MATHEMATICA, Tome 47 (70), N° 2, 2005, pp. 183–194

TWO-WEIGHT NORM INEQUALITY FOR PARABOLIC CALDERON-ZYGMUND OPERATORS

FUAD M. MUSHTAGOV

Abstract. Author establishes the boundedness of parabolic Calderon-Zygmund operators in the weighted L_p spaces on $\mathbb{R}^{n+1}$.

MSC 2000. 35K20, 35B65, 35R05.

Key words. Two-weighted inequality, weighted Lebesgue space, singular integral, parabolic Calderon-Zygmund kernel.

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x' = (x_1, ..., x_n)$, $|x'|^2 = \sum_{i=1}^n x_i^2$ and denote by $x = (x', t) = (x_1, \dots, x_n, t)$ a point in $\mathbb{R}^{n+1}$. An almost everywhere positive and locally integrable function $\omega(t), t \in \mathbb{R}$, will be called a weight. We shall denote by $L_{p,\omega}(\mathbb{R}^{n+1})$ the set of all measurable function f on $\mathbb{R}^{n+1}$ such that the norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^{n+1})} \equiv \|f\|_{p,\omega;\mathbb{R}^{n+1}} = \left(\int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(t) \mathrm{d}x \right)^{1/p}, \qquad 1 \le p < \infty,$$

is finite.

Let us now endow $\mathbb{R}^{n+1}$ with the following parabolic metric introduced by Fabes and Riviére in [4]

$$(1) \qquad d(x,y) = \rho(x-y), \qquad \text{where} \quad \rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}.$$

A ball with respect to the metric d centered at zero and of radius r is just the ellipsoid

$$\mathcal{E}_r(0) = \left\{ x \in R^{n+1} \mid \frac{|x'|^2}{r^2} + \frac{t^2}{r^4} < 1 \right\}.$$

Obviously, the unit sphere with respect to this metric coincides with the unit sphere in $\mathbb{R}^{n+1}$, i.e.

$$\partial\mathcal{E}_1(0) \equiv \Sigma_{n+1} = \left\{ x \in R^{n+1} \mid |x| = \Big(\sum_{i=1}^n x_i^2 + t^2 \Big)^{1/2} = 1 \right\}.$$

Let $\widetilde{d}(x,y) = \widetilde{\rho}(x-y)$, $\widetilde{\rho}(x) = \max(|x'|, |t|^{1/2})$, and I be a *parabolic cylinder* centered at some point x and with radius r, that is $I \equiv I_r(x) = \{y \in \mathbb{R}^{n+1} \colon |x'-y'| < r, |t-\tau| < r^2\}$. It is easy to see that for any ellipsoid $\mathcal{E}_r$ there exist cylinders $\underline{I}$ and $\overline{I}$ with measures comparable to r^{n+2} and such that $\underline{I} \subset \mathcal{E}_r \subset \overline{I}$. Obviously, this implies an equivalence of both metrics and the topologies induced by them. Later we shall use this equivalence without making reference to, except required.

It is worth noting that $\rho(x)$ has been employed in the study of singular integral operators with Calderón-Zygmund kernels of mixed homogeneity (see [4]).

Definition 1. A function K defined on $\mathbb{R}^{n+1}\setminus\{0\}$, is said to be a parabolic Calderon-Zygmund (PCZ) kernel in the space $\mathbb{R}^{n+1}$ if
i) $K\in C^{\infty}(\mathbb{R}^{n+1}\setminus\{0\})$;
ii) $K(rx',r^2t)=r^{-(n+2)}K(x',t)$ for each $r>0$, $x=(x',t)\in\mathbb{R}^{n+1}\setminus\{0\}$;
iii) $\int_{\Sigma_{n+1}}K(x)\mathrm{d}\sigma=0$, where $\mathrm{d}\sigma$ is the element of area of the sphere Σ_{n+1}.

Let K be a parabolic Calderon–Zygmund kernel and T be the corresponding integral operator $Tf(x)=p.v.\int_{\mathbb{R}^{n+1}}K(x-y)f(y)\mathrm{d}y$. We establish boundedness in weighted L_p space parabolic Calderon–Zygmund integral operators.

