MATHEMATICA, Tome 47 (70), N° 2, 2005, pp. 223–230

TOTALLY REFLEXIVE, TOTALLY SYMMETRIC PATTERN ALGEBRAS

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Abstract. A k-ary relation ρ on a set A induces a partition of each power A^n into "patterns" in a natural way. An operation on A is called a ρ-pattern operation if its restriction to each pattern is a projection. We examine functional completeness of algebras with ρ-pattern fundamental operations in the case when ρ is the totally reflexive, totally symmetric relation of A.

MSC 2000. 08A40.
Key words. Majority function, semiprojection, central relation, minimal central relation, functionally completeness.

1. PRELIMINARIES

A finite algebra $\mathbf{A} = (A; F)$ is called *functionally complete* if every (finitary) operation on A is a polynomial operation of $\mathbf{A}$. A n-ary operation f on A is *conservative* if $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$ for all $x_1, \dots, x_n \in A$. An algebra is conservative if its all fundamental operations are conservative.

A possible approach to conservative operations is to consider them as relational pattern functions or ρ-pattern functions. Given a k-ary relation $\rho \subseteq A^k$, two n-tuples $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in A^n$ are of the same pattern with respect to ρ if for all $i_1, \dots, i_k \in \{1, \dots, n\}$, $(x_{i_1}, \dots, x_{ik}) \in \rho$ and $(y_{i_1}, \dots, y_{i_k}) \in \rho$ mutually imply each other. An operation $f : A^n \to A$ is a *ρ-pattern function* if $f(x_1, \dots, x_n)$ always equals some x_i, $i \in \{1, \dots, n\}$ where i depends only on the ρ-pattern of $(x_1, \dots, x_n)$. In fact, any conservative operation is a ρ-pattern function for some ρ — see [11]. If ρ is the equality relation on A, then the pattern functions whose notion was introduced by Quackenbush [5] are ρ-pattern functions. The maximum and minimum operations on chains, and the ternary discriminator, the dual discriminator studied in [2] and [4] are examples for ρ-pattern functions. An algebra is called a ρ-pattern algebra if its operations (or equivalently its term operations) are ρ-pattern functions for the same relation ρ on A. Specifically, if ρ is a totally reflexive, totally symmetric relation then the ρ-pattern algebras are called totally reflexive, totally symmetric pattern algebras. B. Csákány [1] proved that every finite ρ-pattern algebra $(A; f)$ with $|A| \geq 3$ is functionally complete if f is an arbitrary nontrivial ρ-pattern function where ρ is the equality relation on A.

The aim of this paper is to continue research on the functional completeness of finite totally reflexive, totally symmetric pattern algebras. We have already proved a series of facts concerning functional completeness, for the cases when ρ is an equivalence [14], a central relation [10], a graph of a permutation [11],

[13], a bounded partial order [12], and a regular relation [9] on A. These relations are occurring in Rosenberg's primality criterion [7]. We will use the following definitions and results.

An n-ary relation ρ on A is called *central* if $\rho \neq A^n$ and there exists a nonvoid proper subset C of A such that

(a) $(a_1, \dots, a_n) \in \rho$ whenever at least one $a_j \in C$ $(1 \leq j \leq n)$;
(b) ρ is *totally reflexive*, i.e. $(a_1, \dots, a_n) \in \rho$ if $a_i = a_j$ for some $i \neq j$, $(1 \leq i, j \leq n)$,
(c) ρ is *totally symmetric*, i.e. $(a_1, \dots, a_n) \in \rho$ implies $(a_{1\pi}, \dots, a_{n\pi}) \in \rho$ for every permutation π of the indices $1, \dots, n$.

Note that every unary relation C distinct form $\emptyset$ and A is central. A n-ary central relation ρ on A is called *minimal central relation* if $(a_1, \dots, a_n) \notin \rho$ with all pairwise different noncentral elements $a_1, \dots, a_n \in A$. The compatible binary reflexive symmetric relations of $\mathbf{A}$ are called tolerance relations of $\mathbf{A}$. If $\mathbf{A}$ has no nontrivial tolerance then $\mathbf{A}$ is a *tolerance-free* algebra. A ternary operation f on A is a *majority function* if for all $x, y \in A$ $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$ holds. By an n-ary i-th *semiprojection* on $A (n \geq 3, 1 \leq i \leq n)$ we mean an operation f with the following property $f(x_1, \dots, x_n) = x_i$ whenever at least two elements among $x_1, \dots, x_n$ are equal.

