MATHEMATICA, Tome 46(69), N° 2, 2004, pp. 217–220

INVESTIGATION OF INDEX OF COMPOSITION OF ENTIRE FUNCTIONS

ANDRZEJ WRZESIEŃ

Abstract. In this paper we generalize some of the known results about maximum modulus and maximum term of composition of entire function ([4, 5]).

MSC 2000. 30D20.
Key words. Entire functions, growth theory.

1. INTRODUCTION

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. $\mu(r, f) = \max_{n\geq 0}\{|a_n z^n|\}$ is called the maximum term of $f(z)$ on $|z| = r$ and $M(r, f) = \max_{|z|=r} |f(z)|$ – maximum moduls of $f(z)$ on $|z| = r$. Sato [5] introduced the concept of "index" of an entire function.

If

$$\limsup_{r\to\infty} \frac{\log^{[q]} M(r, f)}{\log r} = \rho_f(q), \quad 0 \leq \rho_f(q) \leq \infty, \tag{1}$$

for $q = 2, 3, \ldots$, where

$$\exp^{[0]} x = \log^{[0]} x = x,$$

$$\exp^{[m]} x = \log^{[-m]} x = \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x),$$

for $\log^{[-m-1]} x > 0, m = 0, \pm 1, \pm 2, \ldots$, then $f(z)$ is said to be of index q if $\rho_f(q - 1) = \infty$ and $\rho_f(q) < \infty$.

Analogous to (1) lower index q is introduced by Bajpai, Kapoor and Juneja [1] as

$$\liminf_{r\to\infty} \frac{\log^{[q]} M(r, f)}{\log r} = \lambda_f(q), \quad 0 \leq \lambda_f(q) \leq \infty \tag{2}$$

if $\lambda_f(q - 1) = \infty$ and $\lambda_f(q) \leq \infty$.

We will say that $f(z)$ has index q of order $\rho_f(q)$ and lower index q of order $\lambda_f(q)$.

2. KNOWN RESULTS

LEMMA 1 (Clunie [2]). *Let $f(z)$ and $g(z)$ be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = (1 - \alpha)^2/4\alpha$. Then*

$$M(r, f\circ g) \geq M\Big(c(\alpha)M(\alpha r, g), f\Big). \tag{3}$$

For $\alpha = \frac{1}{2}$ *we have*

$$M(r, f\circ g) \geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right), f\right). \tag{4}$$

LEMMA 2 (Singh [3]). *Let* $f(z)$ *and* $g(z)$ *be two entire functions with* $g(0) = 0$. *Let* α *satisfying* $0 < \alpha < 1$ *and* $c(\alpha) = (1-\alpha^2)/4\alpha$. *Also let* $0 < \delta < 1$, *then*

$$\mu(r, f\circ g) \geq (1-\delta)\mu\Big(c(\alpha)\mu(\alpha\delta r, g), f\Big). \tag{5}$$

And if $g(z)$ *is any entire function, then with* $\alpha = \delta = \frac{1}{2}$, *for sufficiently large values of* r,

$$\mu(r, f\circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right). \tag{6}$$

3. MAIN RESULT

THEOREM 1. *Let* f *and* g *be two entire functions such that*

$$0 < \lambda_f(q) \leq \rho_f(q) < \infty$$

and

$$0 < \lambda_g(q) \leq \rho_g(q) < \infty.$$

Then for every positive constant A *and every real number* x

$$\lim_{r\to\infty} \frac{\log^{[2q-2]} M(r, f\circ g)}{\left\{\log^{[q]} M(r^A, f)\right\}^{1+x}} = \infty \tag{7}$$

and

$$\lim_{r\to\infty} \frac{\log^{[2q-2]} M(r, f\circ g)}{\left\{\log^{[q]} M(r^A, g)\right\}^{1+x}} = \infty. \tag{8}$$