Theorem 1. *Let $p\in(1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(t)$, $\omega_1(t)$ be weight functions on $\mathbb{R}$ and the following three conditions are satisfied:*

(a) *there exists $b>0$ such that*

$$\sup_{|t|/4<|\tau|\leq 4|t|}\omega_1(\tau)\leq b\,\omega(t)\quad \textit{for a.e. } t\in\mathbb{R},$$

(b) $\mathcal{A}\equiv\sup_{\tau>0}\left(\int_{|t|>2|\tau|}\omega_1(t)|t|^{-p}\mathrm{d}\tau\right)\left(\int_{|t|<|\tau|}\omega^{1-p'}(t)\mathrm{d}t\right)^{p-1}<\infty,$

(c) $\mathcal{B}\equiv\sup_{\tau>0}\left(\int_{|t|<|\tau|}\omega_1(t)\mathrm{d}t\right)\left(\int_{|t|>2|\tau|}\omega^{1-p'}(t)|t|^{-p'}\mathrm{d}t\right)^{p-1}<\infty.$

Then there exists a constant c, independent of f, such that

$$\int_{\mathbb{R}^{n+1}}|Tf(x)|^p\omega_1(t)\mathrm{d}x'\mathrm{d}t\leq c\int_{\mathbb{R}^{n+1}}|f(x)|^p\omega(t)\mathrm{d}x'\mathrm{d}t \tag{2}$$

for all $f\in L_{p,\omega}(\mathbb{R}^{n+1})$. Moreover, condition (a) can be replaced by the condition

(a_1) *there exists $b>0$ such that*

$$\omega_1(t)\left(\sup_{|t|/4\leq|\tau|\leq 4|t|}\frac{1}{\omega(t)}\right)\leq b\quad \textit{for a.e. } t\in\mathbb{R}.$$

Proof. For $k\in Z$ we define $E_k=\{x\in R^{n+1}:\,2^k<|t|\leq 2^{k+1}\}$, $E_{k,1}=\{x\in R^{n+1}:\,|t|\leq 2^{k-1}\}$, $E_{k,2}=\{x\in R^{n+1}:\,2^{k-1}<|t|\leq 2^{k+2}\}$, $E_{k,3}=\{x\in R^{n+1}:\,|t|>2^{k+2}\}$. Then $E_{k,2}=E_{k-1}\cup E_k\cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^{n+1})$, we write

$$
\begin{aligned}
|Tf(x)| &= \sum_{k\in Z} |Tf(x)|\chi_{E_k}(x) \le \sum_{k\in Z} |Tf_{k,1}(x)|\,\chi_{E_k}(x)\\
&\quad + \sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(x) + \sum_{k\in Z} |Tf_{k,3}(x)|\,\chi_{E_k}(x)\\
&\equiv T_1 f(x) + T_2 f(x) + T_3 f(x),
\end{aligned}
\tag{3}
$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$. We shall estimate $\|T_1 f\|_{L_{p,\omega_1}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have

$$|\tau| \le 2^{k-1} \le |t|/2.$$

Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $|t-\tau| \ge |t|/2$. Hence

$$
\begin{aligned}
T_1 f(x) &\le c_1 \sum_{k\in Z} \left(\int_{\mathbb{R}^{n+1}} \frac{|f_{k,1}(y)|}{\rho(x-y)^{n+2}} \mathrm{d}y \right) \chi_{E_k}(t)\\
&\le c_1 \int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\rho(x-y)^{n+2}} \mathrm{d}y\\
&\le c_2 \int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\left(|x'-y'| + |t|^{1/2}\right)^{n+2}} \mathrm{d}y'\mathrm{d}\tau
\end{aligned}
$$

for any $x \in E_k$. Using this last inequality we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |T_1 f(x)|^p \omega_1(t) \mathrm{d}x'\mathrm{d}t\\
&\quad \le c_2 \left\{ \int_{\mathbb{R}^{n+1}} \left(\int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\left(|x'-y'| + |t|^{1/2}\right)^{n+2}} \mathrm{d}y'\mathrm{d}\tau \right)^p \omega_1(t)\mathrm{d}x \right\}^{1/p}.
\end{aligned}
$$

