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A SECOND-KIND INTEGRAL EQUATION METHOD FOR STOKES FLOW PAST SMOOTH OBSTACLES IN A CHANNEL

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Abstract. In this paper we obtain a compound double-layer representation for Stokes flow due to the motion of a solid particle in an ambient flow located in a two-dimensional channel. Our indirect method is an extension of the well known Completed Double Layer Boundary Integral Equation Method of Power and Miranda [18] from the case of Stokes flow due to the motion of a solid particle in a viscous incompressible fluid of infinite expanse to the case of Stokes flow in a two-dimensional channel. The problem is reduced to the study of a system of Fredholm integral equations of the second kind. We prove that this system has a unique continuous solution. The numerical results are presented for Stokes flow due to the motion of a circular obstacle in a two-dimensional channel between two parallel solid walls. We also include some conclusions which refer to the effect of the walls on the considered Stokes flow.

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Key words. Stokes flow, completed double layer boundary integral equation method, Green's function, singularities of Stokes flow, boundary element method.

1. INTRODUCTION

The motion of a body of arbitrary shape near a plane wall at small Reynolds number has a long record in the fluid dynamics research. For example, Power, Miranda and Gonzáles [19] proposed a boundary integral formulation for the flow due to the motion of a body of arbitrary shape near a plane wall at small Reynolds number. Their formulation is given in terms of a system of Fredholm integral equations of the first kind, since the solution of the corresponding boundary value problem is sought in the form of a single-layer potential. The problem of determining the Stokes flow due to the translational motion of a solid particle of arbitrary shape near a plane wall was treated by Hsu and Ganatos [7]. They used the boundary integral representation of the velocity field in terms of the boundary velocity and traction. Recently, Phan-Thien et al. [17] as well as Power and Power [20] extended the Completed Double Layer Boundary Integral Equation Method (CDLBIEM) of Power and Miranda [18] to the problem of Stokes flow due to the slow motion of a particle of arbitrary shape near a plane wall (see also [21], Section 6.2.3). Ganatos et al. [4], [5] presented exact solutions for Stokes flow due to the creeping motion of a sphere of arbitrary size and position between two plane parallel walls. Pozrikidis [22] obtained Greeen's function of Stokes flow due to a point force located in a two-dimensional channel. This function vanishes on the walls of the channel. Also, it has been extensively used in the boundary integral formulation of many problems which refer to Stokes flow past or due to the motion of a solid obstacle in a two-dimensional channel (see e.g. [11], [23]). Kohr [8] proposed a boundary integral equation method for asymmetric Stokes flow between two parallel planes. Also, we gave an extension of the CDLBIEM to the problem of Stokes flow past a cylinder of arbitrary cross-section in a half-plane (see [10]). In addition, we have obtained an indirect boundary integral method for Stokes flow due to the translational motion of a solid particle in a two-dimensional channel (see [11]).

Note that Power and Miranda's method [18] removes the marginal eigenvalues of the spectrum of the double-layer integral operator (which appears into a boundary integral representation of Stokes flow in terms of a double-layer potential without any additional term), but leaves unchanged the rest of the eigenvalues. In fact, the CDLBIEM is a completion plus a deflation procedure that leads to a bounded and invertible integral operator (with a spectral radius strictly less than one), and thus iterative solution strategies are guaranteed to converge to a unique solution.

2. THE MATHEMATICAL FORMULATION OF THE PROBLEM

A viscous incompressible flow with velocity and pressure fields $\mathbf{U}_\infty$ and p_∞, located in a two-dimensional channel $\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^2 : \ -h < x_2 < h\}$, is perturbed by the slow motion with velocity $\mathbf{U}$ of a solid particle Ω^1. Let us assume that the boundary C of Ω^1 is a closed Lyapunov curve (i.e. it has a continuously varying normal vector; for details see e.g. [15], Chapter 16), and that the velocity field $\mathbf{U}$ is a continuous vector function on C.

The boundary of the channel $\mathcal{D}$ is determined by the walls L_0 and L_1 given by: $L_0 = \{\mathbf{x} \in \mathbb{R}^2 : \ x_2 = -h\}$, $\quad L_1 = \{\mathbf{x} \in \mathbb{R}^2 : \ x_2 = h\}$.

The velocity field $\mathbf{U}_\infty$ of the undisturbed flow is an admissible solution of the system of continuity and Stokes equations in $\mathcal{D}$, and satisfies the following boundary condition: $\mathbf{U}_\infty(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in L_0 \cup L_1$.

Let us assume that the resulting flow due to the presence of the solid particle in the basic flow is a Stokes flow. Then the velocity and pressure fields of the disturbance flow, $\mathbf{u}$ and p, have to satisfy the Stokes system of equations

$$(1) \qquad \nabla \cdot \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \ -\nabla p(\mathbf{x}) + \mu \nabla^2 \mathbf{u}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \Omega,$$

where Ω is the flow domain whose boundary is determined by L_0, L_1 and C, and μ is the dynamic viscosity of the fluid. In addition, we have to require the boundary conditions

$$(2) \qquad \mathbf{u}(\mathbf{x}) = -\mathbf{U}_\infty(\mathbf{x}) + \mathbf{U}(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in C, \ \mathbf{u}(\mathbf{x}) = \mathbf{0} \ \text{ for } \ \mathbf{x} \in L_0 \cup L_1.$$

Moreover, we assume that the flow fields $\mathbf{u}$ and p vanish at infinity such that

$$(3) \qquad |\mathbf{u}(\mathbf{x})||\nabla \mathbf{u}(\mathbf{x})| = o(|\mathbf{x}|^{-1}), \ |p(\mathbf{x})||\mathbf{u}(\mathbf{x})| = o(|\mathbf{x}|^{-1}) \ \text{ as } \ |\mathbf{x}| \to \infty,$$

where $|\mathbf{x}| = (x_1^2 + x_2^2)^{1/2}$ is the Euclidean distance between the current point $\mathbf{x} = (x_1, x_2)$ in the flow field and the origin of a frame of Cartesian coordinates (x_1, x_2), located midway the walls.

