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COMPACTLY GENERATED SMASHING SUBCATEGORIES

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Abstract. For a smashing subcategory of a compactly generated triangulated category, we give here some necessary and sufficient conditions to be also compactly generated.

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Triangulated categories are a common generalization of the derived categories of an abelian one and the stable homotopy of spectra. The both main examples of triangulated categories are compactly generated. Smashing subcategories naturally arise in the stable homotopy category of spectra, but they are also important in the case of derived categories.

Consider a triangulated category. A set of compact objects generates a smashing subcategory. The generalized smashing conjecture states that every smashing subcategory of a compactly generated triangulated category is also compactly generated. In this generality, the conjecture is known to be false, Keller producing an example of a smashing subcategory which contains no compact objects (see [2]). Still there are some triangulated categories, where the conjecture hold, as example the derived category of an commutative, noetherian ring, as was showing in [6]. A detailed study of this conjecture, and also a proof for an modified version, may be found in [4]. Having in mind the above considerations, it would be very interesting to find necessary and sufficient conditions, for a smashing category to be compactly generated. This is the aim of the present note.

For basic facts about abelian categories, we refer the reader to [8] or [1], and for the general theory of triangulated categories to [7].

A (right) module over a preadditive category $\mathcal{C}$ means, in analogy with the case of ordinary modules over a ring, an additive contravariant functor $M : \mathcal{C}^{\text{op}} \to \mathcal{A}b$. The modules over a category $\mathcal{C}$ will be also called, simply, $\mathcal{C}$-modules. Denote by $\text{Hom}_{\mathcal{C}}(M', M)$ the class of all natural transformations between $\mathcal{C}$-modules M' and M. Provided that $\mathcal{C}$ is skeletally small, the class $\text{Hom}_{\mathcal{C}}(M', M)$ is actually a set for all $\mathcal{C}$-modules M and M', so the class of all modules over $\mathcal{C}$ together with the natural transformations between them form a category, denoted here by $\text{Mod}(\mathcal{C})$, which is Grothendieck by [8, chapter 4, 4.9]. In this category the limits and the colimits are computed pointwise. In the general case, in which $\mathcal{C}$ is not necessary skeletally small, a module M

over the category $\mathcal{C}$ is called *finitely presented*, if there is an exact sequence of functors and natural transformations

$$\mathcal{C}(-,y) \to \mathcal{C}(-,x) \to M \to 0,$$

for suitable $x, y \in \mathcal{C}$. Denote by $\mathrm{mod}(\mathcal{C})$ the class of all finitely presented modules over $\mathcal{C}$. Since by Yoneda lemma $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,x), M) \cong M(x)$, it follows that the natural transformations between two finitely presented $\mathcal{C}$-modules is a set, therefore $\mathrm{mod}(\mathcal{C})$ is a good defined category. If, in addition, $\mathcal{C}$ is triangulated, then the category $\mathrm{mod}(\mathcal{C})$ is an abelian one, by [7, 5.1.10].

In what follows $\mathcal{C}$ will be always a triangulated category (see [7, Definition 1.3.13]) A triangulated subcategory of $\mathcal{C}$ is a full subcategory closed under suspension, which contains the third term of a triangle, whenever it contains the other two. A triangulated subcategory is called *smashing* if the inclusion functor has a right adjoint which preserves coproducts.

For the triangulated category $\mathcal{C}$, denote by $\mathcal{C}_0$ its full subcategory consisting of all compact objects. Recall that an object $c \in \mathcal{C}$ is called *compact* provided that the covariant functor $\mathcal{C}(c,-) : \mathcal{C} \to \mathcal{A}b$ commutes with direct sums. It is well-known, and also easy to see, that $\mathcal{C}_0$ is a *thick* subcategory of $\mathcal{C}$, that means, a triangulated subcategory which is closed under retracts. Throughout of this note we assume $\mathcal{C}$ has arbitrary coproducts, $\mathcal{C}_0$ is a skeletally small category, and it generates $\mathcal{C}$, that is, if $x \in \mathcal{C}$ has the property $\mathcal{C}(c,x) = 0$ for all $c \in \mathcal{C}_0$, then $x = 0$.