For $x = (x', t) \in \mathbb{R}^{n+1}$ let

$$
\begin{aligned}
I(t) &= \int_{\mathbb{R}^n} \left(\int_{|\tau|<|t|/2} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + |t|^{1/2})^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x'\\
&= \int_{\mathbb{R}^n} \left(\int_{|\tau|<|t|/2} \left(\int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + |t|^{1/2})^{n+2}} \mathrm{d}y' \right) \mathrm{d}\tau \right)^p \mathrm{d}x'.
\end{aligned}
$$

Using the Minkowski and Young inequalities we obtain

$$I(t) \le \left[\int_{|\tau|<|t|/2}\left(\int_{\mathbb{R}^n}|f(y',\tau)|^p \mathrm{d}y'\right)^{1/p}\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+|t|^{1/2})^{n+2}}\right)\mathrm{d}\tau\right]^p$$
$$= \left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+|t|^{1/2})^{n+2}}\right)^p$$
$$= \frac{c_3}{|t|^p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+1)^{n+2}}\right)^p$$
$$= \frac{c_4}{|t|^p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p.$$

Integrating over $\mathbb{R}$ we get

$$\int_{\mathbb{R}^{n+1}}|T_1f(x)|^p\omega_1(t)\mathrm{d}x'\mathrm{d}t \le c_5\int_{\mathbb{R}}\omega_1(t)|t|^{-p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\mathrm{d}t.$$

Since $\mathcal{A}<\infty$, the Hardy inequality

$$\int_{\mathbb{R}}\omega_1(t)|t|^{-p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p \le C\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\mathrm{d}\tau$$

holds and $C \le c'\mathcal{A}$ where c' depends only on p. In fact the condition $\mathcal{A}<\infty$ is necessary and sufficient for the validity of this inequality, (see [2], [9]). Hence, we obtain

$$\int_{\mathbb{R}^{n+1}}|T_1f(x)|^p\omega_1(t)\mathrm{d}x'\mathrm{d}t \le c\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\mathrm{d}\tau = c\|f\|^p_{L_{p,\omega}(\mathbb{R}^{n+1})}. \tag{4}$$

Let us estimate $\|T_3f\|_{L_{p,\omega_1}}$. As is easy to verify, for $x\in E_k$, $y\in E_{k,3}$ we have $|\tau|>2|t|$ and $|t-\tau|\ge|\tau|/2$. For $x\in E_k$ we obtain

$$T_3f(x)\le c_2\int_{\mathbb{R}^n}\int_{|\tau|>2|t|}\frac{|f(y)|}{\left(|x'-y'|+|\tau|^{1/2}\right)^{n+2}}\mathrm{d}y'\mathrm{d}\tau.$$

Using this last inequality we have

$$\|T_3f\|_{L_{p,\omega_1}(\mathbb{R}^{n+1})}$$
$$\le c_2\left\{\int_{\mathbb{R}^{n+1}}\left(\int_{\mathbb{R}^n}\int_{|\tau|>2|t|}\frac{|f(y)|}{\left(|x'-y'|+|\tau|^{1/2}\right)^{n+2}}\mathrm{d}y'\mathrm{d}\tau\right)^p\omega(t)\mathrm{d}x\right\}^{1/p}.$$

For $x=(x',t)\in\mathbb{R}^{n+1}$ let

$$I_1(t)=\int_{\mathbb{R}^n}\left(\int_{|\tau|>2|t|}\int_{\mathbb{R}^n}\frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}}\mathrm{d}y\right)^p\mathrm{d}x'$$
$$=\int_{\mathbb{R}^n}\left(\int_{|\tau|>2|t|}\left(\int_{\mathbb{R}^n}\frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}}\mathrm{d}y'\right)\mathrm{d}\tau\right)^p\mathrm{d}x'.$$

Using the Minkowski and Young inequalities we obtain

$$I_1(t)\le\left[\int_{|\tau|>2|t|}\left(\int_{\mathbb{R}^n}|f(y',\tau)|^p\mathrm{d}y'\right)^{1/p}\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+|\tau|^{1/2})^{n+2}}\right)\mathrm{d}\tau\right]^p$$
$$=\left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+|\tau|^{1/2})^{n+2}}\right)^p$$
$$=c_3\left(\int_{|\tau|>2|t|}|\tau|^{-p}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}y'}{(|y'|+1)^{n+2}}\right)^p$$
$$=c_4\left(\int_{|\tau|>2|t|}|\tau|^{-p}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p.$$