Note that the far field conditions (3) are sufficient to deduce the uniqueness of the classical solution of the present boundary value problem.

Remark. Let f and g be two functions defined in a neighbourhood of a point $\mathbf{x}_0$ (which can be ∞). Then the condition $f(\mathbf{x}) = o(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ means that the ratio $|f(\mathbf{x})|/|g(\mathbf{x})|$ tends to zero as $\mathbf{x} \to \mathbf{x}_0$.

3. GREEN'S FUNCTION OF STOKES FLOW

Let $\mathbf{G}(G_{ij})$ denote Green's function of Stokes flow due to a point force acting at a point $\mathbf{x}$ in the unbounded domain $\mathcal{D}$ and let $\mathbf{q}$ be the corresponding pressure vector. Note that $\mathbf{G}$ and $\mathbf{q}$ satisfy the following equations and conditions:

(1) $$\frac{\partial^2 G_{ij}(\mathbf{y}, \mathbf{x})}{\partial y_k \partial y_k} - \frac{\partial q_j(\mathbf{y}, \mathbf{x})}{\partial y_i} = -4\pi\delta_{ij}\delta(\mathbf{y} - \mathbf{x}) \text{ for } -h < y_2 < h$$

(2) $$\frac{\partial G_{ij}(\mathbf{y}, \mathbf{x})}{\partial y_i} = 0 \text{ for } -h < y_2 < h$$

(3) $$G_{ij}(\mathbf{y}, \mathbf{x}) = 0 \text{ for } \mathbf{x} \in L_0 \cup L_1$$

(4) $$G_{ij}(\mathbf{y}, \mathbf{x}) \to 0, \quad q_i(\mathbf{y}, \mathbf{x}) \to 0 \text{ as } |\mathbf{y}| \to \infty,$$

where δ is the two-dimensional delta function or Dirac's distribution, and δ_{ij} is the Kronecker symbol, i.e. $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$.

Note that in eqns (1) and (2) we have used the repeated index summation convention. From now on, we take into account this rule.

Let us define the stress tensor $\mathbf{T}$, associated to Green's function $\mathbf{G}$, as follows: $T_{ijk}(\mathbf{y}, \mathbf{x}) = -\delta_{ik} q_j(\mathbf{y}, \mathbf{x}) + \frac{\partial G_{ij}(\mathbf{y}, \mathbf{x})}{\partial y_k} + \frac{\partial G_{kj}(\mathbf{y}, \mathbf{x})}{\partial y_i}$.

Green's function $\mathbf{G}$ was obtained by Pozrikidis [22] (see also [23], pp. 96–98, or the forthcoming book [12], Chapter 2). The expression of this function is given in [22] and [23].

The function $\mathbf{G}$ satisfies the symmetry property $G_{ij}(\mathbf{y}, \mathbf{x}) = G_{ji}(\mathbf{x}, \mathbf{y})$ and

(5) $$G_{ij}(\mathbf{y}, \mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty, \; G_{ij}(\mathbf{x}, \mathbf{y}) = 0 \text{ for } \mathbf{x} \in L_0 \cup L_1.$$

Also, we mention the following properties (see e.g. [12], Chapter 2):

(6) $$\mathbf{q}(\mathbf{y}, \mathbf{x}) = \mathbf{0} \text{ for } \mathbf{x} \in L_0 \cup L_1, \; T_{ijk}(\mathbf{y}, \mathbf{x}) = 0 \text{ for } \mathbf{x} \in L_0 \cup L_1,$$

(7) $$q_i(\mathbf{x}, \mathbf{y}) \to 0, \, T_{ijk}(\mathbf{y}, \mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty.$$

4. BOUNDARY INTEGRAL REPRESENTATION OF THE SOLUTION

Next, using the CDLBIEM, we determine the disturbance velocity field $\mathbf{u}$ in the form

$$u_i(\mathbf{x}) = \frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y}) + \frac{1}{4\pi\mu}G_{ij}(\mathbf{x},\mathbf{x}_c)\beta_j + \frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial G_{i1}(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3, \quad \mathbf{x}\in\Omega, \tag{1}$$

where β_1, β_2 and γ_3 are real constants, $\Phi = (\phi_1, \phi_2, 0)$ is an unknown continuous vector density on C, $\mathbf{x}_c = (x_{1c}, x_{2c}, 0)$ is an arbitrary point located inside C (for example, $\mathbf{x}_c$ can be chosen as the centre of mass of the solid particle), and $\mathbf{n}$ is the outward unit normal vector to C. The first term on the right-hand side of eqn (1) is a double-layer potential with the density $\Phi/(4\pi)$, the second term represents the velocity field due to a point force located at $\mathbf{x}_c$ and with the strength $\beta = (\beta_1, \beta_2, 0)$, and the third term is the velocity field of a couplet (i.e. a singularity of Stokes flow) located at the point $\mathbf{x}_c$ and having the strength $\gamma = (0, 0, \gamma_3)$. For the following arguments we choose the constants β_1, β_2 and γ_3 in the form

$$\beta_i = \int_C U_k^i(\mathbf{y})\phi_k(\mathbf{y})\mathrm{d}l(\mathbf{y}),\ \gamma_3 = \int_C U_k^3(\mathbf{y})\phi_k(\mathbf{y})\mathrm{d}l(\mathbf{y}), \quad i = 1,2. \tag{2}$$