Consider a smashing subcategory $\mathcal{B}$ of $\mathcal{C}$. Denote by $\mathbf{i} : \mathcal{B} \to \mathcal{C}$ the inclusion functor and by $\mathbf{a} : \mathcal{C} \to \mathcal{B}$ its right adjoint. Since the right (or the left) adjoint of a triangulated functor is also triangulated, by [7, 5.3.6], it follows that $\mathbf{a}$ is so. Moreover, the Brown representability theorem implies that the functor $\mathbf{a}$ has also a right adjoint (see [7, 8.4.4]), which is denoted here by $\mathbf{k}$. Clearly, $\mathbf{k}$ is fully-faithful. We define also

$$\mathfrak{I} = \{\alpha : c \to d \mid c, d \in \mathcal{C}_0 \text{ and } \alpha \text{ factors through some } x \in \mathcal{B}\}.$$

Obviously $\mathfrak{I}$ is a two-sided ideal of maps in $\mathcal{C}$, that is both the sum of two parallel maps in $\mathfrak{I}$ and also the composition (at left or right) of a map in $\mathfrak{I}$ with other arbitrary map remain in $\mathfrak{I}$. We know by [4, Theorem A] that

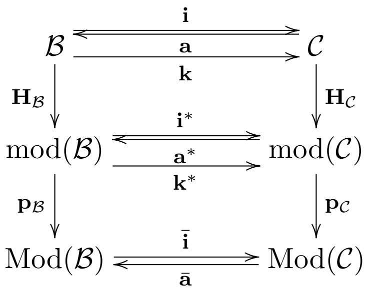
$$\mathcal{B} = \{x \in \mathcal{C} \mid \text{every map } c \to x \text{ with } c \in \mathcal{C} \text{ factors through some } \alpha : c \to d \text{ in } \mathfrak{I}\},$$

and, for any $x \in \mathcal{C}$, $\mathcal{C}(\mathcal{B}, x) = 0$ if and only if $\mathcal{C}(\mathfrak{I}, x) = 0$. Finally, we denote $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{C}_0$. Observe that $\mathcal{B}$ is generated by a set of compact objects if and only if it is compactly generated with a skeleton of $\mathcal{B}_0$ as a set of (compact) generators.

We construct the canonical functor $\mathbf{H}_{\mathcal{C}} : \mathcal{C} \to \mathrm{mod}(\mathcal{C})$, $\mathbf{H}_{\mathcal{C}}(x) = \mathcal{C}(-,x)$, which is an embedding by Yoneda lemma. (It is also called the Yoneda embedding.) Consider also the restriction functor $\mathbf{p}_{\mathcal{C}} : \mathrm{mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C}_0)$, $\mathbf{p}_{\mathcal{C}}(M) = M|_{\mathcal{C}_0}$, and we put $\mathbf{h}_{\mathcal{C}} = \mathbf{p}_{\mathcal{C}} \circ \mathbf{H}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}(\mathcal{C}_0)$. The functors $\mathbf{H}_{\mathcal{B}}$,

$\mathbf{p}_{\mathcal{B}}$ and $\mathbf{h}_{\mathcal{B}}$ are defined similarly. Note that, since $\mathcal{C}$ is compactly generated, it follows that $\mathbf{p}_{\mathcal{C}}$ has a left and a right adjoint, denoted by $\mathbf{L}_{\mathcal{C}}$, respectively $\mathbf{R}_{\mathcal{C}}$. Moreover, the both functors $\mathbf{L}$ and $\mathbf{R}$ are fully-faithful (see also [5]). Obviously, if $\mathcal{B}$ is also compactly generated (by $\mathcal{B}_0$), there is a left adjoint $\mathbf{L}_{\mathcal{B}}$ and a right adjoint $\mathbf{R}_{\mathcal{B}}$ of the functor $\mathbf{p}_{\mathcal{B}}$, the both being fully-faithful.

Note that, with the previous assumptions and notations, there are three unique (up to natural isomorphism) exact functors $\mathbf{i}^*, \mathbf{k}^* : \mathrm{mod}(\mathcal{B}) \to \mathrm{mod}(\mathcal{C})$ and $\mathbf{a}^* : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{B})$ such that $\mathbf{H}_{\mathcal{C}} \circ \mathbf{i} = \mathbf{i}^* \circ \mathbf{H}_{\mathcal{B}}$, $\mathbf{H}_{\mathcal{B}} \circ \mathbf{a} = \mathbf{a}^* \circ \mathbf{H}_{\mathcal{C}}$, respectively $\mathbf{H}_{\mathcal{C}} \circ \mathbf{k} = \mathbf{k}^* \circ \mathbf{H}_{\mathcal{B}}$ (see [7, Lemma 5.3.1]). Moreover $\mathbf{a}^*$ is the right adjoint of $\mathbf{i}^*$ and the left adjoint of $\mathbf{k}^*$, by [7, Lemma 5.3.6]. Since $\mathbf{i}$ restricts to a well-defined functor between the subcategories $\mathbf{B}_0$ and $\mathcal{C}_0$, we have also an adjoint pair $\bar{\mathbf{i}} : \mathrm{Mod}(\mathcal{B}_0) \to \mathrm{Mod}(\mathcal{C}_0)$ and $\bar{\mathbf{a}} : \mathrm{Mod}(\mathcal{C}_0) \to \mathrm{Mod}(\mathcal{B}_0)$. Putting this together we obtain the (not necessary commutative) diagram of categories and functors:



