MATHEMATICA, Tome 47 (70), N° 2, 2005, pp. 151–155

GROUP GRADED ALGEBRAS AND THE RELATIVE FREENESS OF POINTED GROUPS

CAMELIA DICU

Abstract. The main result of Zhou [1] characterizes relative freeness of pointed groups on a G-algebra. We show here that this theorem follows from results on induced modules over group graded algebras.

MSC 2000. 20C20.

Key words. G-algebras, pointed groups, group graded algebras, Green correspondence.

1. INTRODUCTION

1.1. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k of characteristic $p > 0$, and A be an $\mathcal{O}$-algebra. Recall that $\alpha \in \mathcal{P}(A)$ is a point of A if α is a conjugacy class of primitive idempotents of A. Let G be a finite group. A is a G-algebra over $\mathcal{O}$ if there is a group homomorphism $\varphi : G \to \mathrm{Aut}_{\mathcal{O}}(A)$. Throughout this paper, A is a G-algebra. If $a \in A$ and $g \in G$ we write a^g instead of $\varphi(g^{-1})(a)$, and for any $H \leq G$ denote $A^H = \{a \in A \mid a^h = a \text{ for all } h \in H\}$ the set of H-fixed elements of A. The trace map $\mathrm{Tr}_H^G : A^H \to A^G$ maps $a \in A^H$ to $\sum_{g\in[G/H]} a^{g^{-1}}$.

1.2. A pointed group H_β on A is a pair (H, β) where $H \leq G$ and $\beta \in \mathcal{P}(A^H)$. Given two pointed groups G_α and H_θ on A, we say that G_α is projective relative to H_θ if $H \leq G$ and $\alpha \subseteq \mathrm{Tr}_H^G(A^H\theta A^H)$; further G_α is free relative to H_θ if there exist $i \in \alpha$ and $j \in \beta$ such that $i = \mathrm{Tr}_H^G(j)$ and $jj^g = 0$ for any $g \in G \setminus H$. A pointed group P_γ is local if it is not relative projective to any pointed group of Q on A for any $Q < P$. The pointed group P_γ is a defect pointed group of the pointed group G_α if $P_\gamma \leq G_\alpha$, P_γ is local and G_α is relative projective to P_γ.

In general, if j is an idempotent of A^H, $\mathrm{Tr}_H^G(j)$ need not be an idempotent. We say that j has an orthogonal G/H-trace if for any $g \in G \setminus H$ we have $jj^g = 0$. The existence of orthogonal G/H-trace is needed to define induction of divisors. The G-algebra A is called inductively complete if for any pointed group H_β on A, there exist $j \in \beta$ such that j has an orthogonal G/H-trace. Denote $\mathcal{D}(A^H)$ the set of divisors of A^H. The following results are due to Puig [4, Chapter 5].

Theorem 1.3. *Assume that A is an inductively complete G-algebra. Then there exists a unique linear map*

$$\mathrm{ind}_H^G : \mathcal{D}(A^H) \to \mathcal{D}(A^G),$$

mapping $\beta \in \mathcal{P}(A^H)$ *to the divisor* α *containing the idempotent* $\mathrm{Tr}_H^G(j)$*, where* $j \in \beta$ *satisfies* $jj^g = 0$ *for any* $g \in G \setminus H$*.*

THEOREM 1.4. *For any* G*-algebra* A*, there exist an inductively complete* G*-algebra* B *and a divisor* $\omega \in \mathcal{D}(B^G)$ *such that* $A \simeq B_\omega$ *(so in particular,* A *and* B *are Morita equivalent).*

1.5. This paper is a sequel of [2], and our notation is explained there. By using the bijection established in [2, Proposition 2.4] we can interpret a pointed group H_β on A as an isomorphism class of indecomposable R_H-direct summand of A, where $R = A * G$ is the G-graded skew group algebra of A and G. This allows us to consider induction of pointed groups without having to pass to an inductively complete G-algebra as above.