For convenience, we have considered the two-dimensional vectors $\mathbf{x}$, $\mathbf{x}_c$ and Φ as three-dimensional vectors of the form $\mathbf{x} = (x_1, x_2, 0)$, $\mathbf{x}_c = (x_{1c}, x_{2c}, 0)$ and $\Phi = (\phi_1, \phi_2, 0)$. Also, the vector functions $\mathbf{U}^k$, $k = 1, 2, , 3$, represent the velocity fields of the three linearly independent rigid body motions in $\mathbb{R}^2$ and are given by

$$\mathbf{U}^1 = (1,0,0), \quad \mathbf{U}^2 = (0,1,0), \quad \mathbf{U}^3 = (x_2, -x_1, 0). \tag{3}$$

Further, we seek the pressure field p as follows:

$$p(\mathbf{x}) = \frac{\mu}{4\pi}\int_C P_{jk}(\mathbf{x},\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y}) + \frac{1}{4\pi}q_j(\mathbf{x},\mathbf{x}_c)\beta_j + \frac{1}{8\pi}\left[\frac{\partial q_2(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial q_1(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3, \quad \mathbf{x}\in\Omega, \tag{4}$$

where the functions P_{jk} are the components of the pressure tensor $\mathbf{P}$ associated to the stress tensor $\mathbf{T}$, i.e. $\mathbf{P}$ and $\mathbf{T}$ determine a fundamental solution of Stokes flow in $\mathcal{D}$. More exactly, we have (see e.g. [23], pp. 81–82, 93, or [12], Section 3.2):

$$\frac{\partial T_{jik}(\mathbf{y},\mathbf{x})}{\partial x_i} = 0, \text{ and } -\frac{\partial P_{jk}(\mathbf{x},\mathbf{y})}{\partial x_i} + \frac{\partial^2 T_{jik}(\mathbf{y},\mathbf{x})}{\partial x_m \partial x_m} = 0 \tag{5}$$

for $\mathbf{x} \neq \mathbf{y}$.

Now, using the relations (1)–(4), (5), (6), (7), and (5), we deduce that the functions $\mathbf{u}$ and p, given by the boundary integral representations (1) and (4), satisfy the Stokes system of equations (1), $\mathbf{u}$ vanishes on the walls L_0 and L_1, and $\mathbf{u}$ and q vanish at infinity such that the far field conditions (3) are satisfied. On the other hand, the double-layer potential $\frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y})$, $\mathbf{x}\in\mathcal{D}\setminus C$ has a jump across the contour C, given by the following formulas (see e.g. [12], Chapter 3):

$$\lim_{\mathbf{x}\to\mathbf{x}_0\in C}\frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y}) = \pm\frac{1}{2}\phi_i(\mathbf{x}_0)+\frac{1}{4\pi}\int_C^{PV} T_{jik}(\mathbf{y},\mathbf{x}_0)n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y}), \tag{6}$$

where the plus sign applies for the external side of C, in the direction of the unit normal vector, and the minus sign otherwise. Therefore, applying the boundary condition (2) to the flow field given by the boundary integral representation (1), and using the above jump formulas, we obtain the following system of boundary integral equations with unknown Φ:

$$\frac{1}{2}\phi_i(\mathbf{x})+\frac{1}{4\pi}\int_C^{PV} T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}l(\mathbf{y})+\frac{1}{4\pi\mu}G_{ij}(\mathbf{x},\mathbf{x}_c)\beta_j + \frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}}-\frac{\partial G_{i1}(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3 = -U_{i\infty}(\mathbf{x})+U_i(\mathbf{x}),\quad \mathbf{x}\in C. \tag{7}$$

Note that the superscript PV stands for the principal value of the double-layer potential (i.e. the value of the improper but convergent double-layer potential at an arbitrary point $\mathbf{x}$ of C). For simplicity, we next omit this symbol.

Further, we take into account the following decomposition formula:

$$T_{jik}(\mathbf{y},\mathbf{x}) = T^S_{jik}(\mathbf{y}-\mathbf{x})+T^c_{jik}(\mathbf{y},\mathbf{x}), \tag{8}$$

where T^S_{jik} are the components of the stress tensor $\mathbf{T}^S$ associated to the two-dimensional Stokeslet, given by the formula $T^S_{jik} = -4\frac{(y_j-x_j)(y_i-x_i)(y_k-x_k)}{r^4}$, where $r=|\mathbf{y}-\mathbf{x}|$. Also, T^c_{jik} are the components of a continuous matrix function required by the no-slip condition at the walls (i.e. $T_{jik}(\mathbf{y},\mathbf{x})=0$ for $\mathbf{x}\in L_0\cup L_1$). The decomposition formula (8) yields that the kernel of the double-layer integral operator, which appears in the system of eqns (7), is weakly singular, and hence this operator is compact on $C^0(C)$ (the space of continuous vector functions on C). Consequently, Fredholm's alternative applies to the system of eqns (7) (see e.g. [13], Chapter 4). In view of this result, the system of eqns (7) admits a unique continuous solution Φ if and only if the following homogeneous system has only the trivial solution in the space $C^0(C)$:

$$\frac{1}{2}\phi^0_i(\mathbf{x})+\frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi^0_j(\mathbf{y})\mathrm{d}l(\mathbf{y})+\frac{1}{4\pi\mu}G_{ij}(\mathbf{x},\mathbf{x}_c)\beta^0_j + \frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}}-\frac{\partial G_{i1}(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma^0_3 = 0,\quad \mathbf{x}\in C,\ i=1,2, \tag{9}$$

