MATHEMATICA, Tome 47 (70), N° 2, 2005, pp. 137–143

REMARKS ON UHLENBECK'S PERTURBATION METHOD

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Abstract. Let f be a C^2-function on a C^2-Finsler manifold. Perturb it to $f^\varepsilon = f + \varepsilon g$, $\varepsilon > 0$, $g > 0$ and assume that f^ε satisfies the Palais-Smale condition, for all $\varepsilon > 0$. In [6], K. Uhlenbeck found, under suitable hypothesys, a method to extend the critical point theory from f^ε to f. In this paper we give a variant of this perturbation method.

MSC 2000. 58E05.
Key words. Finsler manifold, critical point, Palais-Smale condition.

1. INTRODUCTION

It is known that there are many variational problems for which the Palais-Smale condition cannot be verified. An approximating method would be useful. In [6] K. Uhlenbeck found an elegant method to extend the critical point theory for the perturbed functions to the original one and applied this technique in particular to harmonic mappings. Instead of looking directly for the critical points of f, a small perturbation εg is considered and the critical points of $f^\varepsilon = f + \varepsilon g$ are studied, with the assumption that f^ε satisfies the Palais-Smale condition. If the critical points of f^ε are convergent to the critical points of f as $\varepsilon \longrightarrow 0$, then a Morse theory is given. The proof use the fact that it is possible to associate to f^ε a function which has the same critical set and it satisfies the Palais-Smale condition.

In this paper we give a variant for the perturbation method introduced by K. Uhlenbeck.

2. MORSE THEORY ON FINSLER MANIFOLDS

In this section we recall basic notions and results on critical point theory on Finsler manifolds (see [2]–[4]).

Let $\pi : E \longrightarrow M$ be a Banach vector bundle. Recall that $\|\cdot\| : E \longrightarrow \mathbf{R}_+$ is called a Finsler structure if

(i) $\|\cdot\|$ is continuous;

(ii) $\forall\, p \in M, \|\cdot\|_p := \|\cdot\|\,|_{E_p}$ is an equivalent norm on $E_p := \pi^{-1}(p)$;

(iii) $\forall\, p_0 \in M$, for any neighborhood U of p_0 which trivializes the vector bundle E, $\forall k > 1$, there exists a neighborhood V of p_0 such that

$$\frac{1}{k}\|\cdot\|_p \le \|\cdot\|_{p_0} \le k\|\cdot\|_p, \forall p \in V.$$

A regular C^1-Banach manifold M together with a Finsler structure on its tangent bundle TM is called a Finsler manifold. Any paracompact Banach manifold posses a Finsler structure on its tangent bundle, so it is a Finsler manifold. If M is a Finsler manifold with Finsler structure $\|\cdot\|$, then we can define

$$\|(p, x^*)\| = \sup\{\langle x^*, x\rangle | x \in T_p(M), \|x\|_p \leq 1\}.$$

Particularly, if M is a paracompact Banach manifold and $f \in C^1(M, \mathbf{R})$, then $\|(p, \mathrm{d}f(p))\|$ is well defined. We denote $\|(p, \mathrm{d}f(p))\|$ by $\|\mathrm{d}f(p)\|$. If M is a Finsler manifold and $\sigma \in C^1([0,1], M)$, then $L(\sigma) = \int_0^1 \|(\sigma(t), \sigma'(t))\| dt$. We define a metric d as follows:

$$d(x, y) = \inf\{L(\sigma) | \sigma \in C^1([0,1], M), \sigma(0) = x, \sigma(1) = y\}.$$

The reduced topology is equivalent to the topology on the manifold.

Let M be a C^1-Finsler manifold and $f \in C^1(M, \mathbf{R})$. A point $p \in M$ is called a critical point of f if $\mathrm{d}f(p) = 0$. A real number c is called a critical value of f if $\exists\ p \in M$ such that $\mathrm{d}f(p) = 0$ and $f(p) = c$. We denote

$$K(f) = \{p \in M \mid \mathrm{d}f(p) = 0\}$$

and we call it the critical set of f. If $c \in \mathbf{R}$, then

$$K_c(f) = K(f) \cap f^{-1}(c)$$

is the critical set of level c of f and

$$M_c(f) = \{p \in M | f(p) \leq c\}$$

is the set of sublevel c of f.

