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HEMIVARIATIONAL INEQUALITIES SYSTEMS AND APPLICATIONS

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Abstract. In this paper we trait hemivariational inequality systems. In certain case, this problem can be reduced to study a hemivariational inequality. Several applications are given as Browder and Hartmann-Stampacchia type results and Nash equilibrium point theorems.

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1. INTRODUCTION

Let Ω be a bounded, open subset of $\mathbb{R}^N$ with smooth boundary, and let $K_1, \ldots, K_n$ be subsets of $H_0^1(\Omega)$ $(N, n \geq 1)$. First, we consider the problem:

Find $(u_1, \ldots, u_n) \in K_1 \times \cdots \times K_n$ such that

$$-\triangle u_i + b_i(x)u_i \in -\partial_i j(x, u_1, \ldots, u_n) \text{ in } \Omega, \ \forall i \in \{1, \ldots, n\}, \tag{1}$$

where $b_i \in L^\infty(\Omega, \mathbb{R})$ and $j : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function such that $j(x, \cdot)$ is locally Lipschitz, $\partial_i j$ being the partial Clarke generalized gradient in ith variable, see (3) bellow.

Multiplying by $(v_i - u_i)$ the relation from (1) $(v_i \in K_i)$, integrating over Ω and applying the Green-Gauss formula, we obtain the following problem:

Find $(u_1, \ldots, u_n) \in K_1 \times \cdots \times K_n$ such that for all $v_i \in K_i$ and $i \in \{1, \ldots, n\}$

$$\begin{aligned} &\int_\Omega \nabla u_i \cdot \nabla(v_i - u_i)\mathrm{d}x + \int_\Omega b_i u_i(v_i - u_i)\mathrm{d}x \\ &\qquad + \int_\Omega j_i^0(x, u_1(x), \ldots, u_n(x); v_i(x) - u_i(x))\mathrm{d}x \geq 0, \end{aligned} \tag{HIS^1}$$

where $j_i^0(x, y_1, \ldots, y_n; h_i)$ is the partial Clarke derivative in ith variable of j, see (2), assuming that j satisfies some growth conditions, see (j_i) below.

Therefore, it seems natural to consider the following general problem:
Let $X_1, \ldots, X_n$ be Banach spaces and $T_i : X_i \to L^p(\Omega, \mathbb{R}^k)$, for $i \in \{1, \ldots, n\}$ be linear, continuous operators, $k \geq 1$, $1 \leq p < \infty$, Ω as above. Let K_i be subsets of X_i, $A_i : K_1 \times \cdots \times K_n \to X_i^*$ be given operators, $i \in \{1, \ldots, n\}$ and $j : \Omega \times \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{n} \to \mathbb{R}$ a Carathéodory function such that $j(x, \cdot, \ldots, \cdot)$ is locally Lipschitz, $\forall x \in \Omega$, and satisfies the following assumptions for all $i \in \{1, \ldots, n\}$:

(j$_i$) there exist $h_1^i \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}_+)$ and $h_2^i \in L^\infty(\Omega, \mathbb{R}_+)$ such that

$$|z_i| \leq h_1^i(x) + h_2^i(x)|y|_{\mathbb{R}^{kn}}^{p-1}$$

for almost every $x \in \Omega$, every $y = (y_1, \dots y_n) \in \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{n}$ and $z_i \in \partial_i j(x, y_1, \dots y_n)$.

Our main problem is to study the following general *hemivariational inequality system*:

Find $(u_1, \dots, u_n) \in K_1 \times \cdots \times K_n$ such that for all $v_i \in K_i$ and $i \in \{1, \dots, n\}$

$$(HIS^2) \qquad \langle A_i(u_1, \dots, u_n), v_i - u_i\rangle + \int_\Omega j_i^0(x, T_1u_1(x), \dots, T_iu_i(x), \dots, T_nu_n(x); T_iv_i(x) - T_iu_i(x))\mathrm{d}x \geq 0.$$

We say that $(u_1, \dots, u_n) \in K_1 \times \cdots \times K_n$ is an *equilibrium point* for the above system. Let fix the above notations. $j_i^0(x, y_1, \dots, y_n; h_i)$ is the partial Clarke derivative in ith variable (the Clarke derivative of the locally Lipschitz mapping $j(x, y_1, \dots, y_{i-1}, \cdot, y_{i+1}, \dots, y_n)$ at the point $y_i \in \mathbb{R}^k$ with the direction $h_i \in \mathbb{R}^k$), that is

$$(2) \qquad j_i^0(x, y_1, \dots, y_n; h_i) = \limsup_{\substack{y' \to y_i \\ t \to 0^+}} \frac{1}{t}(j(x, y_1, \dots, y_{i-1}, y' + th_i, y_{i+1}, \dots, y_n) - j(x, y_1, \dots, y_{i-1}, y', y_{i+1}, \dots, y_n)).$$

