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ON A NEW CLASS OF RAPIDLY VARYING FUNCTIONS

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Abstract. We introduce a new class K of the perfect Karamata's kernels which includes already known classes of rapidly varying functions in Karamata's sense. Characterisation theorems for this class are given. It is also shown that the classical weighted means of arbitrary order preserves regular variation of the target function if and only if weights are from the class K.

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1. INTRODUCTION

Karamata's theory of Regular Variation have applications in many branches of Real and Complex Analysis, Probability theory etc.

We begin with some basic definitions.

DEFINITION 1. It is said that a positive, measurable function $\ell(\cdot)$ is slowly varying (at infinity) if

$$\ell(\lambda x) \sim \ell(x) \qquad (x \to \infty),$$

for each $\lambda > 0$.

Some examples of slowly varying functions are

$$1, \quad \log^a x, \quad \log^b(\log x), \quad \exp(\log^c x), \quad \exp\left(\frac{\log x}{\log\log x}\right); \quad a, b \in R, \ 0 < c < 1.$$

The sum, product and quotient of two slowly varying functions are also slowly varying.

Functions of the form

$$f(x) := x^\rho \ell(x)$$

are *regularly varying* with index ρ, $\rho \in R$. An excellent survey on this topic is given in [1] and [2].

1.1. Apart from regular variation we shall deal here with classes of *rapidly varying* functions in the sense of Karamata, i.e. an extension of regular variation to the case $\rho = \infty$. For example, from the property of regularly varying function f with index ρ,

$$\forall \lambda > 0, \quad f(\lambda x)/f(x) \to \lambda^\rho \qquad (x \to \infty),$$

arise a natural extension to the class R_∞ of rapidly varying functions with index $\rho = \infty$.

Definition 2. (de Haan (1970) [1, p. 85]) A positive, measurable function g belongs to the class R_∞ if, as $x \to \infty$,

$$\frac{g(\lambda x)}{g(x)} \to \begin{cases} 0 & 0 < \lambda < 1, \\ 1 & \lambda = 1, \\ \infty & \lambda > 1 \end{cases} .$$

Remark 1. *Note that to establish $g \in R_\infty$, only $g(\lambda x)/g(x) \to \infty$ for $\lambda > 1$ has to be proved.*

According to Heiberg's result [5], $g \in R_\infty$ is bounded away from 0 and ∞ on every finite interval sufficiently far to the right. In the sequel we shall suppose, without loss of generality, that g (and ℓ, as well) is locally bounded on $[1, \infty)$.

1.2. Another extension to the class Θ of rapidly varying functions follows from a variant of Karamata's Theorem [1, pp. 26–30].

Let f be positive and locally integrable on $[1, \infty)$ and $\rho > 0$. Then

$$\tilde{f}(x) := xf(x)/\int_1^x f(t)\mathrm{d}t \to \rho \qquad (x \to \infty),$$

if and only if f varies regularly with index $\rho - 1$.

Therefore we get

Definition 3. *A positive, locally integrable on $[1, \infty)$ function h belongs to the class Θ if*

$$\tilde{h}(x) \to \infty \qquad (x \to \infty).$$

Some important properties of the class Θ are given in [3].

1.3. Those classes of rapidly varying functions are not comparable. Namely, there exist measurable functions f_1, f_2 such that $f_1 \in R_\infty, f_1 \notin \Theta$ and $f_2 \in \Theta, f_2 \notin R_\infty$.

Indeed, we can take $f_1(x) := \mathrm{e}^x$ except at the points $x = \mathrm{e}^n,\ n \in N$, where we put $f_1(\mathrm{e}^n) := \mathrm{e}^{\mathrm{e}^n - 2n}$. Using Definition 2, it is easy to verify that $f_1 \in R_\infty$. But

$$\tilde{f}_1(\mathrm{e}^n) = \mathrm{e}^{\mathrm{e}^n - n}/\int_1^{\mathrm{e}^n} \mathrm{e}^t\mathrm{d}t \to 0 \qquad (n \to \infty).$$

Hence $\liminf_{x\to\infty} \tilde{f}_1(x) = 0$ i.e. $f_1 \notin \Theta$.

For the second example, let $\{p_n\}_{n\geq 1}$ be the sequence of primes and define $f_2(x) := \mathrm{e}^x$ except at the points $x = p_n$ where $f_2(p_n) := \mathrm{e}^{2p_n}$.

Since $\forall x \geq 1,\ f_2(x) \geq \mathrm{e}^x$, we get

$$\tilde{f}_2(x) \geq x\mathrm{e}^x/\int_1^x \mathrm{e}^t\mathrm{d}t \to \infty \qquad (x \to \infty);$$

hence $f_2 \in \Theta$.

But $\liminf_{x\to\infty} f_2(2x)/f_2(x) = 1$, i.e. $f_2 \notin R_\infty$.