Integrating over $\mathbb{R}$ we get

$$\int_{\mathbb{R}^{n+1}}|T_3f(x)|^p\omega_1(t)\mathrm{d}x'\mathrm{d}t\le c_5\int_{\mathbb{R}}\omega_1(t)\left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}|\tau|^{-p}\,\mathrm{d}\tau\right)^p\mathrm{d}t.$$

Since $\mathcal{B}<\infty$, the Hardy inequality

$$\int_{\mathbb{R}}\omega_1(t)\left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\tau^{-p}\,\mathrm{d}\tau\right)^p\le C'\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\mathrm{d}\tau$$

holds and $C'\le c'\mathcal{B}$ where c' depends on n and p. In fact the condition $\mathcal{B}<\infty$ is necessary and sufficient for the validity of this inequality (see [2], [9]). Hence, we obtain

$$\|T_3f\|_{L_{p,\omega_1}(\mathbb{R}^{n+1})}\le c\left(\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\mathrm{d}\tau\right)^{1/p}=c\|f\|^p_{L_{p,\omega}(\mathbb{R}^{n+1})}. \tag{5}$$

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^{n+1})$ boundedness of T [3] we have

$$\int_{\mathbb{R}^{n+1}} |T_2 f(x)|^p \omega_1(t)\mathrm{d}x = \int_{\mathbb{R}^{n+1}} \left(\sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(t)\right)^p \omega_1(t)\mathrm{d}x$$

$$= \int_{\mathbb{R}^{n+1}} \left(\sum_{k\in Z} |Tf_{k,2}(x)|^p\,\chi_{E_k}(t)\right) \omega_1(t)\mathrm{d}x = \sum_{k\in Z}\int_{E_k} |Tf_{k,2}(x)|^p\,\omega_1(t)\mathrm{d}x$$

$$\leq \sum_{k\in Z} \sup_{x\in E_k} \omega_1(t) \int_{\mathbb{R}^{n+1}} |Tf_{k,2}(x)|^p\,\mathrm{d}x \leq \|T\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(t) \int_{\mathbb{R}^{n+1}} |f_{k,2}(x)|^p\,\mathrm{d}x$$

$$= \|T\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(\tau) \int_{E_{k,2}} |f(x)|^p \mathrm{d}x,$$

where $\|T\| \equiv \|T\|_{L_p(\mathbb{R}^{n+1})\to L_p(\mathbb{R}^{n+1})}$. Since, for $x \in E_{k,2}$, $2^{k-1} < |t| \leq 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} \omega_1(\tau) = \sup_{2^{k-1}<|\tau|\leq 2^{k+2}} \omega_1(\tau) \leq \sup_{|t|/4<|\tau|\leq 4|t|} \omega_1(\tau) \leq b\omega(t)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int_{\mathbb{R}^{n+1}} |T_2 f(x)|^p \omega_1(t)\mathrm{d}x \leq$$

$$\leq \|T\|^p b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p \omega(t)\mathrm{d}x \leq c_6 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(t)\mathrm{d}x, \tag{6}$$

where $c_6 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (3), (4), (5), (6) imply (2) which completes the proof. □

Remark 1. Note that, Theorem 2 for singular integral operators with Calderon-Zygmund kernels was proved in [11], if $\omega(x)$,$\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and for singular integral operators, defined on homogeneous groups $\mathbb{G}$ in [10], [6] (see also [7]), if $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{G}$.

Theorem 2. *Let $p \in (1,\infty)$, K be a parabolic Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and let following conditions be satisfied:*

(a') *there exist $b_1 > 0$ such that*

$$\sup_{t/4<\tau\leq 4t} \omega_1(\tau) \leq b_1\,\omega(t) \quad \textit{for a.e. } t > 0,$$

(b') $\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^{\infty} \omega_1(t) t^{-p}\mathrm{d}\tau\right) \left(\int_0^{\tau} \omega^{1-p'}(t)\mathrm{d}t\right)^{p-1} < \infty.$

Then there exists a constant $c > 0$ *such that for all* $f \in L_{p,\omega}(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \omega_1(|t|)\mathrm{d}x'\mathrm{d}t \leq c \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(|t|)\mathrm{d}x'\mathrm{d}t. \tag{7}$$

Proof. Suppose that $f \in L_{p,\omega}(\mathbb{R}^{n+1})$ and ω_1 are positive increasing functions on $(0,\infty)$ and $\omega(t)$,$\omega_1(t)$ satisfied the conditions (a'), (b').