Let $f \in C^1(M, \mathbf{R})$ and $c \in \mathbf{R}$. We say that f satisfies the Palais-Smale condition at level c and we denote it by $(PS)_c$ if any sequence (x_n) in M such that $f(x_n) \longrightarrow c$ and $\mathrm{d}f(x_n) \longrightarrow 0$ has a convergent subsequence. We say that f satisfies the Palais- Smale condition, denoted by (PS), if it satisfies $(PS)_c$ for all $c \in \mathbf{R}$.

Basic tools in critical point theory are deformation theorems (see [1] or [2]); we recall the second deformation lemma:

THEOREM 2.1. *Let M be a C^2-Finsler manifold. Suppose that $f \in C^1(M, \mathbb{R})$ satisfies the $(PS)_c$ condition for all $c \in [a, b]$ and a is the only critical value of f in $[a, b)$. Assume that the connected components of $K_a(f)$ are only isolated points. Then $M_a(f)$ is a strong deformation retract of $M_b(f) \setminus K_b(f)$.*

An operator $L \in B(X)$ (the Banach algebra of all bounded linear operators from a Banach space X into itself) is called hyperbolic if the spectrum $\sigma(L)$ is contained in two compact subsets separated by the imaginary axis.

Let M be a C^2-Finsler manifold and $f \in C^2(M, \mathbb{R})$. Let p_0 be an isolated critical point of f. We say that p_0 is nondegenerate if there exists U a neighborhood of p_0 on which TM is trivialized to a be $U \times X$, such that there exists a hyperbolic operator $L \in B(X)$ which satisfies the following conditions:

(1) $d^2 f(p_0)(Lx, y) = d^2 f(p_0)(x, Ly), \forall x, y \in X$

(2) $d^2 f(p_0)(Lx, x) > 0, \forall x \in X \setminus \{0\}$

(3) $\langle \mathrm{d}f(p), Lx\rangle > 0, \forall p \in U, p = p_0 + x$ in the local coordinates.
The dimension of the negative invariant subspace of L is called the index of p.

The following handle body decomposition theorem gives us a picture of the changes in topological structure as the level sets pass through a critical value such that the corresponding critical points are nondegenerate (see [5] or [1]):

THEOREM 2.2. *Let M be a C^2 Finsler manifold modeled on a Banach space with differentiable norms and let $f \in C^1(M,\mathbb{R})$ satisfying the (PS) condition on M. Assume that c is a isolated critical value of f such that $K_c(f) = \{p_1,\ldots,p_k\}$ consists of only nondegenerate critical points with indices $m_1, \ldots, m_k$ respectively.*

Then there exist $\varepsilon > 0$ and homeomorphisms $h_i : B^{m_i} \to M$ (into), where B^{m_i} is the m_i- dimensional ball, such that $M_{c+\varepsilon}(f)$ can be retracted onto $M_{c-\varepsilon}(f) \cup \bigcup_{i=1}^{k} h_i(B^{m_i})$ and

$$M_{c-\varepsilon}(f) \cap h_i(B^{m_i}) = f^{-1}(c-\varepsilon) \ \cap \ h_i(B^{m_i}) = h_i(\partial B^{m_i})$$

for $i = \overline{1,k}$.

If the assumptions of the previous theorem are safisfied, we say that $M_{c+\varepsilon}(f)$ can be retracted onto $M_{c-\varepsilon}(f)$ with handles adjoined corresponding to the critical points of f in $f^{-1}(c-\varepsilon, c+\varepsilon)$ and the dimensions of the handles correspond to the indices of the critical points.