The following proposition was got in [13] from Rosenberg's fundamental theorem [6].

Proposition 1. *The clone of term operations of every nontrivial finite ρ-pattern algebra* $\mathbf{A}$ *with at least three elements contains a nontrivial binary ρ-pattern function or a ternary majority ρ-pattern function, or a nontrivial ρ-pattern function which is a semiprojection.*

Now we formulate the following theorem (see [13, Proposition 4]) which also will be used.

Theorem 2. *Let* $\mathbf{A}$ *be an at least three element finite simple conservative algebra.* $\mathbf{A}$ *is functionally complete iff*

(1) $\mathbf{A}$ *has no compatible binary central relation preserved by every automorphism,*

(2) $\mathbf{A}$ *has no compatible bounded partial order ρ, such that for every automorphism π of* $\mathbf{A}$ *the relation*

$$\rho^\pi = \{(x\pi, y\pi) : (x, y) \in \rho\}$$

equals ρ or ρ^{-1}.

We need the following observation.

Remark 3. Let ρ be compatible binary central relation of $\mathbf{A}$ preserved by an automorphism π of $\mathbf{A}$. If π has a central element of ρ in one of its cycles, then this cycle only contains central elements of ρ.

2. RESULTS

If ρ is a unary relation on A, then the ρ pattern algebra $\mathbf{A}$ is not functionally complete see [14].

PROPOSITION 4. *Let ρ be an at least 3-ary totally reflexive relation on an at least three element finite set A. The ρ-pattern algebra $\mathbf{A}$ is functionally complete iff $\mathbf{A}$ is tolerance-free.*

Proof. The functionally complete algebras are tolerance-free. If $\mathbf{A}$ is a tolerance-free ρ-pattern algebra where ρ is an at least 3-ary totally reflexive relation then we show that it is functionally complete. The binary ρ-pattern functions are projections on A. If $a, b \in A$, then the patterns (a, a, b), (a, b, a), (b, a, a) are the same with respect to ρ. Thus none of the ρ-pattern functions is a majority function on A. The nontrivial ρ-pattern functions which are semiprojections do not preserve the bounded partial orders on $\mathbf{A}$ (see [3]). Using Proposition 1 we get that the algebra $\mathbf{A}$ is functionally complete. □

THEOREM 5. *Let ρ be an at least binary totally reflexive, totally symmetric relation on an at least three element finite set A. The finite ρ-pattern algebra $\mathbf{A}$ is functionally complete iff $\mathbf{A}$ is tolerance-free.*

Proof. If ρ is the equality relation on A, then our theorem is true (see [1]). If ρ is an at least 3-ary relation on A, then by the Proposition 4 our theorem is true. From now on let ρ be a binary no equality relation on A. The functionally complete ρ-pattern algebras are tolerance-free algebras. Therefore it is enough to prove that every finite tolerance-free ρ-pattern algebra $\mathbf{A}$ is functionally complete. There are two binary nontrivial ρ-pattern functions on A. One of them is

$$f(x,y) = \begin{cases} x, & \text{if} \quad (x,y) \in \rho, \\ y & \text{otherwise.} \end{cases}$$

The other function can be obtained from f by changing x and y. We will show that f does not preserve the bounded partial orders on A. From that we get that the other function does not preserve them either. Let $\leq$ be a bounded partial order on A with the least element 0, and the greatest element 1.

a) Let ρ be a central relation on A. If ρ has at least two central elements then ρ is not simple see [10], and the ρ- pattern algebra $\mathbf{A}$ is not tolarance-free algebra.

b) Let ρ be a central relation on A with a single central element c and $a, b \in A$, $(a, b) \notin \leq$, $(a, b) \notin \rho$. The following matrices will be used

a	a	b	b	0	c	0	c	0	0	c	k
0	b	a	1	1	1	k	k	k	c	0	0
a	b	a	b	1	c	k	c	k	0	c	0

If $c = 0$, then the first, if $c = 1$ then the second matrix shows that f does not preserve ρ. If 0 and 1 are not central elements of ρ, then there is an element

$k \in A$ with $(0, k) \notin \rho$. If $k = 1$, then the third matrix will be used. If $k \neq 0, 1$, then we have the following cases:

(1) k and c are incomparable,

(2) $k \leq c$,

(3) $c < k$.