where $\beta_i^0 = \int_C U_k^i(\mathbf{y})\phi_k^0(\mathbf{y})\mathrm{d}l(\mathbf{y})$, $\gamma_3^0 = \int_C U_k^3(\mathbf{y})\phi_k^0(\mathbf{y})\mathrm{d}l(\mathbf{y})$, $i = 1, 2$. Let us assume that $\Phi^0 = (\phi_1^0, \phi_2^0, 0)$ is a non-trivial continuous solution of the system of eqns (9). Using this solution, we determine the fields $\mathbf{u}^0$ and p^0, as follows:

$$u_i^0(\mathbf{x}) = \frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j^0(\mathbf{y})\mathrm{d}l(\mathbf{y}) + \frac{1}{4\pi\mu}G_{ij}(\mathbf{x},\mathbf{x}_c)\beta_j^0 + \frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial G_{i1}(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3^0 \tag{10}$$

$$p^0(\mathbf{x}) = \frac{\mu}{4\pi}\int_C P_{jk}(\mathbf{x},\mathbf{y})n_k(\mathbf{y})\phi_j^0(\mathbf{y})\mathrm{d}l(\mathbf{y}) + \frac{1}{4\pi}q_j(\mathbf{x},\mathbf{x}_c)\beta_j^0 + \frac{1}{8\pi}\left[\frac{\partial q_2(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial q_1(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3^0 \tag{11}$$

for $\mathbf{x} \in \Omega$. From eqns (1), (2), (5), and (9), as well as the jump formulas (6), we deduce that the fields $\mathbf{u}^0$ and p^0 determine a Stokes flow in the unbounded domain Ω (as well as in the bounded domain Ω^1), and $\mathbf{u}^0$ vanishes on C. Also, $\mathbf{u}^0$ vanishes on L_0 and L_1, in view of the properties (5) and (6). In addition, the fields $\mathbf{u}^0$ and p^0 satisfy the decay conditions (3). According to the uniqueness of the solution of the boundary value problem (1)–(3), we deduce that

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{0}, \quad p^0(\mathbf{x}) = 0 \ \text{ for } \ \mathbf{x} \in \Omega. \tag{12}$$

Consequently, the total force and torque (with respect to the point $\mathbf{x}_c$) exerted by the flow $(\mathbf{u}^0, p^0)$ on the boundary C vanish. On the other hand, it is known that a double-layer potential with a well-defined boundary traction on C does not exert total force and torque on this curve (see e.g. [21], p. 168).

Let us now consider the vector field $\mathbf{w}^0$ with the following components:

$$w_i^0(\mathbf{x}) = \frac{1}{4\pi\mu}G_{ij}(\mathbf{x},\mathbf{x}_c)\beta_j^0 + \frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x},\mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial G_{i1}(\mathbf{x},\mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_3^0 \tag{13}$$

for $\mathbf{x} \in \Omega$. In fact, $\mathbf{w}^0$ is the sum of a two-dimensional point force located at $\mathbf{x}_c$ and with the strength $\beta^0 = (\beta_1^0, \beta_2^0, 0)$, and a two-dimensional couplet acting at $\mathbf{x}_c$ and having the strength $\gamma^0 = (0, 0, \gamma_3^0)$. As we previously mentioned, the point force and the couplet are fundamental solutions (or singularities) of Stokes flow in the layer $\mathcal{D}$. Moreover, the point force exerts a total force on C equal to $-\beta^0$, and zero total torque with respect to the point $\mathbf{x}_c$. The couplet exerts zero total force on any contour (within the domain $\mathcal{D}$) that encloses the point $\mathbf{x}_c$, and a total torque on C with respect to the point $\mathbf{x}_c$ equal to $-\gamma^0$ (for details see e.g. [23], Chapter 7, or [12], Chapter 2). Taking into account these arguments, we conclude that the total force $\mathbf{F}^0$ exerted by the Stokes flow $(\mathbf{u}^0, p^0)$ on the solid obstacle is equal to $-\beta^0$. But $\mathbf{F}^0 = \mathbf{0}$, in view of the equations (12), and hence

$$\int_C U_k^i(\mathbf{y})\phi_k^0(\mathbf{y})\mathrm{d}l(\mathbf{y}) = 0, \quad i = 1, 2. \tag{14}$$

Similarly, the total torque $\mathbf{M}^0 = (0, 0, M^0)$ (with respect to the point $\mathbf{x}_c$) exerted by the Stokes flow $(\mathbf{u}^0, p^0)$ on the solid obstacle is given by $M^0 = -\gamma_3^0$, and thus, in view of eqns (12),

$$\int_C U_k^3(\mathbf{y})\phi_k^0(\mathbf{y})\mathrm{d}l(\mathbf{y}) = 0. \tag{15}$$

According to the properties (14) and (15), we deduce that the homogeneous system (9) takes the simplified form

$$\frac{1}{2}\phi_i^0(\mathbf{x}) + \frac{1}{4\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\phi_j^0(\mathbf{y})\mathrm{d}l(\mathbf{y}) = 0, \quad \mathbf{x} \in C. \tag{16}$$

Let $\mathbf{I} : C^0(C) \to C^0(C)$ be the identity operator and let $\mathbf{K} : C^0(C) \to C^0(C)$ be the double-layer integral operator given by

$$(\mathbf{K\Psi})_i(\mathbf{x}) = \frac{1}{2\pi}\int_C T_{jik}(\mathbf{y},\mathbf{x})n_k(\mathbf{y})\psi_j(\mathbf{y})\mathrm{d}l(\mathbf{y}), \quad \mathbf{x} \in C,\ \mathbf{\Psi} \in C^0(C).$$