The set $\partial_i j(x, y_1, \dots, y_n)$ is the Clarke partial generalized gradient in the ith variable of the mapping $j(x, y_1, \dots, y_{i-1}, \cdot, y_{i+1}, \dots, y_n)$ at the point $y_i \in \mathbb{R}^k$, i.e.

$$(3) \qquad \partial_i j(x, y_1, \dots, y_n) = \{z_i \in \mathbb{R}^k : \langle z_i, h_i\rangle \leq j_i^0(x, y_1, \dots, y_n; h_i), \text{ for all } h_i \in \mathbb{R}^k\}.$$

If $n = 1$, the problem (HIS^2) reduces to a classical hemivariational inequality, see [6], [7], [9], where several applications were given; for example the function u being the temperature in the case of heat conduction problems, pressure and electric potential in problems of hydraulics and electrostatics.

The motivation to study such systems comes from the above examples. In fact, in a given mechanical problem several variables can occur in same time, for example temperature, pressure, etc; the equilibrium of such mechanical systems depending of these functions.

Moreover, the idea of considering such systems is also motivated by the Nash equilibrium theory. Kassay, Kolumbán and Páles in [5] introduced the notion of Nash stationary point, i.e. such point in which a certain kind of derivative is nonnegative. From our problem (HIS^2) we can deduce the existence of (weak) Nash stationary points and Nash equilibrium points for a certain class of functions.

The paper is constructed as follows. In Section 2 we give sufficient conditions to obtain equilibrium points for (HIS^2). In Section 3 we give some applications: Nash equilibrium type theorem, Browder and Hartman-Stampacchia type theorems from variational inequalities, deducing also easily the Brouwer fixed point theorem.

2. EXISTENCE RESULTS FOR (HIS^2)

In order to obtain solution for (HIS^2), we need some preliminary notions and results.

Let X, Y be two Banach spaces, K a subset of X and $A : K \to Y^*$ be an operator.

Definition 1. $A : K \to Y^*$ is said to be generalized w^*-demicontinuous if for any sequence $\{u_n\} \subset K$ converging to u (in the strong topology), the sequence $\{A(u_n)\}$ converges to $A(u)$ in the w^*-topology of Y^*.

Remark 1. If $X = Y$, the generalized w^*-demicontinuity reduces to the classical w^*-demicontinuity.

Lemma 1. [3] *Let X be a Hausdorff topological vector space, K a subset of X and for each $x \in K$, let $S(x)$ be a closed subset of X, such that*
(i) *there exists $x_0 \in K$ such that the set $S(x_0)$ is compact;*
(ii) *S is KKM-mapping, i.e. for each $x_1, x_2, \ldots, x_n \in K$,*

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} S(x_i),$$

where co stands for the convex hull operator. Then

$$\bigcap_{x \in K} S(x) \neq \emptyset.$$

First, we establish a similar result as in [9] for hemivariational inequalities, which will be used later to obtain solution for our hemivariational inequality system.

Therefore, let us put ourselves within the framework of [9], i.e. let X be a Banach space, $T : X \to L^p(\Omega, \mathbb{R}^k)$, be a linear and continuous operator where $1 \le p < \infty$, $k \ge 1$; $\Omega \subset \mathbb{R}^N$ being bounded and open. Let K be a subset of X and $A : K \to X^*$ an operator and $j = j(x, y) : \Omega \times \mathbb{R}^k \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz with respect to the second variable $y \in \mathbb{R}^k$, satisfying the following assumption
(j) there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}_+)$ and $h_2 \in L^\infty(\Omega, \mathbb{R}_+)$ such that

$$|z| \le h_1(x) + h_2(x)|y|_{\mathbb{R}^k}^{p-1}$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $z \in \partial j(x, y)$.

In the sequel, we give a non-compact variant of [9, Theorem 1], using a coercivity assumption.