1.4. Karamata's Theorem states that, for $\rho > 0$,

$$\int_1^x t^{\rho-1}\ell(t)\mathrm{d}t \sim \ell(x)\int_1^x t^{\rho-1}\mathrm{d}t \quad (x\to\infty). \tag{A}$$

In accordance with (A), we define the class K of rapidly varying functions as

Definition 4. *A positive, locally integrable on* $[1,\infty)$, *kernel* $C(\cdot)$ *belongs to the class* K *if the asymptotic relation*

$$\int_1^x f(t)C(t)\mathrm{d}t \sim f(x)\int_1^x C(t)\mathrm{d}t \qquad (x\to\infty),$$

takes place for every regularly varying function $f(\cdot)$ *of arbitrary index.*

We named K as the class of *perfect Karamata's kernels* in honor of the hundreth anniversary of Karamata's birthday. Besides, although the classes R_∞ and Θ are uncomparable, the class K includes both of them (Theorem 3, below).

1.5. Another remarkable property (Theorem 1) is the fact that the classical weighted means preserves regular variation of the target function if and only if the weight function belongs to the class K.

Recall that the classical weighted means $M^{(r)}(p,f,x)$ of order $r,\ r\in R$ with weight function $p(\cdot)$, are defined by

Definition 5. [4, p. 74]

$$M^{(r)}(p,f,x) := \left(\frac{\int_1^x p(t)(f(t))^r\mathrm{d}t}{\int_1^x p(t)\mathrm{d}t}\right)^{1/r}, \quad r\in R/\{0\};$$

$$M^{(0)}(p,f,x) := \exp\frac{\int_1^x p(t)\log f(t)\mathrm{d}t}{\int_1^x p(t)\mathrm{d}t}.$$

1.6. Finding a representation form for the classes of rapidly varying functions is an open and difficult problem since their structure is very ambiguous. For instance, we proved in [3] that the class Θ is not closed under multiplication. In this sense is the following observation.

Remark 2. *From Definition* 4, *it follows that if a function* f *is in the class* K, *it is still in* K *if changed in a denumerable number of points.*

2. RESULTS

Let us start with the following

Theorem 1. *For a positive, measurable weight function* $p(\cdot)$, *the asymptotic equivalence*

$$M^{(r)}(p,f,x)\sim f(x) \qquad (x\to\infty),$$

holds for each $r\in R$ *and arbitrary regularly varying function* $f(\cdot)$ *if and only if* $p\in K$.

The next is a *Characterization Theorem* for the class K.

THEOREM 2. *The following assertions are equivalent*

(i) $p \in K$;
(ii) $\int_1^x p(t)\mathrm{d}t \in \Theta$;
(iii) $\int_1^x p(t)\mathrm{d}t \in R_\infty$;
(iv) $\frac{\int_1^x p(t)\log f(t)\mathrm{d}}{\int_1^x p(t)\mathrm{d}t} - \log f(x) \to 0 \ (x \to \infty)$,

where $f(\cdot)$ *is a regularly varying function of arbitrary index.*

Another property mentioned above is

THEOREM 3. *The classes* R_∞ *and* Θ *are proper subclasses of the class* K.

3. PROOFS

3.1. PROOF OF THEOREM 1. That the condition $p \in K$ is necessary, one can see putting $r = 1$. Then, by Definition 5.,

$$M^{(1)}(p, f, x) := \int_1^x p(t)f(t)\mathrm{d}t / \int_1^x p(t)\mathrm{d}t \sim f(x) \quad (x \to \infty).$$

Hence, by Definition 4, it follows that $p \in K$.

Suppose now $p \in K$ and note that regular variation of $f(\cdot)$ implies regular variation of $(f(\cdot))^r$ for each fixed $r \in R/\{0\}$.

Therefore, by Definition 4 again, we get

$$\int_1^x p(t)(f(t))^r\mathrm{d}t \sim (f(x))^r \int_1^x p(t)\mathrm{d}t \quad (x \to \infty),$$

and

$$M^{(r)}(p, f, x) \sim (f^r(x))^{1/r} \sim f(x) \quad \forall r \in R/\{0\}. \tag{1}$$

For the case $r = 0$ we need the following lemma.

LEMMA 1. ([4], pp. 76) *The weighted mean* $M^{(r)}(p, f, x)$ *of order* $r \in R$ *is a strictly increasing function of* r.

Applying this lemma and (1), we obtain

$$f(x) \sim M^{(-1)}(p, f, x) < M^{(0)}(p, f, x) < M^{(1)}(p, f, x) \sim f(x) \quad (x \to \infty).$$

Hence

$$M^{(0)}(p, f, x) \sim f(x) \qquad (x \to \infty), \tag{2}$$

and the proof is done.

3.2. PROOF OF THEOREM 2. The fact that $\int_1^x p(t)\mathrm{d}t \in \Theta$ if and only if $p \in K$ is proved in [3]. We shall prove here that (ii) $\Longleftrightarrow$ (iii).