It can be proved that the null space of the operator $\mathbf{I}+\mathbf{K}$ is three-dimensional, and that a basis of this space is determined by the functions $\mathbf{U}^i$, $i = 1, 2, 3$, given by eqns (3) (see e.g. [12], Section 3.5). Therefore, there exist some real constants η_1, η_2 and η_3 such that the function Φ^0, which satisfies the system of equations (16), can be written as follows:

$$\Phi^0 = \sum_{k=1}^{3} \eta_k \mathbf{U}^k \ \text{ on } \ C. \tag{17}$$

Now, from the relations (14), (15) and (17), we obtain the following homogeneous linear algebraic system with the unknowns η_1, η_2 and η_3:

$$\sum_{k=1}^{3} \eta_k \int_C U_i^k(\mathbf{x})U_i^j(\mathbf{x})\mathrm{d}l(\mathbf{x}) = 0, \quad j = 1, 2, 3. \tag{18}$$

Since the functions $\mathbf{U}^1$, $\mathbf{U}^2$ and $\mathbf{U}^3$ are linearly independent, the determinant of the linear algebraic system (18) is non-zero, and hence this system has only the trivial solution. Therefore, in view of (17), we deduce that $\Phi^0 = \mathbf{0}$ on C. However, this result is a contradiction to the assumption $\Phi^0 \neq \mathbf{0}$ on C. Hence the homogeneous system of eqns (9) admits only the trivial solution in the space $C^0(C)$. Finally, applying Fredholm's alternative, we obtain the following result:

Theorem 4.1 *The non-homogeneous system of Fredholm integral equations of the second kind* (7) *has a unique continuous solution* Φ *on* C. *Moreover, the boundary integral representations* (1) *and* (4), *obtained with this density, determine the unique classical solution* $(\mathbf{u}, p)$ *of the boundary value problem consisting of the system of eqns* (1) *subject to the boundary and far field conditions* (2)–(3).

In addition, the total force $\mathbf{F} = (F_1, F_2, 0)$ exerted by the flow $(\mathbf{u}+\mathbf{U}_\infty, p+p_\infty)$ on the solid obstacle is given by $F_i = -\beta_i$, i.e.,

$$F_i = -\int_C \phi_j(\mathbf{y}) U_j^i(\mathbf{y}) \mathrm{d}l(\mathbf{y}), \quad i = 1, 2. \tag{19}$$

5. NUMERICAL RESULTS

An apparent difficulty in the computation of the numerical solution of the system of eqns (7) is that the kernels of the involved double-layer potentials are weakly singular. In order to remove this difficulty we use the fact that $\mathbf{U}^1$, $\mathbf{U}^2$ and $\mathbf{U}^3$ are the linearly independent eigenfunctions corresponding to the eigenvalue -1 of the double-layer integral operator $\mathbf{K}$. Therefore, we have the following property (see e.g. [12], Section 3.5): $\frac{1}{4\pi}\int_C T_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) \mathrm{d}l(\mathbf{y}) = -\frac{1}{2}\delta_{ij}, \quad \mathbf{x} \in C$, which yields that the system of equations (7) can be written in the form

$$\frac{1}{4\pi}\int_C T_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y})(\phi_j(\mathbf{y}) - \phi_j(\mathbf{x})) \mathrm{d}l(\mathbf{y}) + \frac{1}{4\pi\mu} G_{ij}(\mathbf{x}, \mathbf{x}_c)\beta_j$$
$$+\frac{1}{8\pi\mu}\left[\frac{\partial G_{i2}(\mathbf{x}, \mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial G_{i1}(\mathbf{x}, \mathbf{x}_c)}{\partial x_{2c}}\right]\gamma_j = -U_{i\infty}(\mathbf{x}) + U_i(\mathbf{x}), \quad \mathbf{x} \in C. \tag{1}$$

Further, according to the decomposition formula (8), we deduce that the continuity behaviour of the double-layer potential on the left-hand side of eqn (1) is provided only by the stress tensor $\mathbf{T}^S$. Moreover, taking into account the property

$$\begin{aligned} T_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) &= T^S_{jik}(\mathbf{y} - \mathbf{x}) n_k(\mathbf{y}) + T^c_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) \\ &= -\frac{4(y_j - x_j)(y_i - x_i)(y_k - x_k) n_k(\mathbf{y})}{r^4} + T^c_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) \\ &= -4\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_i}\frac{\cos(\mathbf{n}(\mathbf{y}), \mathbf{y} - \mathbf{x})}{r} + T^c_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) \\ &= -4\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_i}\frac{\partial \ln r}{\partial n(\mathbf{y})} + T^c_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}), \end{aligned} \tag{2}$$

it follows that the double-layer integral in eqn (1) can be written in the form

$$\begin{aligned} &\int_C T_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y})(\phi_j(\mathbf{y}) - \phi_j(\mathbf{x})) \mathrm{d}l(\mathbf{y}) \\ &= -4\int_C (\phi_j(\mathbf{y}) - \phi_j(\mathbf{x}))\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_i} \mathrm{d}\alpha(\mathbf{y}) \\ &+ \int_C T^c_{jik}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y})(\phi_j(\mathbf{y}) - \phi_j(\mathbf{x})) \mathrm{d}l(\mathbf{y}), \quad \mathbf{x} \in C, \end{aligned} \tag{3}$$

since $\frac{\partial \ln r}{\partial n(\mathbf{y})}\mathrm{d}l(\mathbf{y}) = \mathrm{d}\alpha(\mathbf{y})$ is the differential of plane angle at the point $\mathbf{y}$. Consequently, the first integral on the right-hand side of eqn (3) is a proper integral in the new variable of integration (the second integral is a proper integral, too) and hence the singularities of the double-layer integrals in the system of eqns (1) can be simply removed by setting their integrands equal to zero when $\mathbf{y} = \mathbf{x}$.