In case (1) the fourth matrix, in case (2) the fifth matrix, and in case (3) the sixth matrix show that f does not preserve $\leq$.

c) If ρ is not a central relation on A, then the following matrices will be used

$$\begin{array}{cc} 1 & 1 \\ 0 & \mathrm{a} \\ \hline 1 & \mathrm{a} \end{array} \quad \begin{array}{cc} 0 & \mathrm{a} \\ 1 & 1 \\ \hline 1 & \mathrm{a} \end{array} \quad \begin{array}{cc} \mathrm{d} & 1 \\ \mathrm{e} & \mathrm{e} \\ \hline \mathrm{d} & \mathrm{e} \end{array} \quad \begin{array}{cc} \mathrm{e} & 1 \\ \mathrm{d} & \mathrm{d} \\ \hline \mathrm{e} & \mathrm{d} \end{array}$$

If $(0, 1) \in \rho$, then there exists element a with $(a, 1) \notin \rho$. Now the first matrice shows that f does not preserve ρ. If $(0, 1) \notin \rho$ and there exists element a with $a \neq 1$ $(a, 1) \in \rho$, then the second matrix does the work. If $(0, 1) \notin \rho$ and there does not exist element a with $a \neq 1$, $(a, 1) \in \rho$, then there are elements d, e with $d \neq e$, $(d, e) \in \rho$. In this case the third matrix with $(d, e) \notin\leq$ and the fourth matrix with $(e, d) \notin\leq$ show that f does not preserve $\leq$. Therefore the nontrivial binary ρ-pattern functions do not preserve the bounded partial orders if ρ is a binary no equality relation on A.

If ρ is a binary no equality relation on A with $a \neq b$, $(a, b) \in \rho$, then the patterns (a, a, b), (a, b, a) and (a, b, b) are the same with repect to ρ. In this case none of the ρ-pattern functions is a majority function. We have already mentioned that the nontrivial semiprojections do not preserve the bounded partial orders on A. The tolerance-free algebras are simple and have no compatible binary central relations. Therefore using Theorem 2 we get from Proposition 1 that $\mathbf{A}$ is functionally complete. □

Using Theorem 2 and Theorem 5 we can formulate the following corollary.

COROLLARY 6. *Let ρ be at least binary totally reflexive, totally symmetric relation on an at least three element finite set A. The finite simple ρ-pattern algebra $\mathbf{A}$ is functionally complete iff $\mathbf{A}$ has no compatible binary central relation preserved by every automorphism.*

CLAIM 7. *Let ρ be an at least binary central relation on a finite set A. Then the ρ-pattern algebra (A, f) is not functionally complete if f is a binary ρ-pattern function.*

Indeed, if ρ is a binary central relation on A, then let C be the center of ρ. The binary ρ-pattern functions preserve the nontrivial equivalence with blocks C, $A \backslash C$. Therefore the algebra $(A; f)$ is not functionally complete.

If ρ is an at least 3-ary central relation on A, then the binary ρ-pattern functions are projections.

Claim 8. *Let ρ be an at least binary central relation on a finite set A. Then there exists a nontrivial ρ-pattern function f which is a semiprojection and the algebra $(A; f)$ is not functionally complete.*

Indeed, if ρ is a binary central relation on A, then the function

$$f(x_1, x_2, x_3) = \begin{cases} x_3, & \text{if} \quad (x_1, x_2), (x_2, x_3) \in \rho \text{ and } \ (x_1, x_3) \notin \rho \\ x_1 & \text{otherwise} \end{cases}$$

is a nontrivial ρ-pattern function, which is a semiprojection.

If ρ is an at least 3-ary central relation on A, then the function

$$g_n(x_1, \dots, x_n) = \begin{cases} x_n, & \text{if} \quad (x_1, \dots, x_n) \in \rho, \\ x_1 & \text{otherwise} \end{cases}$$

is a nontrivial ρ-pattern function which is also a semiprojection. It is easy to see that f and g preserve the nontrivial equivalence with block C, $A \setminus C$ where C is the center of ρ. The proof is completed.