Let G_α and H_β be two pointed groups on A, and let Ai, $i \in \alpha$ and Aj, $j \in \theta$ be the corresponding indecomposable R and R_H-modules. We have that G_α is free relative to H_β if and only if $Ai \simeq R \otimes_{R_H} Aj$. Moreover, by using the characterization of relative projectivity in [2], we see that P_γ is a defect pointed group of G_α if and only if P is a vertex of Ai and Ae is a source of Ai, where $e \in \gamma$. Thus the pointed group version of the Green correspondence can be easily deduced from the version for group graded algebras (see [3, Theorem 1.4.23]).

1.6. We fix a strongly G-graded $\mathcal{O}$-algebra R. If $K$and H are subgroups of G, we denote by $[H\backslash G/K]$ a set of representatives for the double cosets of (H, K) in G, and if $K \leq H$ and V is an R_K-module, we denote $\mathrm{Ind}_K^H V = R_H \otimes_{R_K} V$.

By the above remarks, the following theorem, applied to the G-graded algebra $A * G$, implies [1, Theorem 1.6]. We shall give a module theoretic proof using Green's theory of vertices and sources and the Green correspondence for group graded algebras.

THEOREM 1.7. *Let* R *be a strongly* G*-graded algebra,* P *a* p*-subgroup of* G *and* H *a subgroup of* G *containing* P*. Let* U *an indecomposable* R*-module with vertex* P *and* U' *an indecomposable* R_H*-module with vertex* P*. Let* $\overline{U} \in R_{N_G(P)}\text{-mod}$ *and* $\overline{U'} \in R_{N_H(P)}\text{-mod}$ *be the Green correspondents of* U *and* U' *respectively.*

Then $U \simeq \mathrm{Ind}_H^G U'$ *if and only if* $\overline{U} \simeq \mathrm{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$*, and for any* $Q < P$ *and any* $t \in [N_G(P)\backslash G/H]$ *satisfying* $Q \leq {}^tH$*,* $\mathrm{Ind}_{N_{^tH}(Q)}^{N_G(Q)} \mathrm{Res}_{N_{^tH}(Q)}^{^tH} {}^tU'$ *has no indecomposable* $R_{N_G(Q)}$*-summand with vertex* Q*.*

2. PROOF OF THEOREM 1.7

For the proof we need two preliminary results. The first is a generalization of a theorem of Burry (see [5, Theorem 2.9]). Recall that if V is an R-module and $V_1, \ldots, V_r$ a complete set of nonisomorphic indecomposable direct summands of V such that $V \simeq \bigoplus_{i=1}^r n_i V_i$, then $n_i \in \mathbb{N}$ is called the multiplicity of V_i in the module V.

PROPOSITION 2.1. Let R be a strongly G-graded algebra, P a p-subgroup of G and H a subgroup of G containing P. Let f_G be the Green correspondence with respect to $(G, P, N_G(P))$, and denote

$$I_P = \{t \in [N_G(P)\backslash G/H] \mid P \leq {}^tH\}.$$

a) If V is an R_H-module, then f_G induces a multiplicity-preserving bijection between the nonisomorphic indecomposable direct summands of $\operatorname{Ind}_H^G V$ with vertex P and the nonisomorphic indecomposable direct summands of $\bigoplus_{t\in I_P} \operatorname{Ind}_{N_{{}^tH}(P)}^{N_G(P)} \operatorname{Res}_{N_{{}^tH}(P)}^{{}^tH} {}^tV$ with vertex P.

b) If V is an indecomposable R_H-module with vertex P, then f_G induces a multiplicity-preserving bijection between the nonisomorphic indecomposable direct summand of $\operatorname{Ind}_H^G V$ with vertex P and the nonisomorphic indecomposable direct summands of $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \operatorname{Res}_{N_H(P)}^H V$ with vertex P.