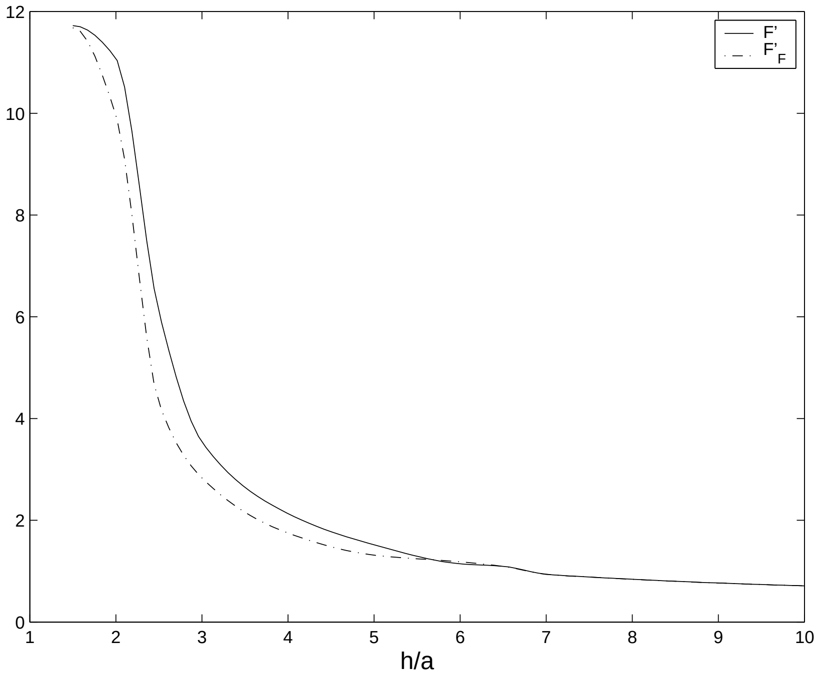

Fig. 5.1 – Dependence of F' and respectively F'_F on the ratio h/a: parallel motion.

In order to reduce the system of eqns (1) to a linear algebraic system, we use the boundary element method. We approximate the Lyapunov contour C by a closed polygonal line consisting of N equal segments L_j, $j = 1, \ldots, N$. We assume that on each segment L_k the functions ϕ_i are constant and equal to their values $\phi_i^k = \phi_i(\mathbf{x}_k)$ at the middle point $\mathbf{x}_k$ of this segment. Further, taking into account the relations (2) and supposing that the discretized form of the system of eqns (1) is satisfied at each point $\mathbf{x}_r$, $r = 1, \ldots, N$, we obtain the following linear algebraic system with unknowns ϕ_j^m, $j = 1, 2$, $m = 1, \ldots, N$:

$$\begin{aligned} &\frac{1}{4\pi} \sum_{m=1}^{N} (\phi_j^m - \phi_j^r) \int_{L_m} T_{jik}(\mathbf{y}, \mathbf{x}_r) n_k(\mathbf{y}) \mathrm{d}l(\mathbf{y}) \\ &+ \frac{1}{4\pi\mu} G_{ik}(\mathbf{x}_r, \mathbf{x}_c) \sum_{m=1}^{N} \phi_j^m \int_{L_m} U_j^k(\mathbf{y}) \mathrm{d}l(\mathbf{y}) \\ &+ \frac{1}{8\pi\mu} \left[\frac{\partial G_{i2}(\mathbf{x}_r, \mathbf{x}_c)}{\partial x_{1c}} - \frac{\partial G_{i1}(\mathbf{x}_r, \mathbf{x}_c)}{\partial x_{2c}} \right] \sum_{m=1}^{N} \phi_j^m \int_{L_m} U_j^3(\mathbf{y}) \mathrm{d}l(\mathbf{y}) \\ &= -U_{i\infty}(\mathbf{x}_r) + U_i(\mathbf{x}_r), \quad r = 1, \ldots, N, \, i = 1, 2. \end{aligned} \tag{4}$$

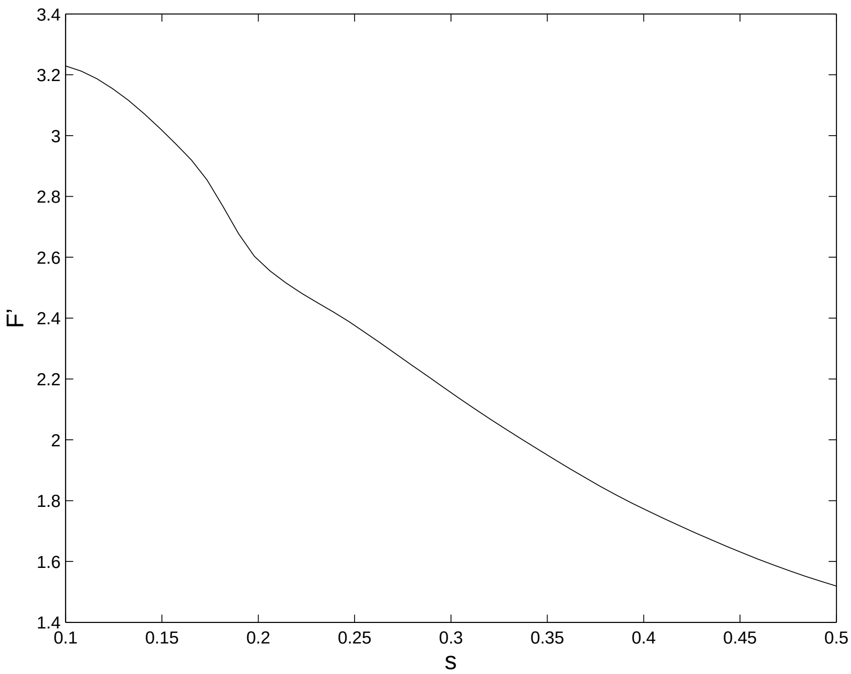

Fig. 5.2 – Dependence of the dimensionless drag force F' on the dimensionless distance s for $h/a = 5$.