Proposition 9. *Let ρ be an at least binary minimal central relation with the center $\{c\}$ on an finite set A. The finite simple ρ-pattern algebra $\mathbf{A}$ is functionally complete iff $\mathbf{A}$ has no compatible binary minimal central relation with the center $\{c\}$.*

Proof. The algebra $\mathbf{A}$ is a nontrivial algebra. From the Theorem 5 we get that $\mathbf{A}$ has no compatible bounded partial order. The central element c is the fixed point of the automorphisms of ρ. These automorphisms are also the automorphisms of $\mathbf{A}$. They can preserve only that binary minimal central relation whose center is $\{c\}$. Using the Theorem 2 and Remark 3 we get the proof.

Define the following subsets of A^n:

$A_1 := \{(c, a_2, \dots, a_n) \in A^n : c, a_2, \dots, a_n$ are pairwise different elements$\}$,

$\vdots$

$A_n := \{(a_1, a_2, \dots, c) \in A^n : a_1, a_2, \dots, c$ are pairwise different elements$\}$,

$B := \{(a_1, \dots, a_n) \in A^n : \exists i, j, (1 \leq i \neq j \leq n)$ with $a_i = a_j\}$,

$D := \{(a_1, \dots, a_n) \in A^n : a_1, a_2, \dots, a_n$ are pairwise different noncentral elements$\}$. □

Theorem 10. *Let ρ be an at least binary minimal central relation with the center $\{c\}$ on a finite set A. If f is a n-ary nontrivial ρ-pattern function which is an i-th semiprojection on A then the algebra $(A; f)$ is functionally complete unless*

(a) $f(x_1, \dots, x_n) = c \quad iff \quad x_i = c;$ *or*

$$f(x_1, \dots, x_n) = \begin{cases} x_i, & if \quad (x_1, \dots, x_n) \in B \ \ or \ \ (x_1, \dots, x_n) \in D, \\ c & otherwise; \end{cases}$$

or

(b) $f(x_1,\ldots,x_n)=f_M(x_1,\ldots,x_n)$ *where*

(c) $$f_M(x_1,\ldots,x_n)=\begin{cases} x_i, & \mathit{if}\ (x_1,\ldots,x_n)\in B\ \mathit{or}\ (x_1,\ldots,x_n)\in D\ \ \mathit{or}\\ & \quad (x_1,\ldots,x_n)\in A_j,\\ & j\in M\subseteq\{1,\ldots,i-1,i+1,\ldots,n\}\\ c & \mathit{otherwise}.\end{cases}$$

Proof. We need the following matrices:

$$
\begin{array}{cc}
a_1 & a_1\\
\vdots & \vdots\\
a_i & a_n\\
\vdots & \vdots\\
c & c\\
\vdots & \vdots\\
a_n & a_n\\
\hline
c & a_n
\end{array}
\qquad
\begin{array}{cc}
a_1 & c\\
\vdots & \vdots\\
a_i & a_i\\
\vdots & \vdots\\
c & c\\
\vdots & \vdots\\
a_n & a_n\\
\hline
c & a_i
\end{array}
\qquad
\begin{array}{cc}
a_1 & a_1\\
\vdots & \vdots\\
c & c\\
\vdots & \vdots\\
a_k & a_k\\
\vdots & \vdots\\
a_n & a_1\\
\hline
a_k & c
\end{array}
\qquad
\begin{array}{cc}
a_1 & c\\
\vdots & \vdots\\
c & c\\
\vdots & \vdots\\
a_k & a_k\\
\vdots & \vdots\\
a_n & a_n\\
\hline
a_k & c
\end{array}
\qquad
\begin{array}{cc}
c & a_1\\
\vdots & \vdots\\
a_i & a_i\\
\vdots & \vdots\\
c & a_k\\
\vdots & \vdots\\
a_n & a_n\\
\hline
a_i & a_k
\end{array}
\qquad
\begin{array}{cc}
a_1 & c\\
\vdots & \vdots\\
a_i & a_i\\
\vdots & \vdots\\
c & c\\
\vdots & \vdots\\
a_k & c\\
\vdots & \vdots\\
a_n & c\\
\hline
a_k & a_i
\end{array}
\qquad
\begin{array}{cc}
a_1 & c\\
\vdots & \vdots\\
c & a_i\\
\vdots & \vdots\\
a_k & c\\
\vdots & \vdots\\
a_n & c\\
\hline
a_k & a_i.
\end{array}
$$

If (a) comes true then it is easy to prove that f preserves the equivalence ε on A with the two blocks $\{c\}$, $A\backslash\{c\}$. In this case the algebra $(A;f)$ is not functionally complete.