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)\mathrm{d}\lambda,$$

where $\omega_1(0+) = \lim_{t\to 0}\omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty}\varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [8], [7], [1], [5] for details).

We have

$$\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \omega_1(|t|)\mathrm{d}x'\mathrm{d}t = \omega_1(0+)\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \mathrm{d}x + \int_{\mathbb{R}^{n+1}} |Tf(x)|^p \left(\int_0^{|t|}\psi(\lambda)\mathrm{d}\lambda\right)\mathrm{d}x = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \neq 0$ by the boundedness of T in $L_p(\mathbb{R}^{n+1})$ thanks to (a')

$$\begin{aligned} J_1 &\leq \|T\|^p \omega_1(0+)\int_{\mathbb{R}^{n+1}} |f(x)|^p \mathrm{d}x \\ &\leq \|T\|^p \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega_1(|t|)\mathrm{d}x'\mathrm{d}t \leq b_1\|T\|^p \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(|t|)\mathrm{d}x'\mathrm{d}t. \end{aligned}$$

After changing the order of integration in J_2 we have

$$\begin{aligned} J_2 &= \int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^n}\int_{|t|>\lambda} |Tf(x)|^p \mathrm{d}x'\mathrm{d}t\right)\mathrm{d}\lambda \\ &\leq 2^{p-1}\int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^n}\int_{|t|>\lambda} |T(f\chi_{\{|t|>\lambda/2\}})(x)|^p \mathrm{d}x'\mathrm{d}t \right. \\ &\quad \left. + \int_{\mathbb{R}^n}\int_{|t|>\lambda} |T(f\chi_{\{|t|\leq\lambda/2\}})(x)|^p \mathrm{d}x'\mathrm{d}t\right)\mathrm{d}\lambda = J_{21} + J_{22}. \end{aligned}$$

Using the boundedness of T in $L_p(\mathbb{R}^{n+1})$ we obtain

$$J_{21} \le c_7 \int_0^\infty \psi(t) \left(\int_{\mathbb{R}^n} \int_{|\tau|>\lambda/2} |f(y',\tau)|^p \mathrm{d}y' \mathrm{d}\tau \right) \mathrm{d}t$$
$$= c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \left(\int_{2|\tau|}^\infty \psi(\lambda)\mathrm{d}\lambda \right) \mathrm{d}y \le c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \omega_1(2|\tau|)\mathrm{d}y'\mathrm{d}\tau$$
$$\le b_1 c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \omega(|\tau|)\mathrm{d}y'\mathrm{d}\tau.$$

Let us estimate J_{22}. For $|t| > \lambda$ and $|\tau| \le \lambda/2$ we have

$$|t|/2 \le |t-\tau| \le 3|t|/2,$$

and so

$$J_{22} \le c_8 \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_{|t|>\lambda} \left(\int_{\mathbb{R}^n} \int_{|\tau|\le 2\lambda} \frac{|f(y)|}{\rho(x-y)^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x \right) \mathrm{d}\lambda \le$$
$$c_9 \int_0^\infty \psi(\lambda) \left(\int_{|t|>\lambda} \int_{\mathbb{R}^n} \left(\int_{|\tau|\le\lambda/2} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|t|^{1/2})^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x'\mathrm{d}t \right) \mathrm{d}\lambda.$$

For $x = (x', t) \in \mathbb{R}^{n+1}$ let

$$J(t,\lambda) = \int_{\mathbb{R}^n} \left(\int_{|\tau|\le\lambda/2} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|t|^{1/2})^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x'$$
$$= \int_{\mathbb{R}^n} \left(\int_{|\tau|\le\lambda/2} \left(\int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|t|^{1/2})^{n+2}} \mathrm{d}y' \right) \mathrm{d}\tau \right)^p \mathrm{d}x'.$$

