K[\beta]\,(K(\alpha))$ if and only if*

(16)
$$\frac{\Gamma(e-1)\Gamma(B)\max\{a_n\}_{n\in N}}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}\left[\frac{abcd}{(1-\alpha)(B-1,2)}+\left(\frac{3-\alpha}{1-\alpha}\right)\frac{1}{B}+1\right]\leq 2,$$

where $B = e-a-b-c-d$.

Proof. We replace (A,n) with $(A,n)\max\{a_n\}_{n\in N}$ in the proof of Theorem 3 and a simple calculus implies (16). □

THEOREM 7. *Let $\overline{HO}_{a,b,c,d,e}(g)(z) = z(2 - HO'_{a,b,c,d,e}(g)(z))$, where $g(z) = z - \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \in A$ and $HO_{a,b,c,d,e}(g)(z)$ defined in (13), for $\beta = 1 - \alpha$. Then the following conditions are true, where $HO'_{a,b,c,d,e}(g)(z)$ denotes $\frac{\mathrm{d}}{\mathrm{d}z}(HO)$.*

(I) *$\overline{HO}_{a,b,c,d,e}(g)(z) \in S^*[\beta](S^*(\alpha))$ if and only if (14) holds.*

(II) *$\overline{HO}_{a,b,c,d,e}g(z) \in K[\beta](K(\alpha))$ if and only if (16) holds.*

Proof. By using (15), we have

(17)
$$\begin{aligned}\overline{HO}_{a,b,c,d,e} &= z\left(1-\frac{d}{dz}\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}a_n z^n\right)\\ &= z-\sum_{n=2}^{\infty}\frac{n(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!}a_n z^n.\end{aligned}$$

Hence part (I) holds. Similarly, part (II) is true by the form (17) and Theorems 6. □

5. INTEGRAL OPERATOR

THEOREM 8. *Let $a,b,c,d,e > 0$ and $e > a+b+c+d$. Then a sufficient condition for $\int_0^z F(a,b,c,d;e;\lambda)\mathrm{d}\lambda$ to be in S^* is that*

$$\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} \leq 2. \tag{18}$$

Proof. Suppose that

$$F(a,b,c,d;e;z) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)(c,n)(d,n)}{(e,n)n!}z^n.$$

Then

$$\int_0^z F(a,b,c,d;e;\lambda)\mathrm{d}\lambda = z+\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)n!}z^n.$$

From Theorem (III), $\int_0^z F(a,b,c,d;e;\lambda)\mathrm{d}\lambda \in S^*$ if

$$\sum_{n=2}^{\infty} n\left(\frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)n!}\right) \leq 1. \tag{19}$$

We denote by $L(a,b,c,d,e)$ the left side of inequality (19). Therefore

$$\begin{aligned} L(a,b,c,d,e) &= \sum_{n=2}^{\infty} \frac{(a,n-1)(b,n-1)(c,n-1)(d,n-1)}{(e,n-1)(n-1)!} \\ &= F(a,b,c,d;e;1) - 1 = \frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} - 1, \end{aligned} \tag{20}$$

where $a,b,c,d,e > 0$ and $e > a+b+c+d$. Then (20) is bounded above by 1 if and only if (18) holds. □

REFERENCES

[1] Aouf, M.K., Hossen, H.M. and Lashin, A.Y., *Convex subclass of starlike functions*, Kyungpook Math. J., **40** (2000), 287–297.

[2] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[3] Exton, H., *Handbook of Hypergeometric Integrals*, Series: Math. and its Applications, Halsted Press [John Wiley and Sons], Chichester, New York, Brisbane, Toronto, 1978.

[4] Jahangiri, M., *A new criterion for close-to-convexity of partial sums of certain hypergeometric functions*, J. Appl. Math. Stochastic Anal., **102** (2) (1997), 197–202.

[5] Ponnusamy, S., *Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane*, Integral Transform. Spec. Funct., **8** (1999), 121–138.

[6] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51** (1975), 109–116.

[7] Silverman, H., *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172** (1993), 574–581.

[8] Silverman, H., *Starlikeness properties for convolutions involving hypergeometric series*, Ann. Univ. M. Curie-Sklodowska Sect. A, **52** (1998), 141–155.

[9] Srivastava, H.M. and Owa, S. (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, Hongkong, 1992.

Received September 19, 2004

Urmia University
Urmia, Iran
E-mail: Marzieh_Ghaedi@yahoo.com

Department of Mathematics
Fergusson College
411004 Pune, India
E-mail: kulkarni_ferg@yahoo.com