LEMMA 1. *If $\mathcal{B}$ is $\mathcal{B}_0$-generated, then for all $x \in \mathcal{B}$ there are objects $b_\lambda \in \mathcal{B}_0$, $\lambda \in \Lambda$, and a map $\coprod b_\lambda \to x$, such that the induced sequence*

$$\mathcal{C}\left(-, \coprod b_\lambda\right)|_{\mathcal{C}_0} \to \mathcal{C}(-, x)|_{\mathcal{C}_0} \to 0$$

is exact.

Proof. Let $x \in \mathcal{B}$. Since $\mathcal{B}$ is $\mathcal{B}_0$-generated, there is a map $\coprod b_\lambda \to x$, with $b_\lambda \in \mathcal{B}_0$, $\lambda \in \Lambda$, such that the induced sequence

$$\mathcal{B}\left(-, \coprod b_\lambda\right)|_{\mathcal{B}_0} \to \mathcal{B}(-, x)|_{\mathcal{B}_0} \to 0$$

is exact. Indeed, it is enough to choose

$$\Lambda = \{(b, \xi) \mid b \in \mathcal{B}_0 \text{ such that } \mathcal{B}(b, x) \neq 0 \text{ and } \xi \in \mathcal{B}(b, x)\}.$$

This sequence may be rewritten as

$$\mathbf{h}_{\mathcal{B}}(\coprod b_\lambda) \to \mathbf{h}_{\mathcal{B}}(x) \to 0.$$

Since $\bar{\mathbf{i}}$ preserves colimits, having a right adjoint $\bar{\mathbf{a}}$, we obtain another exact sequence

$$(\bar{\mathbf{i}} \circ \mathbf{h}_{\mathcal{B}})(\coprod b_\lambda) \to (\bar{\mathbf{i}} \circ \mathbf{h}_{\mathcal{B}})(x) \to 0.$$

Further, if $\mathcal{B}$ is $\mathcal{B}_0$-generated then, by [4, Proposition 2.6], $\bar{\mathbf{i}} \circ \mathbf{h}_{\mathcal{B}} \cong \mathbf{h}_{\mathcal{C}} \circ \mathbf{i}$, and $\mathbf{i}$ is coproduct preserving, since it has a right adjoint. So the last exact sequence is naturally isomorphic to

$$\begin{array}{ccccc} \mathbf{h}_{\mathcal{C}}(\coprod \mathbf{i}(b_\lambda)) & \longrightarrow & \mathbf{h}_{\mathcal{C}}(\mathbf{i}(x)) & \longrightarrow & 0 \\ \| & & \| & & \\ \mathcal{C}\left(-, \coprod b_\lambda\right)|_{\mathcal{C}_0} & \longrightarrow & \mathcal{C}(-,x)|_{\mathcal{C}_0} & \longrightarrow & 0. \end{array}$$

□

Before to give the next lemma, let we make some notations:

$$\mathcal{B}_0^\top = \{\phi \in \operatorname{Hom}\mathcal{C} \mid \mathcal{C}(b,\phi) = 0 \text{ for all } b \in \mathcal{B}_0\}$$

and

$$\mathfrak{I}^\top = \{\phi \in \operatorname{Hom}\mathcal{C} \mid \mathcal{C}(\alpha,\phi) = 0 \text{ for all } \alpha \in \mathfrak{I}\}.$$

Clearly, $\mathfrak{I}^\top \subseteq \mathcal{B}_0^\top$ since $1_b \in \mathfrak{I}$ for all $b \in \mathcal{B}_0$. Finally denote:

$$\Phi(\mathcal{B}_0) = \{M \in \operatorname{mod}(\mathcal{C}) \mid M \cong \operatorname{im}\mathbf{H}_{\mathcal{C}}(\phi) \text{ for some } \phi \in \mathcal{B}_0^\top\}$$

and

$$\Phi(\mathfrak{I}) = \{M \in \operatorname{mod}(\mathcal{C}) \mid M \cong \operatorname{im}\mathbf{H}_{\mathcal{C}}(\phi) \text{ for some } \phi \in \mathfrak{I}^\top\}.$$

LEMMA 2. *With the above notations, the following equality holds:*

$$\operatorname{Ker}(\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*) = \Phi(\mathcal{B}_0).$$

Proof. First, note that, for all $M \in \operatorname{mod}(\mathcal{C})$, $\mathbf{a}^*(M) = M \circ \mathbf{i} = M|_{\mathcal{B}}$, so $\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*(M) = \mathbf{p}_{\mathcal{B}}(M|_{\mathcal{B}}) = M|_{\mathcal{B}_0}$.