3. UHLENBECK'S PERTURBATION METHOD

Let f be a given function, which does not satisfy the Palais-Smale condition. Perturb it to $f^\varepsilon = f + \varepsilon g$ and assume that for any small $\varepsilon > 0$, the function f^ε satisfies the Palais-Smale condition. K. Uhlenbeck [6] found sufficient conditions in order to extend the critical point theory from perturbed functions f^ε to f.

For the following abstract results, see [6] and [1].

LEMMA 3.1. *Let M be a C^2-Finsler manifold and let $f, g \in C^1(M,\mathbf{R})$ such that f is bounded below and $g > 0$. For $\varepsilon \in (0, 1]$ define $f^\varepsilon = f + \varepsilon g$. Suppose that $\|\mathrm{d}g\|$ is bounded on sets on which g is bounded and f^ε satisfies $(PS)_c$, for some c.*

Then $h = \frac{g}{c-f}$ satisfies $(PS)_{\varepsilon^{-1}}$ and $K_{\varepsilon^{-1}}(h) = K_c(f^\varepsilon)$.

COROLLARY 3.1. *In the above assumptions, if $K_c(f^\varepsilon) = \emptyset, \forall \varepsilon \in (0, \varepsilon_0]$, then $M_c(f^{\varepsilon_0})$ is a strong deformation retract of $M_c(f)$.*

Morse theory for Finsler manifolds together with the above corollary are used to prove the following theorem:

THEOREM 3.1. *Let M be a C^2-Finsler manifold modeled on a separable Banach space with differentiable norms. Let $f, g \in C^2(M,\mathbf{R})$ satisfying the following assumptions:*

(i) *f is bounded below, $g > 0$ and $f^\varepsilon = f + \varepsilon g$ satisfies (PS), for all $\varepsilon > 0$;*

(ii) *$\|\mathrm{d}g\|$ is bounded on sets on which g is bounded;*

(iii) *the critical set $\bigcup_{0<\varepsilon\leq\varepsilon_0} K(f^\varepsilon) \cap (f^\varepsilon)^{-1}[a, b]$ has compact closure in M, for some ε_0;*

(iv) *a and b are not critical values of f and the critical points of f with values in (a, b) are nondegenerate.*

Then $M_a(f)$ with handles adjoined corresponding to the critical points of f with values in (a, b) is a deformation retract of $M_b(f)$. The dimensions of the handles correspond to the dimensions of the indices of the critical points.

4. A REMARK

We prove the following property:

Proposition 4.1. *Let M be a C^2-Finsler manifold and $f, g \in C^1(M, \mathbf{R})$. For $\varepsilon > 0$ define $f^\varepsilon = f + \varepsilon g$, where $g > 0$. Define $h = \frac{g}{c-f}$, c being a real number. Suppose that $\frac{1}{c-f}$ is bounded and $\|\mathrm{d}g\|$ is bounded.*

Then $K_{\varepsilon^{-1}}(h) = K_c(f^\varepsilon)$. If h satisfies $(PS)_{\varepsilon^{-1}}$, then f^ε satisfies $(PS)_c$.

Proof. If $x_0 \in K_{\varepsilon^{-1}}(h)$, then $h(x_0) = \varepsilon^{-1}$ and $\mathrm{d}h(x_0) = 0$. It follows that $f^\varepsilon(x_0) = c$, $\mathrm{d}f^\varepsilon(x_0) = 0$ and this implies $x_0 \in K_c(f^\varepsilon)$. Conversely, $x_0 \in K_c(f^\varepsilon)$ implies $x_0 \in K_{\varepsilon^{-1}}(h)$.