Assume firstly $\int_1^x p(t)\mathrm{d}t \in \Theta$. Since, for $\lambda > 1$,

$$\begin{aligned}\int_1^{\lambda x} \mathrm{d}t \int_1^t p(u)\mathrm{d}u &> \int_x^{\lambda x} \mathrm{d}t \int_1^t p(u)\mathrm{d}u \\ &> \int_1^x p(u)\mathrm{d}u \int_x^{\lambda x} \mathrm{d}t \\ &= x(\lambda - 1) \int_1^x p(t)\mathrm{d}t,\end{aligned}$$

we obtain

$$\frac{\int_1^{\lambda x} p(t)\mathrm{d}t}{\int_1^x p(t)\mathrm{d}t} > \frac{\lambda - 1}{\lambda} \cdot \frac{\lambda x \int_1^{\lambda x} p(t)\mathrm{d}t}{\int_1^{\lambda x} \mathrm{d}t \int_1^t p(u)\mathrm{d}u} \to \infty \qquad (x \to \infty).$$

Hence, by Remark 1 and Definition 2, $\int_1^x p(t)\mathrm{d}t \in R_\infty$.

Suppose now that $\int_1^x p(t)\mathrm{d}t \in R_\infty$. We get

$$\frac{\int_1^x \mathrm{d}t \int_1^t p(u)\mathrm{d}u}{x \int_1^x p(t)\mathrm{d}t} = \int_1^x \frac{\int_1^{x/t} p(u)\mathrm{d}u}{\int_1^x p(u)\mathrm{d}u} \frac{\mathrm{d}t}{t^2} \to 0 \qquad (x \to \infty),$$

by dominated convergence, since the integrand is bounded by t^{-2} and, by Definition 2, tends pointwise to zero.

Therefore, by Definition 3, it follows $\int_1^x p(t)\mathrm{d}t \in \Theta$.

Now we shall prove $(iv) \iff (i)$, i.e. that the asymptotic relation

$$\frac{\int_1^x p(t) \log f(t)\mathrm{d}t}{\int_1^x p(t)\mathrm{d}t} - \log f(x) \to 0 \qquad (x \to \infty), \tag{3}$$

holds for each regularly varying function $f(\cdot)$ of arbitrary index if and only if $p \in K$.

Indeed, taking logarithm on both sides of (2), we see that the condition $p \in K$ is sufficient. To prove that it is also necessary, put $f(t) = t$. By partial integration we get

$$\int_1^x p(t) \log t \ \mathrm{d}t = \log x \int_1^x p(t)\mathrm{d}t - \int_1^x \frac{\mathrm{d}t}{t} \int_1^t p(u)\mathrm{d}u.$$

Hence, by (3)

$$\int_1^x \frac{\mathrm{d}t}{t} \int_1^t p(u)\mathrm{d}u / \int_1^x p(t)\mathrm{d}t \to 0 \qquad (x \to \infty),$$

i.e., by Definition 3, it follows $\int_1^x p(t)\mathrm{d}t/x \in \Theta$.

But ([3], Proposition 1), if for some $a_0 \in R$, $x^{a_0} g(x) \in \Theta$, then also $x^a g(x) \in \Theta$ for each $a \in R$.

Therefore $\int_1^x p(t)\mathrm{d}t \in \Theta$, i.e. by (ii) it follows $p \in K$.

3.3. Proof of Theorem 3. We prove first that $\Theta \subset K$. Let $p \in \Theta$. Since

$$\frac{D(\int_1^x \mathrm{d}t \int_1^t p(u)\mathrm{d}u)}{D(x\int_1^x p(t)\mathrm{d}t)} = \frac{1}{1 + xp(x)/\int_1^x p(t)\mathrm{d}t} \to 0 \qquad (x \to \infty),$$

by [3], Lemma 1 and L' Hospital's rule, it follows

$$\frac{\int_1^x \mathrm{d}t \int_1^t p(u)\mathrm{d}u}{x\int_1^x p(t)\mathrm{d}t} \to 0 \qquad (x \to \infty).$$

Hence, by Definition 3 $\int_1^x p(u)\mathrm{d}u \in \Theta$, i.e. by Theorem 2, part (ii), $p \in K$ and we conclude that $\Theta \subseteq K$.

Now, take an arbitrary $f_0 \in K$. Define $f_1(n) := \int_1^n f_0(t)\mathrm{d}t/n$ for $n = 1, 2, \cdots$ and $f_1 = f_0$ elsewhere. Then, by Remark 2, $f_1 \in K$ but, since $\tilde{f}_1(n) = 1$, $\ f_1 \notin \Theta$. Hence, the class Θ is a proper subclass of K.

For $p \in R_\infty$, it is an easy exercise to show that also $\int_1^x p(t)\mathrm{d}t \in R_\infty$. By Theorem 2, part (iii), it follows that $p \in K$; hence $R_\infty \subseteq K$.

Suppose $p_0 \in K$. Define $p_1(2^n) = 2^n$, $\ n = 1, 2, \cdots$ and $p_1 = p_0$ elsewhere. Then, by Remark 2., $p_1 \in K$ but, since $\liminf_{x\to\infty} \frac{p_1(2x)}{p_1(x)} = 2$, $\ p_1 \notin R_\infty$, i.e. $R_\infty \subset K$.

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