If $M \in \operatorname{mod}(\mathcal{C})$ has the property $M \cong \operatorname{im}\mathbf{H}_{\mathcal{C}}(\phi)$ for some $\phi \in \mathcal{B}_0^\top$, then $M(b) \cong \mathcal{C}(b,\phi) = 0$ for all $b \in \mathcal{B}_0$. Consequently, the restriction of M to $\mathcal{B}_0$ is 0, what means $M \in \operatorname{Ker}(\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*)$.

Conversely, let $M \in \operatorname{Ker}(\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*)$. Since $M \in \operatorname{mod}(\mathcal{C})$, then there is a presentation

$$\mathbf{H}_{\mathcal{C}}(x) \to \mathbf{H}_{\mathcal{C}}(y) \to M \to 0.$$

By Yoneda lemma, the morphism $\mathbf{H}_{\mathcal{C}}(x) \to \mathbf{H}_{\mathcal{C}}(y)$ is of the form $\mathbf{H}_{\mathcal{C}}(\xi)$ for some $\xi : x \to y$. Fitting ξ into a triangle

$$x \xrightarrow{\xi} y \xrightarrow{\phi} z \to \Sigma x,$$

and using the fact that the functor $\mathbf{H}_{\mathcal{C}}$ is homological, we find a morphism $M \to \mathbf{H}_{\mathcal{C}}(z)$ making commutative the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}\mathbf{H}_{\mathcal{C}}(\xi) & \longrightarrow & \mathbf{H}_{\mathcal{C}}(y) & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im}\mathbf{H}_{\mathcal{C}}(\xi) & \longrightarrow & \mathbf{H}_{\mathcal{C}}(y) & \xrightarrow{\mathbf{H}_{\mathcal{C}}(\phi)} & \mathbf{H}_{C}C(z) & & . \end{array}$$

The ker-coker lemma implies that $M \to \mathbf{H}_{\mathcal{C}}(z)$ is a monomorphism. This means $M \cong \operatorname{im} \mathbf{H}_{\mathcal{C}}(\phi)$. Moreover, we have

$$0 = M(b) \cong \operatorname{im} \mathbf{H}_{\mathcal{C}}(\phi)(b) = \operatorname{im} \mathcal{C}(b, \phi),$$

for all $b \in \mathcal{B}_0$, so $\phi \in \mathcal{B}_0^{\top}$. □

Corollary 1. *The following statements hold:*

(a) $\Phi(\mathcal{B}_0)$ *is a Serre subcategory of* $\operatorname{mod}(\mathcal{C})$*;*
(b) *If* $\mathcal{B}$ *is* $\mathcal{B}_0$*-generated, then* $\Phi(\mathcal{B}_0)$ *localizing and colocalizing, and the both quotient categories* $\operatorname{mod}(\mathcal{C})/\Phi(\mathcal{B}_0)$ *and* $\Phi(\mathcal{B}_0)\backslash\operatorname{mod}(\mathcal{C})$ *are equivalent to* $\operatorname{Mod}(\mathcal{B}_0)$*.*

Proof. (a) It is obvious, since the functors $\mathbf{p}_{\mathcal{B}}$ and $\mathbf{a}^*$ are exact.

(b) If $\mathcal{B}$ is compactly generated, then the functors $\mathbf{i}^* \circ \mathbf{L}_{\mathcal{B}}$ and $\mathbf{k}^* \circ \mathbf{R}_{\mathcal{B}}$ are the left, respectively right, adjoint of the functor $\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*$, so the conclusion follows. □

Now we are ready to state the main result of this note:

Theorem 1. *With the above notations, the following statement are equivalent:*

(i) $\mathcal{B}$ *is* $\mathcal{B}_0$*-generated (or equivalent,* $\mathcal{B}$ *is compactly generated);*
(ii) $\mathfrak{I} = \{\alpha : c \to d \mid c, d \in \mathcal{C}_0$ *and* α *factors through some* $b \in \mathcal{B}_0\}$*;*
(iii) $\mathfrak{I}^{\top} = \mathcal{B}_0^{\top}$*;*
(iv) $\Phi(\mathcal{B}_0) = \operatorname{Ker}(\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*)$*.*