If (a) is not true, we have two cases. In the first case there exists pattern $(a_1,\ldots,a_i,\ldots,c,\ldots,a_n)\in A_j$ with $j\neq i$, $f(a_1,\ldots,a_i,\ldots,c,\ldots,a_n)=c$. Let ε be an arbitrary nontrivial equivalence on A. We show that f does not preserve ε. If $\{c\}$ is a block of ε, then there exist a_i, $a_n\in A$ with $a_i\neq a_n$, $(a_i,a_n)\in\varepsilon$. In this case the first matrix shows that f does not preserve ε. Now let $a_1,a_i,c\in A$ be with $c\neq a_1$, $(c,a_1)\in\varepsilon$, $(c,a_i)\notin\varepsilon$. Now the second matrix will be used. In the second case we can suppose that there is not pattern $(a_1,\ldots,c,\ldots,a_n)\in A_j$ with $1\leq j\leq n$, $f(a_1,\ldots,c,\ldots,a_n)=c$. Suppose that $f(a_1,\ldots,c,\ldots,a_k,\ldots,a_n)=a_k$ if $(a_1,\ldots,c,\ldots,a_k,\ldots,a_n)\in A_i$. Let ε be an arbitrary nontrivial equivalence on A. We show that f does not preserve ε. If ε has a block with a single element c and $a_1,a_n\in A$ with $a_1\neq a_n$, $(a_1,a_n)\in\varepsilon$, then the third matrix does the work. If $a_1,a_k,c\in A$ with $a_1\neq c$, $(c,a_1)\in\varepsilon$, $(c,a_k)\notin\varepsilon$, then in this case the fourth matrix shows that f does not preserve ε. We proved that the algebra $(A;f)$ is simple if (a) is not true.

If (b) or (c) comes true, then it is easy to see that f preserves the binary minimal central relation τ on A with the center $\{c\}$. c.

If (b) and (c) do not come true, then we show that f does not preserve τ. In the first case we can suppose that

$$f(a_1,\ldots,a_k,\ldots,a_n)=a_k,\quad k\neq i,$$

with $(a_1, \dots, a_k, \dots, a_n) \in D$. Then we get from the fifth matrix that f does not preserve τ. In the second case we suppose that there exists $j \neq k$, $(1 \leq j, k \leq n)$ for which $(a_1, \dots, c, \dots, a_k, \dots, a_n) \in A_j$ with

$$f(a_1, \dots, c, \dots, a_k, \dots, a_n) = a_k, \quad k \neq i.$$

If $j \neq i$ and $(a_1, \dots, a_i, \dots, c, \dots, a_k, \dots, a_n) \in A_j$ then the sixth matrix shows that f does not preserve τ. If $j = i$, then in case $(a_1, \dots, c, \dots, a_k, \dots, a_n) \in A_j$ we use the seventh matrix. The proof is completed. □

THEOREM 11. *Let ρ be an at least binary minimal central relation with the center $\{c\}$ on a finite set A. The ρ-pattern algebra* **A** *is functionally complete iff*

(1) **A** *has no compatible equivalence with blocks $\{c\}$, $A\backslash\{c\}$,*
(2) **A** *has no compatible binary minimal central relation with the center $\{c\}$.*

Proof. We can see in the proof of Claim 7 and Theorem 10 that one of the term operations of **A** which can generate minimal clone preserves the equivalence with blocks $\{c\}$, $A\backslash\{c\}$ or does not preserve the nontrivial equivalences on A. Therefore one of the term operations of **A** preserves the equivalence with blocks $\{c\}$, $A\backslash\{c\}$ or does not preserve the nontrivial equivalences on A. Using the Proposition 9 we get our theorem. □

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Received September 18, 2004

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