Let $(x_n) \subset M$ arbitrary such that $f^\varepsilon(x_n) \longrightarrow c$ and $\mathrm{d}f^\varepsilon(x_n) \longrightarrow 0$. Then $h(x_n) - \varepsilon^{-1} = \frac{g(x_n)}{c-f(x_n)} - \varepsilon^{-1} = \frac{f^\varepsilon(x_n)-c}{\varepsilon[c-f(x_n)]}$ and $h(x_n) \longrightarrow \varepsilon^{-1}$. The differential of h at x_n has the representation

$$\begin{aligned}\mathrm{d}h(x_n) =& g(x_n)(c - f(x_n))^{-2}(\mathrm{d}f(x_n) + h(x_n)^{-1}\mathrm{d}g(x_n))\\ =& h(x_n)(c - f(x_n))^{-1}(\mathrm{d}f^\varepsilon(x_n) + (h(x_n)^{-1} - \varepsilon)\mathrm{d}g(x_n))\end{aligned}$$

and this implies

$$\|\mathrm{d}h(x_n)\| \leq |h(x_n)| \cdot |c - f(x_n)|^{-1}(\|\mathrm{d}f^\varepsilon(x_n)\| + |h(x_n)^{-1} - \varepsilon| \cdot \|\mathrm{d}g(x_n)\|)$$

Then $\|\mathrm{d}h(x_n)\| \longrightarrow 0$; because h verifies $(PS)_{\varepsilon^{-1}}$, there is a converging subsequence (x_{n_k}) of (x_n). We conclude that f^ε verifies $(PS)_c$. □

5. A VARIANT OF THE PERTURBATION METHOD

It is possible to give another characterization for the critical set of f^ε, in similar hypothesis. For instance, we can consider a small perturbation $f^\varepsilon = f + \varepsilon g$ for a given function f by using a positive function g. We want to obtain information about the critical set of f^ε, so it is natural to consider the set of some level c for this function, $\{x \in M | f^\varepsilon(x) = c\}$; then $\varepsilon = \frac{c-f(x)}{g(x)}$ and we define $h = \frac{c-f}{g}$.

First of all, we remark that h and f^ε have the same critical set.

PROPOSITION 5.1. *Let M be a C^2-Finsler manifold and let $f, g \in C^1(M, \mathbf{R})$ such that $g > 0$. Define $f^\varepsilon = f + \varepsilon g$, where $\varepsilon > 0$ and $h = \frac{c-f}{g}$, c being a real number.*

Then $K_\varepsilon(h) = K_c(f^\varepsilon)$.

Proof. Assume that $x_0 \in K_c(f^\varepsilon)$. Then $f^\varepsilon(x_0) = c$ and $\mathrm{d}f^\varepsilon(x_0) = 0$. By computation we obtain $h(x_0) = \frac{c-f(x_0)}{g(x_0)} = \frac{c-f^\varepsilon(x_0)+\varepsilon g(x_0)}{g(x_0)} = \varepsilon$ and

$$\begin{aligned}\mathrm{d}h(x_0) &= -\,\mathrm{d}f(x_0)g(x_0)^{-1} - (c - f(x_0))g(x_0)^{-2}\mathrm{d}g(x_0)\\ &= -\,\mathrm{d}f(x_0)g(x_0)^{-1} - h(x_0)g(x_0)^{-1}\mathrm{d}g(x_0)\\ &= -\,g(x_0)^{-1}\mathrm{d}f^\varepsilon(x_0) = 0.\end{aligned}$$

It follows that $x_0 \in K_\varepsilon(h)$. Conversely, $x_0 \in K_\varepsilon(h)$ implies $f^\varepsilon(x_0) = c$ and $\mathrm{d}f^\varepsilon(x_0) = 0$. Then $x_0 \in K_c(f^\varepsilon)$. □

We give now sufficient conditions such that h satisfies the Palais-Smale condition at some level.

PROPOSITION 5.2. *Let M be a C^2-Finsler manifold and let $f, g \in C^1(M, \mathbf{R})$ such that f is bounded and $g > 0$. Define $f^\varepsilon = f + \varepsilon g$, where $\varepsilon > 0$, and $h = \frac{c-f}{g}$, c being a real number. Assume that $\|\mathrm{d}g\|$ is bounded on sets on which g is bounded and f^ε satisfies $(PS)_c$.*

Then h satisfies $(PS)_\varepsilon$.