In addition, we mention the property

$$(\phi_j^m - \phi_j^r) \int_{L_m} T_{jik}(\mathbf{y}, \mathbf{x}_r) n_k(\mathbf{y}) \mathrm{d}l(\mathbf{y}) = 0 \ \text{ for } \ m = r, \tag{5}$$

in view of the above removal procedure of excluding the singularities of the double-layer potential.

On the other hand, for $m \neq r$, the integrals $\int_{L_m} T_{jik}(\mathbf{y}, \mathbf{x}_r) n_k(\mathbf{y}) \mathrm{d}l(\mathbf{y})$ can be numerically computed by using a Gaussian quadrature formula.

Now, from eqns (19) and the above discretization procedure we obtain

$$F_i = -\sum_{m=1}^{N} \phi_j^m \int_{L_m} U_j^i(\mathbf{y}) \mathrm{d}l(\mathbf{y}), \quad i = 1, 2. \tag{6}$$

In the following numerical results the maximum value of N has been chosen equal to 50.

• Let us now consider the case of a circular obstacle of radius a translating parallel to the walls (i.e. $\mathbf{U} = (U, 0, 0)$) in a quiescent incompressible Newtonian fluid (i.e. $\mathbf{U}_\infty = \mathbf{0}$). Assume that the centre of the circular obstacle is located midway between the walls. Figure 5.1 shows the dependence of the drag force $F' = F/(4\pi\mu a U)$ on the ratio h/a. The results yield that F' decreases when the ratio h/a increases.

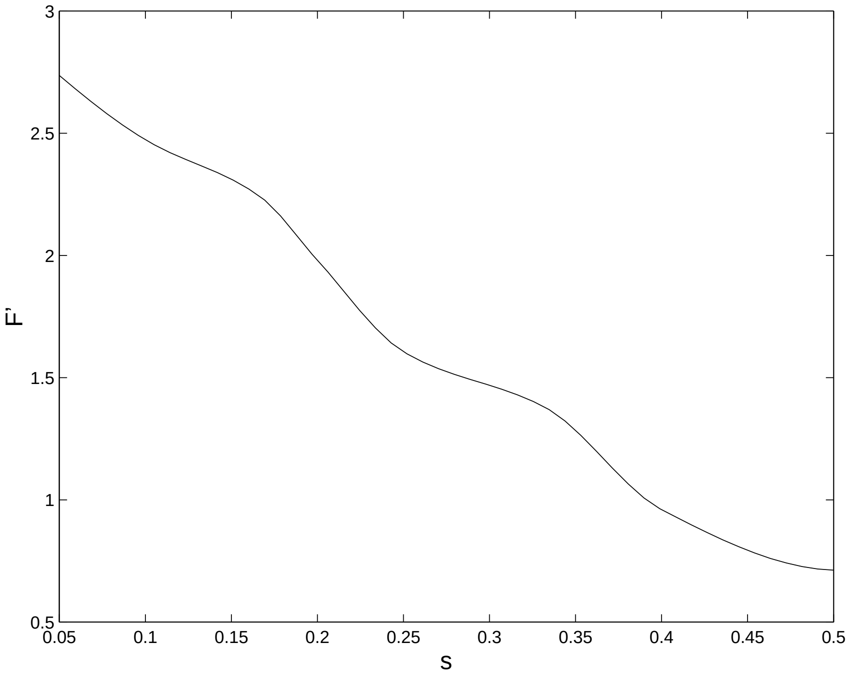

Fig. 5.3 – Dependence of the dimensionless drag force F' on the dimensionless distance s for $h/a = 10$.

For large values of h/a, our numerical results can be compared with those obtained by Faxen by means of the following formula (see [3]; [6], Chapter 7):

$$(7) \qquad F'_F \equiv \frac{F_F}{4\pi\mu a U} = \frac{1}{\ln(h/a) - 0.9157 + 1.7244(a/h)^2 - 1.7302(a/h)^4},$$

where F_F denotes the magnitude of the dimensional force provided by Faxen. All of these results are presented in Figure 5.1. We conclude that for $h/a > 6$ our numerical results are in good agreement with those provided by the formula (7).

• Let us now assume that the centre of the circular obstacle is not located midway between the walls, and let d' be the distance from it to the lower wall L_0. Also, let s be the dimensionless distance given by

$$(8) \qquad s = \begin{cases} \dfrac{d'}{2h} & \text{if} \quad a \le d' \le h \\ 1 - \dfrac{d'}{2h} & \text{if} \quad h \le d' \le 2h - a. \end{cases}$$

Figures 5.2 and 5.3 present the dependence of the dimensionless drag force $F' = F/(4\pi\mu a U)$ on the dimensionless distance s, for some value of the ratio $h/a \in \{5, 10\}$. Each of these cases shows that for a given ratio h/a the smallest value of F' is obtained for the highest value of s, i.e. when the centre of the