Proof. a) Let $V_1, \dots, V_r$ be nonisomorphic indecomposable R-modules such that $\operatorname{Ind}_H^G V \simeq \bigoplus_{i=1}^r n_i V_i$. We may assume that for an $s \leq r$, all $V_1, \dots, V_s$ have vertex P. Then for any $i \in \{1, \dots, s\}$, $\operatorname{Res}_{N_G(P)}^G V_i$ has a unique indecomposable direct summand with vertex P, namely $f_G(V_i)$. If $s < j \leq r$, then V_j doesn't have vertex P, so by the Burry-Carlson theorem (see [5, Theorem 2.6(ii)]), $\operatorname{Res}_{N_G(P)}^G V_j$ has no indecomposable direct summand with vertex P. Then f_G induces a multiplicity-preserving bijection between the nonisomorphic indecomposable direct summands with vertex P of $\operatorname{Ind}_H^G V$ and $\operatorname{Res}_{N_G(P)}^G \operatorname{Ind}_H^G V$. Note that by the Mackey decomposition,

$$\operatorname{Res}_{N_G(P)}^G \operatorname{Ind}_H^G V \simeq \bigoplus_{x\in[N_G(P)\backslash G/H]} \operatorname{Ind}_{{}^xH\cap N_G(P)}^{N_G(P)} \operatorname{Res}_{{}^xH\cap N_G(P)}^{{}^xH} {}^xV.$$

Thus it suffices to show that if M is an indecomposable $R_{N_G(P)}$-module with vertex P and $M | \operatorname{Ind}_{{}^xH\cap N_G(P)}^{N_G(P)} \operatorname{Res}_{{}^xH\cap N_G(P)}^{{}^xH} {}^xV$, then $P \leq {}^xH$. But this follows from the fact that M is relatively ${}^xH \cap N_G(P)$-projective and M has vertex P, so P is $N_G(P)$-conjugate to a subgroup of ${}^xH \cap N_G(P)$. Therefore $P \leq {}^xH$ and the assertion follows.

b) By a), it suffices to show that if for $t \in I_P$, $\operatorname{Ind}_{N_{{}^tH}(P)}^{N_G(P)} \operatorname{Res}_{N_{{}^tH}(P)}^{{}^tH} {}^tV$ has an indecomposable direct summand M with vertex P, then $t \in N_G(P)H$. But since $M | \operatorname{Ind}_{N_{{}^tH}(P)}^{N_G(P)} \operatorname{Res}_{N_{{}^tH}(P)}^{{}^tH} {}^tV$ we may choose an indecomposable $R_{N_{{}^tH}(P)}$-module W such that $W | \operatorname{Res}_{N_{{}^tH}(P)}^{{}^tH} {}^tV$ and $M | \operatorname{Ind}_{N_{{}^tH}(P)}^{N_G(P)} W$. Since M is a summand of $\operatorname{Ind}_{N_{{}^tH}(P)}^{N_G(P)} W$ and M has vertex P, P is contained in a vertex Q of W. If U is a source of ${}^tV$, we have that $W | \operatorname{Res}_{N_{{}^tH}(P)}^{{}^tH} \operatorname{Ind}_{{}^tP}^{{}^tH} U$. By the Mackey decomposition it is easy to deduce that Q is contained in a ${}^tH$-conjugate of ${}^tP$. Then P is ${}^tH$-conjugate to ${}^tP$, hence $th \in N_G(P)$ for some $h \in H$, and the result is established. □

PROPOSITION 2.2. With notations of Theorem 1.7, if $U \simeq \operatorname{Ind}_H^G U'$ then $\overline{U} \simeq \operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$.