Proof. Let (x_n) be an arbitrary sequence in M such that $h(x_n) \longrightarrow \varepsilon$ and $\mathrm{d}h(x_n) \longrightarrow 0$. We obtain $f^\varepsilon(x_n) - c = (\varepsilon - h(x_n))g(x_n) \longrightarrow 0$ because f is bounded and $g(x_n) < (1 + \frac{1}{\varepsilon})(c - f(x_n))$. For $\mathrm{d}f^\varepsilon(x_n)$ we can write

$$\mathrm{d}f^\varepsilon(x_n) = \mathrm{d}f(x_n) + \varepsilon\mathrm{d}g(x_n) = -g(x_n)\mathrm{d}h(x_n) + (\varepsilon - h(x_n))\mathrm{d}g(x_n).$$

By using the inequality

$$\|\mathrm{d}f^\varepsilon(x_n)\| \le \|\mathrm{d}h(x_n)\| \cdot g(x_n) + |\varepsilon - h(x_n)| \cdot \|\mathrm{d}g(x_n)\|$$

and the assumption that $\|\mathrm{d}g\|$ is bounded on sets on which g is bounded, it follows that $\|\mathrm{d}f^\varepsilon(x_n)\| \longrightarrow 0$.

$(PS)_c$- condition for f^ε assures the existence of a convergent subsequence (x_{n_k}) of (x_n); we conclude that h satisfies $(PS)_\varepsilon$. □

If the perturbed function f^ε satisfies the Palais- Smale condition at some level, in some hypothesys, the same holds for h.

PROPOSITION 5.3. *Let M be a C^2-Finsler manifold and let $f, g \in C^1(M, \mathbf{R})$. Define $f^\varepsilon = f + \varepsilon g$, for $\varepsilon > 0$, and $h = \frac{c-f}{g}$, where c is a real number. Assume that $g > 0$, $\frac{1}{g}$ is bounded and $\|\mathrm{d}g\|$ is bounded.*

If h satisfies $(PS)_\varepsilon$, then f^ε satisfies $(PS)_c$.

Proof. Let (x_n) be a sequence in M with properties $f^\varepsilon(x_n) \longrightarrow c$ and $\mathrm{d}f^\varepsilon(x_n) \longrightarrow 0$. Then $h(x_n) - \varepsilon = \frac{c-f(x_n)}{g(x_n} - \varepsilon = \frac{c-f^\varepsilon(x_n)}{g(x_n)} \longrightarrow 0$, $\frac{1}{g}$ being bounded. We can write

$$\begin{aligned}\mathrm{d}h(x_n) &= -\,\mathrm{d}f(x_n)g(x_n)^{-1} - (c - f(x_n))g(x_n)^{-2}\mathrm{d}g(x_n)\\ &= -\,g(x_n)^{-1}(\mathrm{d}f(x_n) + h(x_n)\mathrm{d}g(x_n))\\ &= -\,g(x_n)^{-1}(\mathrm{d}f^\varepsilon(x_n) + (h(x_n) - \varepsilon)\mathrm{d}g(x_n)).\end{aligned}$$

It follows that

$$\|\mathrm{d}h(x_n)\| \le g(x_n)^{-1}(\|\mathrm{d}f^\varepsilon(x_n)\| + |h(x_n) - \varepsilon| \cdot \|\mathrm{d}g(x_n)\|).$$

By using the assumptions $\|\mathrm{d}f^\varepsilon(x_n)\| \longrightarrow 0$ and $h(x_n) \longrightarrow \varepsilon$ we obtain that $\|\mathrm{d}h(x_n)\| \longrightarrow 0$.

Because h satisfies $(PS)_\varepsilon$, there exist a converging subsequence (x_{n_k}) of (x_n); then f^ε satisfies $(PS)_c$. □

For simplicity, for $f : M \longrightarrow \mathbf{R}$ and c real number we will use the notation $\widetilde{M}_c(f) = \{x \in M \mid f(x) \ge c\}$.

COROLLARY 5.1. *Suppose all hypothesis of Proposition 5.2 satisfied and* $K_c(f^\varepsilon) = \emptyset$, *for all* $\varepsilon \in (0, \varepsilon_0]$.