Proof. If x is such that $1 + x \leq 0$ then the theorem is obvious. So we suppose that $1 + x > 0$. For all sufficiently large values of r we get from (4)

$$M(r, f\circ g) \geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right), f\right)$$

and

$$
\begin{aligned}
\log^{[2q-1]} M(r, f\circ g) &= \log^{[q-1]} \left(\log^{[q]} M(r, f\circ g)\right) \\
&\geq \log^{[q-1]} \left(\log^{[q]} M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right), f\right)\right) \\
&> \log^{[q-1]} \left((\lambda_f(q)-\varepsilon)\log\left(\frac{1}{8}M\left(\frac{r}{2}, g\right)\right)\right) \\
&= \log^{[q-1]} \left((\lambda_f(q)-\varepsilon)\log\frac{1}{8} + (\lambda_f(q)-\varepsilon)\log M\left(\frac{r}{2}, g\right)\right) \\
&> \log^{[q-1]} \left((\lambda_f(q)-\varepsilon)\log\frac{1}{8} + (\lambda_f(q)-\varepsilon)\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g(q)-\varepsilon}\right) \\
&\geq \log^{[q-1]} \left(\left(1+0(1)\right)(\lambda_f(q)-\varepsilon)\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g(q)-\varepsilon}\right) \\
&\geq \log^{[q-1]} \left(\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g(q)-2\varepsilon}\right) = \log\left(\frac{r}{2}\right)^{\lambda_g(q)-2\varepsilon} \\
&= (\lambda_g(q)-2\varepsilon)\cdot\log\frac{r}{2},
\end{aligned} \tag{9}
$$

where $0 < \varepsilon < \min\{\lambda_f(q), \lambda_g(q)\}$.

Also, for all sufficiently large values of r

$$\log^{[q]} M(r, f) < \left(\rho_f(q)+\varepsilon\right)\log r$$

and

$$\left\{\log^{[q]} M(r^A, f)\right\}^{1+x} < A^{1+x}(\log r)^{1+x}\left(\rho_f(q)+\varepsilon\right)^{1+x}. \tag{10}$$

From (9) and (10) it follows that

$$\lim_{r\to\infty} \frac{\log^{[2q-1]} M(r, f\circ g)}{\left\{\log^{[q]} M(r^A, f)\right\}^{1+x}} = 0$$

and consequently

$$\lim_{r\to\infty} \frac{\log^{[2q-2]} M(r, f\circ g)}{\left\{\log^{[q]} M(r^A, f)\right\}^{1+x}} = \infty$$

Statement (8) follows similarly. □

Remark 1.

(i) For $q = 2$ and $x = 0$ this theorem is due to Singh and Baloria [4]
(ii) For $A = 1$, $q = 2$ and $x = 0$ this theorem is due to song and Xang [6].

Remark 2. From proof of this theorem we can conclude that

$$\liminf_{r\to\infty} \frac{\log^{[2q-1]} M(r, f\circ g)}{\log^{[q]} M(r, f)} \geq \frac{\lambda_g(q)}{\rho_f(q)}.$$

This result is sharp. For $f(z) = g(z) = \exp^{[q-1]} z$ we get the equality.

REFERENCES

[1] Bajpai, S.K., Kapoor, G.P. and Juneja, O.P., *On entire functions of fast growth*, Trans. Amer. Math. Soc., **203** (1975), 275–297.

[2] Clunie, J., *The composition of entire and meromorphic functions*, Macintyre Memorial Volume, Ohio University Press, (1970), 75–92.

[3] Singh, A.P., *On maximum term of composition of entire functions*, Proc. Nat. Acad. Sci. India, **59**(A), I (1989), 103–115.

[4] Singh, A.P. and Baloria M.S., *On the maximum modulus and maximum term of composition of entire functions*, Indian J. Pure Appl. Math., **22**(12) (1991), 1019–1026.

[5] Sato D., *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., **69** (1963), 411–414.

[6] Song Guo-Dong and Yang Chung-Chun, *Further growth properties of composition of entire and meromorphic functions*, Indian J. Pure Appl. Math., **15**(1) (1984), 67–82.

Received September 5, 2003

Institute of Mathematics
Technical University of Łódź
PL-90-924 Łódź
Al. Politechniki 11, Poland
E-mail: awrzesien@im0.p.lodz.pl