Proof. Observe that $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$ is relatively P-projective since $\overline{U'}$ has vertex P. Denote by M a source of $\overline{U'}$. We first prove that $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$ has no indecomposable direct summand with vertex $Q < P$. Suppose $W | \operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$ is an indecomposable direct summand which is relatively Q-projective, for some $Q < P$. Therefore $W | \operatorname{Ind}_Q^{N_G(P)} W'$ for some R_Q-module W', so we have that $W | \operatorname{Ind}_P^{N_G(P)} \operatorname{Ind}_Q^P W'$. By the Mackey decomposition we have

$$\operatorname{Res}_P^{N_G(P)} W | \bigoplus_{g \in [N_G(P)/P]} {}^g(\operatorname{Ind}_Q^P W'),$$

where ${}^g(\operatorname{Ind}_Q^P W')$ is a relative Q-projective module. Thus any indecomposable direct summand of $\operatorname{Res}_P^{N_G(P)} W$ is relatively Q-projective. But this is not possible, since if $V | \operatorname{Res}_P^{N_G(P)} W$ is an indecomposable direct summand, then $V | \operatorname{Res}_P^{N_G(P)} \operatorname{Ind}_P^{N_G(P)} M$, hence V is isomorphic to ${}^gM$ for some $g \in [N_G(P)/P]$, and therefore has vertex P.

We now apply Proposition 2.1 b) to the indecomposable R_H-module U'. It follows that $f_G(U) = \overline{U}$ is the unique indecomposable direct summand of $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \operatorname{Res}_{N_H(P)}^H U'$ with vertex P. But any indecomposable summand with vertex P of $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$ is a direct summand of $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \operatorname{Res}_{N_H(P)}^H U'$, then is isomorphic to $\overline{U}$ and $\overline{U}$ is the only direct summand of $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$ with vertex P and has multiplicity 1. This implies that $\overline{U} \simeq \operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$. □

Proof of Theorem 1.7. Assume that $U \simeq \operatorname{Ind}_H^G U'$. Proposition 2.2 implies that $\overline{U} \simeq \operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'}$. For any $Q < P$, by Proposition 2.1 a) applied to the R_H-module U' instead of V and Q instead of P, we have that for any $t \in I_Q$, $\operatorname{Ind}_{N_{{}^tH}(Q)}^{N_G(Q)} \operatorname{Res}_{N_{{}^tH}(Q)}^{{}^tH} {}^tU'$ has no indecomposable $R_{N_G(Q)}$-summand with vertex Q.

Conversely, by the Green correspondence, $\operatorname{Res}_{N_H(P)}^H \overline{U}$ has a unique indecomposable direct summand with vertex P, namely $\overline{U'}$. But $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \overline{U'} \simeq \overline{U}$ so $\operatorname{Ind}_{N_H(P)}^{N_G(P)} \operatorname{Res}_{N_H(P)}^H U'$ has a unique indecomposable direct summand with vertex P, namely $\overline{U}$. By Proposition 2.1 b), $\operatorname{Ind}_H^G U'$ has a unique indecomposable direct summand with vertex P and multiplicity 1, namely $f_G^{-1}(\overline{U}) = U$.

But our hypothesis and Proposition 2.1(i) imply that $\mathrm{Ind}_H^G U'$ has no indecomposable direct summands with vertex Q, for any $Q < P$. Therefore $U \simeq \mathrm{Ind}_H^G U'$ and the theorem is proved.

REFERENCES

[1] Zhou, Y., *Relative freeness of pointed groups*, Algebra Colloquium, **10** (2003), 445–450.
[2] Dicu, C. and Marcus, A., *Group graded algebras and the relative projectivity of pointed groups*, to appear.
[3] Marcus, A., *Representation Theory of Group Graded Algebras*, Nova Science Publishers, Commack, NY, 1999
[4] Puig, L., *Blocks of Finite Groups: The Hyperfocal Subalgebra of a Block*, Springer-Verlag, Berlin, 2002.
[5] Karpilovsky, G., *Induced modules over group graded algebras*, North-Holland Mathematics Studies, 1990.
[6] Thevenaz, J., *G-Algebras and Modular Representation Theory*, Clarendon Press, Oxford, 1995.

Received September 29, 2004

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu Nr. 1
400084 Cluj-Napoca, Romania
E-mail: cdicu@math.ubbcluj.ro