Theorem 1. *Let K be a closed, convex subset of a Banach space X and let j satisfy the condition* (j)*. If the operator $A : K \to X^*$ is w^*-demicontinuous and in addition*
(HC^1) there exist $K_0 \subset K$ compact and $w_0 \in K_0$ such that

$$\langle Av, w_0 - v\rangle + \int_\Omega j^0(x, Tv(x); Tw_0(x) - Tv(x))\mathrm{d}x < 0, \ \forall v \in K \setminus K_0,$$

then there exists $u \in K$ such that for every $v \in K$

$$(P) \qquad \langle Au, v - u\rangle + \int_\Omega j^0(x, Tu(x); Tv(x) - Tu(x))\mathrm{d}x \geq 0.$$

Proof. For any $v \in K$, set

$$S(v) = \{u \in K : \ \langle Au, v - u\rangle + \int_\Omega j^0(x, Tu(x); Tv(x) - Tu(x))\mathrm{d}x \geq 0\}.$$

Clearly, $S(v) \neq \emptyset$, $\forall v \in K$, since $v \in S(v)$. Moreover, $S(v)$ is closed, see [9, p. 47]. It's easy to verify that S is KKM-mapping, using the linearity of T and the fact that $z \mapsto j^0(x, y; z)$ is sublinear. Finally, from (HC^1) we have that $S(w_0) \subset K_0$. Since K_0 is compact, it follows that $S(w_0)$ is also compact. From Lemma 1 we obtain that $\bigcap_{v\in K} S(v) \neq \emptyset$, which means that (P) has at least a solution. □

Remark 2. If $K \subset X$ is compact in Theorem 1 the hypothesis (HC^1) can be omitted and we obtain exactly Theorem 1 from [9].

Definition 2. [2, p. 39] Let V be a Banach space, $f : V \to \mathbb{R}$ be a locally Lipschitz function. f is said to be regular at $v \in V$ if for all $h \in V$ the usual one-sided directional derivative $f'(v; h)$ exists and $f'(v; h) = f^0(v; h)$. f is regular if it is regular in every point $v \in V$.

Lemma 2. *Let $j : \Omega \times \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{n} \to \mathbb{R}$ be a function such that $j(x, \cdot, \ldots, \cdot)$ is locally Lipschitz and regular for all $x \in \Omega$. Then, for all $(y_1, \ldots, y_n) \in \mathbb{R}^{kn}$ and $(h_1, \ldots, h_n) \in \mathbb{R}^{kn}$ we have:*

(i) $\partial j(x, y_1, \ldots, y_n) \subseteq \partial_1 j(x, y_1, \ldots, y_n) \times \cdots \times \partial_n j(x, y_1, \ldots, y_n)$;
(ii) $j^0(x, y_1, \ldots, y_n; h_1, \ldots, h_n) \leq \sum_{i=1}^n j_i^0(x, y_1, \ldots, y_n; h_i)$;
(iii) $j^0(x, y_1, \ldots, y_i, \ldots, y_n; 0, \ldots, h_i, \ldots, 0) \leq j_i^0(x, y_1, \ldots, y_n; h_i)$.

Proof. For (i), see [2, Proposition 2.3.15].

(ii) Choose $z \in \partial j(x, y_1, \ldots, y_n)$ such that $j^0(x, y_1, \ldots, y_n; h_1, \ldots, h_n) = \langle z, (h_1, \ldots, h_n)\rangle$ (such a z exists). Due to (i), we have $z = (z_1, \ldots, z_n)$, for all $z_i \in \partial_i j(x, y_1, \ldots, y_n)$. Therefore, we get $j^0(x, y_1, \ldots, y_n; h_1, \ldots, h_n) = \sum_{i=1}^n \langle z_i, h_i\rangle \leq \sum_{i=1}^n j_i^0(x, y_1, \ldots, y_n; h_i)$.

(iii) Since $j(x,\cdot,\dots,\cdot)$ is regular, we have

$$\begin{aligned} j^0(x,&y_1,\dots,y_i,\dots,y_n;0,\dots,h_i,\dots,0)\\ &= j^{'}(x,y_1,\dots,y_i,\dots,y_n;0,\dots,h_i,\dots,0)\\ &= \lim_{t\to 0^+}\frac{j(x,y_1,\dots,y_{i-1},y_i+th_i,y_{i+1},\dots,y_n)-j(x,y_1,\dots,y_n)}{t}\\ &\le \limsup_{\substack{y'\to y_i\\ t\to 0^+}}\frac{1}{t}(j(x,y_1,\dots,y_{i-1},y'+th_i,y_{i+1},\dots,y_n)\\ &\qquad -j(x,y_1,\dots,y_{i-1},y',y_{i+1},\dots,y_n))\\ &= j_i^0(x,y_1,\dots,y_n;h_i). \end{aligned}$$

□

Now, we give the main result of this paper. Let $X_i,\ K_i,\ T_i,\ A_i$ for $i\in\{1,\dots,n\}$ and j as in the introduction.