Using the Minkowski and Young inequalities we obtain

$$J(t,\lambda) \le \left[\int_{|\tau|\le\lambda/2} \left(\int_{\mathbb{R}^n} |f(y',\tau)|^p \mathrm{d}y' \right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y'}{(|y'|+|t|^{1/2})^{n+2}} \right) \mathrm{d}\tau \right]^p$$
$$= \left(\int_{|\tau|\le\lambda/2} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y'}{(|y'|+|t|^{1/2})^{n+2}} \right)^p$$
$$= \frac{c_3}{|t|^p} \left(\int_{|\tau|\le\lambda/2} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y'}{(|y'|+1)^{n+2}} \right)^p$$
$$= \frac{c_4}{|t|^p} \left(\int_{|\tau|\le\lambda/2} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p.$$

Integrating in $(0,\infty)\times\{t\in\mathbb{R}:|t|>\lambda\}$ we get

$$J_{22}\le c_{10}\int_0^\infty \psi(\lambda)\left(\int_{|t|>\lambda}\left(\int_{|\tau|\le\lambda/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\frac{\mathrm{d}t}{|t|^p}\right)\mathrm{d}\lambda$$
$$=c_{11}\int_0^\infty \psi(\lambda)\lambda^{-p+1}\left(\int_{|\tau|\le\lambda/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\mathrm{d}\lambda.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-p+1}\mathrm{d}\tau=p\int_{2t}^\infty\psi(\tau)\mathrm{d}\tau\int_\tau^\infty\lambda^{-p}\mathrm{d}\lambda$$
$$=p\int_{2t}^\infty\lambda^{-p}\mathrm{d}\lambda\int_{2t}^\lambda\psi(\tau)\mathrm{d}\tau\le p\int_{2t}^\infty\lambda^{-p}\omega_1(\lambda)\mathrm{d}\lambda,\quad t>0.$$

The Hardy inequality

$$\int_0^\infty\psi(\lambda)\lambda^{-p+1}\left(\int_{|\tau|\le\lambda/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}\mathrm{d}\tau\right)^p\mathrm{d}\lambda\le C\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(|\tau|)\mathrm{d}\tau$$

for $p\in(1,\infty)$ is characterized by the condition $C\le c'\mathcal{A}''$ (see [2], [9]), where

$$\mathcal{A}''\equiv\sup_{\tau>0}\left(\int_{2\tau}^\infty\psi(t)t^{-p+1}\mathrm{d}t\right)\left(\int_0^\tau\omega^{1-p'}(t)\mathrm{d}t\right)^{p-1}<\infty.$$

Condition (b') of the theorem guarantees that $\mathcal{A}''<p\mathcal{A}'<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\le c_{12}\int_0^\infty\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\mathrm{d}\tau\le c_{12}\int_{\mathbb{R}^{n+1}}|f(x)|^p\omega(|t|)\mathrm{d}x'\mathrm{d}t.$$

Combining the estimates of J_1 and J_2, we get (7) for $\omega_1(t)=\omega_1(0+)+\int_0^t\psi(\tau)\mathrm{d}\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (7). The theorem is proved.

THEOREM 3. *Let $p\in(1,\infty)$, K be a parabolic Calderon–Zygmund kernel and T be the corresponding operator. Moreover, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and condition (a') and*

$$(c')\qquad \mathcal{B}'\equiv\sup_{\tau>0}\left(\int_0^\tau\omega_1(t)\mathrm{d}\tau\right)\left(\int_{2\tau}^\infty\omega^{1-p'}(t)t^{-p'}\mathrm{d}t\right)^{p-1}<\infty.$$

be satisfied. Then inequality (7) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty\psi(\tau)\mathrm{d}\tau$, where $\omega_1(+\infty)=\lim_{t\to\infty}\omega_1(t)$ and $\omega_1(t)\ge 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [8], [7], [1], [5] for details). We have

$$\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \omega_1(|t|) \mathrm{d}x'\mathrm{d}t = \omega_1(+\infty) \int_{\mathbb{R}^{n+1}} |Tf(x)|^p \mathrm{d}x$$
$$+ \int_{\mathbb{R}^{n+1}} |Tf(x)|^p \left(\int_{|t|}^{\infty} \psi(\tau)\mathrm{d}\tau \right) \mathrm{d}x = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of T in $L_p(\mathbb{R}^{n+1})$

$$I_1 \le \|T\| \omega_1(+\infty) \int_{\mathbb{R}^{n+1}} |f(x)|^p \mathrm{d}x$$
$$\le \|T\| \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega_1(|t|) \mathrm{d}x'\mathrm{d}t \le b_1 \|T\| \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(|t|) \mathrm{d}x'\mathrm{d}t.$$

