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REGULAR REFRACTION PROPERTY OF THE PARABOLIC LENS

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Abstract. We show that any parabolic lens has the regular refraction property of index γ (with respect to its focus) if and only if $0 \le \gamma \le \sqrt{2}$.

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Key words. Starlike curve, convex curve, parabolic lens, regular.

1. Preliminaries

Let Γ be a smooth curve with a parametrization $z = z(t)$, $t \in [a, b]$. We suppose that Γ is a directed arc, the direction being that determined as t increases.

The arc Γ is said to be starlike (with respect to the point $\alpha \notin \Gamma$) if $\arg[z(t)-\alpha]$ is a nondecreasing function of t, i.e. if

$$\frac{\mathrm{d}}{\mathrm{d}t}\arg[z(t)-\alpha] \ge 0,\ t \in [a, b].$$

For simplicity we will suppose that $\alpha = 0$. Let Γ be starlike and let R be the radius vector from the origin to the point $z(t) \in \Gamma$. Let N be the outer normal to Γ at the point $z(t)$ and denote by ω the angle between N and R. Let consider the vector V starting from $z(t)$ such that

$$\sin\psi = \gamma \sin\omega, \tag{1}$$

where ψ is the angle between N and V and γ is a given positive number.

From the optical point of view, we remark that if Γ separates two media with different indices of refraction and R and V are the trajectories of light in these two media, then (1) is the well-known refraction law.

We say that the curve Γ has the regular refraction property of index, written $\Gamma \in RP(\gamma)$, iff the argument of the vector V is a nondecreasing function of t on $[a, b]$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\arg V(t) \ge 0,\ t \in [a, b] \tag{2}$$

The above definition was given in [1] and [2] in the case $\gamma \in [0, 1]$.

If we let $\chi = \arg V$ and $\varphi = \arg z(t)$, then

$$\chi = \varphi + \omega - \psi. \tag{3}$$

We have

$$\varphi' = \operatorname{Im}\frac{z'}{z}, \qquad \omega' = \operatorname{Im}\left[\frac{z''}{z'} - \frac{z'}{z}\right], \qquad \psi' = \frac{\gamma\cos\omega}{(1-\gamma^2+\gamma^2\cos^2\omega)^{1/2}}\omega'$$

Hence, by using (3), the inequality (2) is equivalent to

$$\gamma\left(\operatorname{Im}\frac{z'}{z}\right)^2+\left[\sqrt{\Delta}-\gamma\operatorname{Im}\frac{z'}{z}\right]\operatorname{Im}\frac{z''}{z'}\geq 0,\ t\in[a,b], \tag{4}$$

where

$$\Delta=(1-\gamma^2)\left|\frac{z'}{z}\right|^2+\gamma^2\left(\operatorname{Im}\frac{z''}{z'}\right)^2. \tag{5}$$

Since the curvature of Γ at the point $z(t)$ is given by

$$K(t)=\frac{1}{|z'|}\operatorname{Im}\frac{z''}{z'},$$

we deduce that if the curve Γ is convex and $\gamma\in[0,1]$ then Γ has the regular refraction property of index γ, i.e. $\Gamma\in RP(\gamma)$.

If $\gamma>1$, then we have to put first the condition $\Delta\geq 0$, where Δ is given by (5), and then to check the inequality (4).

2. Main result

For a given starlike curve Γ a natural problem is to find the largest interval $[\gamma_0,\gamma_1]$, $\gamma_0<1<\gamma_1$, such that $\Gamma\in RP(\gamma)$ for all $\gamma\in[\gamma_0,\gamma_1]$. We shall call this interval *regular refraction interval of the curve* Γ. If Γ is convex then $\gamma_0=0$ and we have to find the maximum value of γ_1 such that $\Gamma\in RP(\gamma)$ for all $\gamma\in[0,\gamma_1]$. We solved this problem in the case of the ellipse [3]. Now we will find the regular refraction interval for a segment of parabola. We shall call *parabolic lens* the segment of the parabola $y^2=2px$, corresponding to $0\leq x\leq p/2$.

THEOREM 1. *The regular refraction interval of a parabolic lens (with respect to its focus) is given by* $[0,\sqrt{2}]$.

Proof. Without loss of generality we can suppose that the parabolic lens is given by the equation

$$z=-2t+\mathrm{i}\left(1-t^2\right),\ t\in[-1,1]. \tag{6}$$

From (6) we easily deduce

$$\operatorname{Im}\frac{z'}{z}=\frac{2}{1+t^2}$$

and

$$\left|\frac{z'}{z}\right|^2=\frac{4}{1+t^2}.$$

We also have

$$\frac{z''}{z'}=\frac{\mathrm{i}}{1+\mathrm{i}t}$$

and

$$\operatorname{Im}\frac{z''}{z'}=\frac{1}{1+t^2}.$$

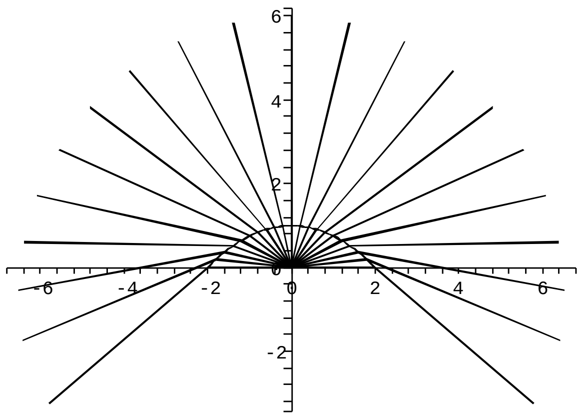
From (5) we deduce $\Delta = \frac{4}{1+t^2}\left[1-\gamma^2\frac{t^2}{1+t^2}\right]$ and the condition $\Delta \geq 0$ becomes $\gamma^2 \leq 1+\frac{1}{t^2}$, $t \in [-1,1]$.

Hence $\Delta \geq 0$, for all $t \in [-1,1]$, if and only if $\gamma \leq \sqrt{2}$. On the other hand, the inequality (4) is equivalent to

$$\frac{4\gamma}{(1+t^2)^2} + \left[\sqrt{\Delta} - \frac{2\gamma}{1+t^2}\right]\frac{1}{1+t^2} = \frac{1}{1+t^2}\left[\sqrt{\Delta} + \frac{2\gamma}{1+t^2}\right] \geq 0,$$

which holds for all $t \in [-1,1]$. Hence Γ has the regular refraction property of index γ if and only if $0 \leq \gamma \leq \sqrt{2}$. □

The regular refraction property of the lens is illustrated, for $\gamma = \sqrt{2}$, in the following figure:



REFERENCES

[1] Mocanu, P. T., *Conformal mappings and refraction law,* Babeş-Bolyai Univ. Fac. of Math. and Phys., The XVIII-th National Conf. on Geometry and Topology, Preprint **2** (1989), 113–116.

[2] Mocanu, P. T., *C^1-maps and refraction law,* Bull. St. Inst. Polytechnic, Cluj-Napoca, **33** (1990), 11–14.

[3] Mocanu, P. T., *Regular refraction property of the ellipse,* Bull. Math. Soc. Sc. Math. Roumanie, **45 (93)**, (2002), 39–43.

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