THEOREM 2. *Assume that K_i are closed and convex, A_i are generalized w^*-demicontinuous and j satisfy* (j_i) *for all $i\in\{1,\dots,n\}$. In addition, if $j(x,\cdot,\dots,\cdot)$ is regular for all $x\in\Omega$ and*
(HC^2) there exist $K_0^i\subset K_i$ compact and $w_i^0\in K_0^i$ such that for all $(v_1,\dots,v_n)\in K_1\times\dots\times K_n\setminus K_0^1\times\dots\times K_0^n$

$$\sum_{i=1}^n\Big[\langle A_i(v_1,\dots,v_n),w_i^0-v_i\rangle+$$

$$+\int_\Omega j_i^0(x,T_1v_1(x),\dots,T_iv_i(x),\dots,T_nv_n(x);T_iw_i^0(x)-T_iv_i(x))\mathrm{d}x\Big]<0.$$

Then (HIS^2) has at least an equilibrium point.

Proof. We will verify the hypotheses from Theorem 1. Let $X:=X_1\times\dots\times X_n,\ K:=K_1\times\dots\times K_n$. We define $A:K\to X^*\simeq X_1^*\times\dots\times X_n^*$ by

$$\langle A(u_1,\dots,u_n),(v_1,\dots,v_n)\rangle=\sum_{i=1}^n\langle A_i(u_1,\dots,u_n),v_i\rangle.$$

It is easy to see that A is w^*-demicontinuous. Moreover, let

$$T:=(T_1,\dots,T_n):X\to L^p(\Omega,\mathbb{R}^k)\times\dots\times L^p(\Omega,\mathbb{R}^k)\simeq L^p(\Omega,\mathbb{R}^{nk}),$$

where $T(u_1,\dots,u_n)=(T_1u_1,\dots,T_nu_n),\ u_i\in X_i$. Clearly, T is well-defined, continuous and linear. With the above notations, $K_0=K_0^1\times\dots\times K_0^n$ and $w_0=(w_1^0,\dots,w_n^0)$ satisfy (HC^1) from Theorem 1. In fact, since $j(x,\cdot,\dots,\cdot)$ is regular for all $x\in\Omega$ and T is linear, from Lemma 2 (ii) and (HC^2) we have for all $v=(v_1,\dots,v_n)\in K\setminus K_0$ that the expression

$$\langle Av,w_0-v\rangle+\int_\Omega j^0(x,Tv(x);Tw_0(x)-Tv(x))\mathrm{d}x=\sum_{i=1}^n\langle A_i(v_1,\dots,v_n),w_i^0-v_i\rangle$$

$$+\int_\Omega j^0(x,T_1v_1(x),\ldots,T_nv_n(x);T_1w_1^0(x)-T_1v_1(x),\ldots,T_nw_n^0(x)-T_nv_n(x))\mathrm{d}x$$

is strictly negative.

Hypothesis (j) follows with the choice $h_1(x) := \sum_{i=1}^n h_1^i(x) \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}_+)$ and $h_2(x) := \sum_{i=1}^n h_2^i(x) \in L^\infty(\Omega,\mathbb{R}_+)$, using Lemma 2 ($i$) and ($\mathrm{j}_i$) for all $i \in \{1,\ldots,n\}$, with $k := kn$.

Therefore, there exists $u = (u_1,\ldots,u_n) \in K_1 \times \cdots \times K_n$ such that for all $v = (v_1,\ldots,v_n) \in K_1 \times \cdots \times K_n$ the expression

$$\sum_{i=1}^n \langle A_i(u_1,\ldots,u_n), v_i - u_i\rangle$$

$$+\int_\Omega j^0(x,T_1u_1(x),\ldots,T_nu_n(x);T_1v_1(x)-T_1u_1(x),\ldots,T_nv_n(x)-T_nu_n(x))\mathrm{d}x$$

is positive. Let us fix an $i \in \{1,\ldots,n\}$ and put $v_j := u_j,\ j \neq i$ in the above inequality. Due to Lemma 2 (iii), we obtain that for all $v_i \in K_i$, the expression

$$\langle A_i(u_1,\ldots,u_n), v_i - u_i\rangle + \int_\Omega j_i^0(x,T_1u_1(x),\ldots,T_nu_n(x);T_iv_i(x)-T_iu_i(x))\mathrm{d}x$$

is positive. Since $i \in \{1,\ldots,n\}$ was arbitrary, the proof of the theorem is complete. □

Due to Remark 2, we have the following

Theorem 3. *Assume that K_i are compact and convex, A_i are generalized w^*-demicontinuous and j satisfy* (j_i) *for all $i \in \{1,\ldots,n\}$. If $j(x,\cdot,\ldots,\cdot)$ is regular for all $x \in \Omega$, then* (HIS^2) *has at least an equilibrium point.*

3. APPLICATIONS

First, we obtain a result from the theory of variational inequality systems.