Proof. (i)⇒(ii). Let $\alpha : c \to d$ be a map in $\mathfrak{I}$. By the definition of $\mathfrak{I}$, α factors trough an object $x \in \mathcal{B}$. By Lemma 1, there is an exact sequence

$$\mathcal{C}(-, \coprod b_\lambda)|_{\mathcal{C}_0} \to \mathcal{C}(-, x)|_{\mathcal{C}_0} \to 0,$$

for some objects $b_\lambda \in \mathcal{B}_0$. Since $\mathbf{h}_{\mathcal{C}}(c)$ is projective in $\operatorname{Mod}(\mathcal{C}_0)$, it follows that the map $\mathbf{h}_{\mathcal{C}}(c) \to \mathbf{h}_{\mathcal{C}}(x)$ extends making commutative the diagram:

$$\begin{array}{ccccc} & & \mathbf{h}_{\mathcal{C}}(c) & & \\ & \swarrow & \downarrow & & \\ \mathbf{h}_{\mathcal{C}}(\coprod b_\lambda) & \longrightarrow & \mathbf{h}_{\mathcal{C}}(x) & \longrightarrow & 0. \end{array}$$

The map $\mathbf{h}_{\mathcal{C}}(c) \to \mathbf{h}_{\mathcal{C}}(\coprod b_\lambda)$ is induced by a morphism $c \to \coprod b_\lambda$ in $\mathcal{C}$. Since c is compact, we deduce that this last morphism factors trough a finite coproduct $b = b_{\lambda_1} \amalg \cdots \amalg b_{\lambda_n}$. Consequently, the morphism $c \to x$ factors trough $b \in \mathcal{B}_0$, so α does the same, which proves the direct inclusion in (ii). As the converse inclusion is obvious, the implication is shown.

(ii)⇒(i). Let $x \in \mathcal{B}$ such that $\mathcal{C}(b, x) = \mathcal{B}(b, x) = 0$ for all $b \in \mathcal{B}_0$. If $c \to x$ is a map, with $c \in \mathcal{C}_0$, then it factors trough some $\alpha : c \to d$ in $\mathfrak{I}$. By hypothesis, α also factors trough some $b \in \mathcal{B}_0$, so $c \to x$ is zero, since the composite map

$b \to d \to x$ has the same property. We deduce $x = 0$, because $\mathcal{C}$ is compactly generated.

(ii)⇒(iii). The inclusion $\mathfrak{I}^{\top} \subseteq \mathcal{B}_0^{\top}$ is always true, as we already noticed. Let $\phi : y \to z$ be a map in $\mathcal{B}_0^{\top}$. If $\alpha : c \to d$ belongs to $\mathfrak{I}$, then it factors by an object $b \in \mathcal{B}_0$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(d,y) & \xrightarrow{\mathcal{C}(\alpha,\phi)} & \mathcal{C}(c,z) \\ \downarrow & & \uparrow \\ \mathcal{C}(b,y) & \xrightarrow{\mathcal{C}(b,\phi)} & \mathcal{C}(b,z) \end{array}$$

of abelian groups. From $\mathcal{C}(b,\phi) = 0$, we infer $\mathcal{C}(\alpha,\phi) = 0$, showing that $\phi \in \mathfrak{I}^{\top}$.

(iii)⇒(iv). This is obvious, having in mind Lemma 2.

(iv)⇒(i). Consider $x \in \mathcal{B}$, with the property $\mathcal{C}(b,x) = \mathcal{B}(b,x) = 0$ for all $b \in \mathcal{B}_0$. We deduce $\mathcal{C}(-,x) \in \operatorname{Ker}(\mathbf{p}_{\mathcal{B}} \circ \mathbf{a}^*)$. By hypothesis, $\mathcal{C}(-,x)$ is isomorphic to $\operatorname{im}\mathbf{H}_{\mathcal{C}}(\phi)$ for some $\phi \in \mathfrak{I}^{\top}$. Then, for all $\alpha \in \mathfrak{I}$, we have

$$\mathcal{C}(\alpha,x) \cong \operatorname{im}\mathbf{H}_{\mathcal{C}}(\phi)(\alpha) = \operatorname{im}\mathcal{C}(\alpha,\phi) = 0,$$

so $\mathcal{C}(\mathfrak{I},x) = 0$, or equivalently, $\mathcal{C}(\mathcal{B},x) = 0$. Because $x \in \mathcal{B}$, it follows $x = 0$, so $\mathcal{B}$ is compactly generated. □

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