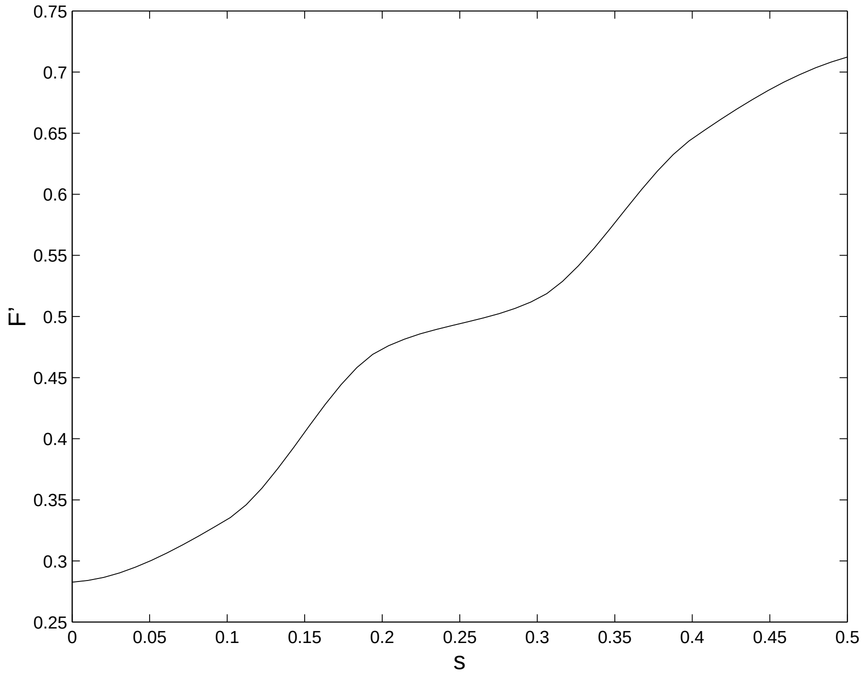

Fig. 5.4 – Dependence of the dimensionless drag force F' on the dimensionless distance s for $c = 10$.

circular obstacle is located midway between the walls. Also, the drag force F' becomes highest when the obstacle approaches one of the walls.

Figure 5.4 shows the dependence of the dimensionless drag force $F' = F/(4\pi\mu aU)$ on the dimensionless distance $s = c\dfrac{a}{2h}$, for $a \leq d' \leq h$ and $c = d'/a$ fixed. We deduce that F' increases as s increases (i.e. $2h/a$ decreases) and becomes smallest for $s = 0$, i.e. when the distance $2h$ between the walls tends to infinity. Consequently, the presence of the upper wall L_1 plays an key role in the behaviour of the dimensionless drag force F', which takes the smallest value (for c fixed) when the boundary of the flow domain consists only of the contour C and a single wall L_0.

• Let us now consider the case of a circular obstacle of radius a translating in a direction perpendicular to the walls in a quiescent incompressible Newtonian fluid. For $s = 1/2$, i.e. when the centre of the circular obstacle is located midway between the walls, and for large values of the ratio h/a, our numerical results can be compared with those obtained by Westberg by means of the following formula (see [6], Chapter 7):

$$(9) \qquad F'_W = \frac{F_W}{4\pi\mu aU} = \frac{1}{\ln(h/a) - 0.62026 + 1.04207(a/h)^2}.$$

Note that F_W is the magnitude of the dimensional force provided by Westberg, and U is the magnitude of the velocity of the obstacle. Figure 5.5 shows the

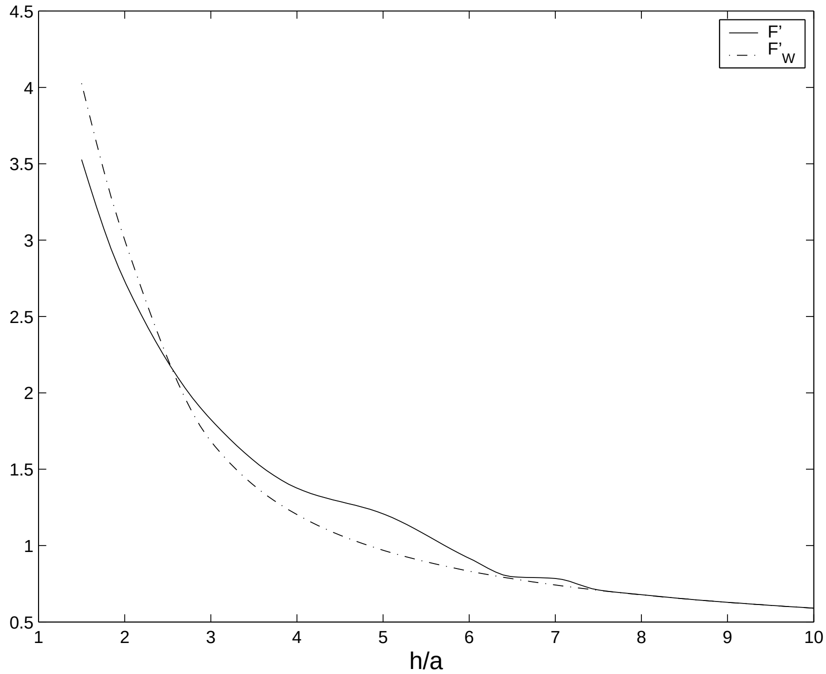

Fig. 5.5 – Dependence of F' and respectively F'_W on the ratio h/a: perpendicular motion

dependence of the drag force $F' = F/(4\pi\mu aU)$ on the dimensionless distance h/a, as well as the corresponding results provided by the Westberg formula (9). We conclude that for $h/a > 6$ our numerical results are in good agreement with those provided by the Westberg formula (9). In addition, the dimensionless drag force $F' = F/(4\pi\mu aU)$ decreases when the dimensionless distance h/a increases. We remark that in each of the above cases a higher number of discretization elements yields comparable results with those obtained for $N = 50$.

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