After changing the order of integration in J_2 we have

$$I_2 = \int_0^{\infty} \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_{|t|<\lambda} |Tf(x)|^p \mathrm{d}x'\mathrm{d}t \right) \mathrm{d}\lambda$$
$$\le 2^{p-1} \int_0^{\infty} \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_{|t|<\lambda} |T(f\chi_{\{|t|<2\lambda\}})(x)|^p \mathrm{d}x'\mathrm{d}t \right.$$
$$\left. + \int_{\mathbb{R}^n} \int_{|t|<\lambda} |T(f\chi_{\{|t|\ge 2\lambda\}})(x)|^p \mathrm{d}x'\mathrm{d}t \right) \mathrm{d}\lambda = I_{21} + I_{22}.$$

Using the boundedness of T in $L_p(\mathbb{R}^{n+1})$ we obtain

$$I_{21} \le c_7 \int_0^{\infty} \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_{|\tau|<2\lambda} |f(y',\tau)|^p \mathrm{d}y'\mathrm{d}\tau \right) \mathrm{d}\lambda$$
$$= c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \left(\int_{|\tau|/2}^{\infty} \psi(\lambda)\mathrm{d}\lambda \right) \mathrm{d}y$$
$$\le c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \omega_1(|\tau|/2) \mathrm{d}\tau\mathrm{d}y'$$
$$\le b_1 c_7 \int_{\mathbb{R}^{n+1}} |f(y)|^p \omega(|\tau|) \mathrm{d}\tau\mathrm{d}y'.$$

Let us estimate J_{22}. For $|t| < \lambda$ and $|\tau| \ge 2\lambda$ we have

$$|\tau|/2 \le |t-\tau| \le 3|\tau|/2,$$

and so

$$J_{22} \le c_8 \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_{|t|<\lambda} \left(\int_{\mathbb{R}^n} \int_{|\tau|\ge 2\lambda} \frac{|f(y)|}{\rho(x-y)^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x \right) \mathrm{d}\lambda$$

$$\le c_9 \int_0^\infty \psi(\lambda) \left(\int_{|t|<\lambda} \int_{\mathbb{R}^n} \left(\int_{|\tau|\ge 2\lambda} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x'\mathrm{d}t \right) \mathrm{d}\lambda.$$

For $x=(x',t)\in\mathbb{R}^{n+1}$ let

$$J(t,\lambda) = \int_{\mathbb{R}^n} \left(\int_{|\tau|\ge 2\lambda} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}} \mathrm{d}y \right)^p \mathrm{d}x'$$

$$= \int_{\mathbb{R}^n} \left(\int_{|\tau|\ge 2\lambda} \left(\int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}} \mathrm{d}y' \right) \mathrm{d}\tau \right)^p \mathrm{d}x'.$$

Using the Minkowski and Young inequalities we obtain

$$J(t,\lambda) \le \left[\int_{|\tau|\ge 2\lambda} \left(\int_{\mathbb{R}^n} |f(y',\tau)|^p \mathrm{d}y' \right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y'}{(|y'|+|\tau|^{1/2})^{n+2}} \right) \mathrm{d}\tau \right]^p$$

$$= c_3 \left(\int_{|\tau|\ge 2\lambda} |\tau|^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y'}{(|y'|+1)^{n+2}} \right)^p$$

$$= c_4 \left(\int_{|\tau|\ge 2\lambda} |\tau|^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p.$$

