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AN APPLICATION OF CERTAIN INTEGRAL OPERATOR

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Abstract. The integral operator $I^n f(z)$ was introduced by Salagean for functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the unit disc $U = \{z : |z| < 1\}$. The object of the present paper is to give an application of $I^n f(z)$ to Miller and Mocanu's theorem.

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1. INTRODUCTION

Let A denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$ which are analytic in the disc $U = \{z : |z| < 1\}$. For a function $f(z)$ belonging to the class A, we define the integral operator $I^n f(z)$, $n \in N_0 = \{0, 1, 2, \dots\}$, by $I^0 f(z) = f(z)$, $I^1 f(z) = If(z) = \int_0^z f(t)\, t^{-1} \mathrm{d}t$, and $I^n f(z) = I\left(I^{n-1} f(z)\right) \cdot$ The integral operator $I^n f(z)$ was introduced by Salagean [2].

For our purpose, we introduced

Definition 1. *Let H be the set of complex valued functions $h(r,s,t)$;*

$$h(r,s,t) : C^3 \to C \qquad (C \text{ is the complex plane})$$

such that

(i) *$h(r,s,t)$ is continuous in a domain $D \subset C^3$,*
(ii) *$(0,0,0) \in D$ and $|h(0,0,0)| < 1$,*
(iii) *$\left|h\left(\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta} + L\right)\right|$ whenever $\left(\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta} + L\right) \in D$ such that $\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta} L\right) \geq m(m-1)$ for real θ and real $m \geq 1$.*

2. MAIN RESULT

We begin with the ststement of the following lemma due to Miller and Mocanu [1].

Lemma 1. *Let a function $w(z) \in A$ with $w(z) \neq 0$ in U. If $z_o = r_o \mathrm{e}^{\mathrm{i}\theta}$ $(0 < r_o < 1)$ and*

$$|w(z_o)| = \max_{|z| \leq |z_o|} |w(z)|,$$

then

$$z_o w'(z_o) = m w(z_o), \tag{2.1}$$

and

$$\text{Re}\left\{1+\frac{z_o w''(z_o)}{w'(z_o)}\right\} \geq m, \tag{2.2}$$

where m *is real and* $m \geq 1$.

Making use of the above lemma, we prove

THEOREM 2. *Let* $h(r,s,t) \in H$, *and let* $f(z)$ *belonging to* A *satisfy*

$$\left(I^n f(z), I^{n-1} f(z), I^{n-2} f(z)\right) \in D \subset C^3, \tag{2.3}$$

and

$$\left|h\left(I^n f(z), I^{n-1} f(z), I^{n-2} f(z)\right)\right| < 1, \tag{2.4}$$

for $n \geq 2$ *and* $z \in U$. *Then we have*

$$|I^n f(z)| < 1 \qquad (z \in U)\cdot \tag{2.5}$$

Proof. We define the function $w(z)$ by

$$w(z) = I^n f(z) \qquad (n \in N_0), \tag{2.6}$$

for $f(z) \in A$, we have $w(z) \in A$ and $w(z) \neq 0\,(z \in U)$. Note that

$$z\left(I^n f(z)\right)' = I^{n-1} f(z)\cdot \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$I^{n-1} f(z) = z w'(z), \tag{2.8}$$

and

$$I^{n-2} f(z) = z w'(z) + z^2 w''(z)\cdot \tag{2.9}$$

If $z_o = r_o e^{i\theta} \qquad (0 < r_o < 1)$ and

$$|w(z_o)| = \max_{|z| \leq |z_o|} |w(z)| = 1\cdot \tag{2.10}$$

Letting $w(z_o) = e^{i\theta}$ and using (2.1), we see that

$$I^n f(z_o) = w(z_o) = e^{i\theta}, \tag{2.11}$$

$$I^{n-1} f(z_o) = m e^{i\theta}, \text{ and } I^{n-2} f(z_o) = m e^{i\theta} + L, \tag{2.12}$$

where $L = z_o^2 w''(z_o)$ and $m \geq 1$.

Further, an application of (2.2) gives

$$\text{Re}\left\{\frac{z_o w''(z_o)}{w'(z_o)}\right\} = \text{Re}\left\{\frac{z_o^2 w''(z_o)}{m e^{i\theta}}\right\} \geq (m-1), \tag{2.13}$$

or

$$\text{Re}\left\{e^{-i\theta} L\right\} \geq m(m-1)\cdot \tag{2.14}$$

Since $h(r,s,t) \in H$, we have

$$\left|h\left(I^n f(z_o), I^{n-1} f(z_o), I^{n-2} f(z_o)\right)\right| = \left|h\left(\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta}, m\mathrm{e}^{\mathrm{i}\theta} + L\right)\right| > 1, \tag{2.15}$$

which contradicts condition (2.4). Therfore, we conclude that

$$|w(z)| \quad = \quad |I^n f(z)| < 1, \tag{2.16}$$

for all $z \in U$. This completes the assertion of the theorem.

Corollary 3. *Let* $h_1(r,s,t) = s$, *let* $f(z) \in A$ *satisfy the conditions* (2.3) *and* (2.4) *for* $n \geq 2$ *and* $z \in U$. *Then*

$$\left|I^{n+i} f(z)\right| \quad < \quad 1 \qquad (i \geq 0, n \geq 2, z \in U) \cdot \tag{2.17}$$

Proof. Since $h_1(r,s,t) = s \in H$, so with the aid of the theorem, we have

$$\left|I^{n-1} f(z)\right| < 1 \Rightarrow |I^n f(z)| < 1 \qquad (n \geq 2)$$

$$\Rightarrow \quad \left|I^{n+i} f(z)\right| < 1 \qquad (i \geq 0) \cdot \tag{2.18}$$

REFERENCES

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