Then $\widetilde{M}_c(f)$ *is a strong deformation retract of* $\widetilde{M}_c(f^{\varepsilon_0})$.

Proof. We apply Proposition 5.1 and $K_c(f^\varepsilon) = K_\varepsilon(h)$. Using $K_c(f^\varepsilon) = \emptyset$, for all $\varepsilon \in (0, \varepsilon_0]$, it follows that $K_\varepsilon(h) = \emptyset$, for all $\varepsilon \in (0, \varepsilon_0]$ and we conclude that h has no critical values in the interval $(0, \varepsilon_0]$. But h satisfies $(PS)_\varepsilon$, $\forall\, \varepsilon \in (0, \varepsilon_0]$, so we can apply the second deformation lemma and we obtain that $M_0(h)$ is a strong deformation retract of $M_{\varepsilon_0}(h)$. On the other hand,

$$M_0(h) = \{x \in M \mid h(x) \le 0\} = \{x \in M \mid\ f(x) \ge c\} = \widetilde{M}_c(f)$$

and

$$M_{\varepsilon_0}(h) = \{x \in M \mid h(x) \le \varepsilon_0\} = \{x \in M \mid f^{\varepsilon_0}(x) \ge c\} = \widetilde{M}_c(f^{\varepsilon_0}).$$

The conclusion follows. □

Now we prove the main result of this section:

THEOREM 5.1. *Let* M *be a* C^2*-Finsler manifold modeled on a separable Banach space with differentiable norms. Let* $f, g \in C^2(M, \mathbf{R})$ *satisfying the following assumptions:*

(i) f *is bounded,* $g > 0$ *and* $f^\varepsilon = f + \varepsilon g$ *satisfies* (PS), *for all* $\varepsilon > 0$;

(ii) $\|\mathrm{d}g\|$ *is bounded on sets on which* g *is bounded;*

(iii) *the critical set* $\bigcup\limits_{0<\varepsilon\le\varepsilon_0} K(f^\varepsilon) \cap (f^\varepsilon)^{-1}[a, b]$ *has compact closure in* M, *for some* ε_0;

(iv) a *and* b *are not critical values of* f *and the critical points of* f *with values in* (a, b) *are nondegenerate.*

Then $\widetilde{M_b}(f)$ with handles adjoined corresponding to the critical points of f with values in (a,b) is a deformation retract of $\widetilde{M_a}(f)$. The dimensions of the handles correspond to the dimensions of the indices of the critical points.

Proof. Remark that $\widetilde{M_c}(f) = M_{-c}(-f)$ and $\widetilde{M_c}(f^{\varepsilon_0}) = M_{-c}(-f^{\varepsilon_0})$, for any real number c.

By using Corollary 5.1, $\widetilde{M_a}(f)$ is a strong deformation retract of $\widetilde{M_a}(f^{\varepsilon_0})$ and $\widetilde{M_b}(f)$ is a strong deformation retract of $\widetilde{M_b}(f^{\varepsilon_0})$.

f^{ε_0} satisfies the Palais- Smale condition and the same is true for $-f^{\varepsilon_0}$. Then we can apply Theorem 2.2 and it follows that $M_{-b}(-f^{\varepsilon_0}) \cup \bigcup_{i=1}^{k} h_i(B^{m_i})$ is a deformation retract of $M_{-a}(-f^{\varepsilon_0})$, where $h_i(B^{m_i})$ denotes the attached handle. This is equivalent with the fact that $\widetilde{M_b}(f^{\varepsilon_0}) \cup \bigcup_{i=1}^{k} h_i(B^{m_i})$ is a deformation retract of $\widetilde{M_a}(f^{\varepsilon_0})$.

We obtain that $\widetilde{M_b}(f)$ with handles adjoined corresponding to the critical points of f with values in (a,b) is a deformation retract of $\widetilde{M_a}(f)$. □

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Received November 27, 2004

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