Integrating in $(0,\infty)\times\{t\in\mathbb{R} : |t|<\lambda\}$ we get

$$J_{22} \le c_{10} \int_0^\infty \psi(\lambda) \left(\int_{|t|<\lambda} \left(\int_{|\tau|\ge 2\lambda} |\tau|^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p \mathrm{d}t \right) \mathrm{d}\lambda$$

$$= 2c_{10} \int_0^\infty \psi(\lambda)\lambda \left(\int_{|\tau|\ge 2\lambda} |\tau|^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau \right)^p \mathrm{d}\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda(\int_{|\tau|\ge 2\lambda} |\tau|^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} \mathrm{d}\tau)^p \mathrm{d}\lambda \le C \int_{\mathbb{R}} \|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n} \omega(|\tau|)\mathrm{d}\tau$$

for $p\in(1,\infty)$ is characterized in [2] and [9] by the condition $C\le c''\mathcal{B}''$, where

$$\mathcal{B}'' \equiv \sup_{\tau>0} \left(\int_0^\tau \psi(t)\mathrm{d}\tau \right) \left(\int_{2\tau}^\infty \omega^{1-p'}(t)t^{-p'}\mathrm{d}t \right)^{p-1} < \infty.$$

Note that

$$\int_0^t \psi(\lambda)\lambda\mathrm{d}\lambda \;=\; \int_0^t \psi(\lambda)\mathrm{d}\lambda \int_0^\lambda \mathrm{d}\tau \;=\; \int_0^t \mathrm{d}\tau \int_\tau^t \psi(\lambda)\mathrm{d}\lambda \;\le\; \int_0^t \omega_1(\tau)\mathrm{d}\tau.$$

Condition (c') of the theorem guarantees that $\mathcal{B}'' \le \mathcal{B}' < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c_{11} \int_0^\infty \|f(\cdot,t)\|^p_{p,\mathbb{R}^n} \omega(t)\mathrm{d}t \le c_{11} \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(|t|)\mathrm{d}x'\mathrm{d}t.$$

Combining the estimates of J_1 and J_2, we get (7) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)\mathrm{d}\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (7). The theorem is proved.

Remark 2. Note that, the weighted pair $(\omega(t), \omega_1(t))$ satisfying conditions (b) or (c) is equivalent to the weighted pair $(\omega(|t|), \omega_1(|t|))$ satisfying conditions (b') or (c') respectively.

Acknowledgements. The author would like to express their thanks to Prof. V.I. Burenkov and Prof. V.S. Guliyev for many helpful discussions about this subject.

REFERENCES

[1] Aliev, I.A. and Gadjiev, A.D. *Weighted estimates of multidimensional singulars generated by the generalized shift operator.* English, translated into Russian Acad. Sci. Sb. Math., **77** (1994), 37–55.

[2] Bradley, J.S., *The Hardy's inequalities with mixed norms.* Canad. Math. Bull., **21** (1978), 405–408.

[3] Bramanti, M. and Cerutti, M., *$W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients.* Commun. in Part. Diff. Equations, **18** (1993), 1735–1763.

[4] Fabes, E.B. and Riviere, N., *Singular integrals with mixed homogeneity*, Studia Math. **27** (1966), 19–38.

[5] Genebashvili, I., Gogatishvili, A., Kokilashvili, V. and Krbec, M., *Weight theory for integral transforms on spaces of homogeneous type.* Pitman monographs and surveys in pure and applied mathematics (1998).

[6] Guliev, V.S., *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications.* Baku. (1999), 1–332.

[7] Guliev, V.S. *Two-weighted inequalities for integral operators in L_p-spaces, and applications.* Trudy Math. Inst. Steklov, **204** (1993), 113–136. English trans. in Proc. Steklov Inst. Math., **204** (1994), 97–116.

[8] Guseinov, E.G. *Singular integrals in the spaces of functions summable with monotone weight.* (Russian) Mat. Sb. 132(174) (1977), 28–44.

[9] Kokilashvilii, V.M., *On Hardy's inequalities in weighted spaces.* (Russian) Bull. Acad. Sci. Georgian SSR, **96** (1979), 37–40.

[10] Kokilashvili, V.M. and Meskhi, A., *Two-weight inequalities for singular integrals defined on homogeneous groups.* Proc. Razmadze Math.Inst., **112** (1997), 57–90.

[11] Rakotondratsimba, Y., *Two-weight norm inequality for Calderon-Zygmund operators,* Acta Math. Hungar., **80** (1988), 39–54.

Received October 18, 2000

Azerbaijan National Academy of Sciences,
Institute of Mathematics and Mechanics,
"F. Agaev" Str., bl. 10, Baku, Azerbaijan
E-mail: vagif@guliyev.com