Theorem 4. *Assume that K_i are compact and convex subsets of the Banach spaces X_i and that $A_i : K_1\times\cdots\times K_n \to X_i^*$ are generalized w^*-demicontinuous for all $i \in \{1,\ldots,n\}$. Then there exists $(u_1,\ldots,u_n) \in K_1\times\cdots\times K_n$ such that*

$$\langle A_i(u_1,\ldots u_n), v_i - u_i\rangle \geq 0,\ \forall v_i \in K_i,\ \forall i \in \{1,\ldots,n\}.$$

Proof. In Theorem 3 we substitute $j := 0$. □

Remark 3. From the above theorem we obtain Browder and Hartman-Stampacchia type results, see [1] and [4] respectively, taking $n := 1$.

Moreover, we can easily deduce the well-known Brouwer's fixed point theorem.

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a convex, compact set and $f : K \to K$ be a continuous function. Then f has at least a fixed point.*

Proof. Let $X_1 = X_2 := \mathbb{R}^n$, $K_1 = K_2 := K$, $A_1(u,v) = u - f(v)$ and $A_2(u,v) = v - u$. Clearly, A_1 and A_2 are generalized w^*-demicontinuous, due to the continuity of f. Therefore, from Theorem 4 follows that there exists $(u_1, u_2) \in K \times K$ such that for every $v, w \in K$

$$\langle u_1 - f(u_2), v - u_1 \rangle \geq 0 \text{ and } \langle u_2 - u_1, w - u_2 \rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ is the euclidian inner product. Substituting $w := u_1$ in the second inequality, we obtain $u_1 = u_2$. Putting $v := f(u_1)$ in the first inequality, we obtain $f(u_1) = u_1$, i.e. u_1 is a fixed point of f. □

Now, we are interested in finding weak Nash stationary and Nash equilibrium points. Let $X_1, \ldots, X_n$ be Banach spaces and $K_i \subset X_i$ be nonempty compact and convex sets for each $i \in \{1, \ldots, n\}$. Let $D_1, \ldots, D_n$ open, convex sets such that $K_i \subset D_i$, and $f_i : K_1 \times \cdots \times D_i \times \cdots \times K_n \to \mathbb{R}$, $i \in \{1, \ldots, n\}$.

Definition 3. (i) An element $(u_1, \ldots, u_n) \in K_1 \times \cdots \times K_n$ is called Nash equilibrium point of functions $f_1, \ldots, f_n$ if for each $i \in \{1, \ldots, n\}$

$$f_i(u_1, \ldots, v_i, \ldots, u_n) \geq f_i(u_1, \ldots, u_i, \ldots, u_n), \ \forall v_i \in K_i.$$

(ii) Assume that the partial derivatives $\partial_i f_i$ exist on $K_1 \times \cdots \times D_i \times \cdots \times K_n$ for each $i \in \{1, \ldots, n\}$. An element $(u_1, \ldots, u_n) \in K_1 \times \cdots \times K_n$ is called weak Nash stationary point of functions $f_1, \ldots, f_n$ if for each $i \in \{1, \ldots, n\}$

$$\partial_i f_i(u_1, \ldots, u_n)(v_i - u_i) \geq 0, \ \forall v_i \in K_i.$$

Remark 4. The first notion is due to Nash, see [8]; the second one is similar to the notion introduced by Kassay, Kolumbán and Páles, see [5].

Theorem 5. *Let $K_1, \ldots, K_n$ be compact, convex subsets in the Banach spaces $X_1, \ldots, X_n$. Let $D_1, \ldots, D_n$ be open, convex sets such that $K_i \subset D_i$ and $f_i : K_1 \times \cdots \times D_i \times \cdots \times K_n \to \mathbb{R}$ functions such that the partial derivatives $\partial_i f_i$ exist and are continuous on $K_1 \times \cdots \times D_i \times \cdots \times K_n$ for each $i \in \{1, \ldots, n\}$. Then there exists at least one weak Nash stationary point of functions $f_1, \ldots, f_n$.*

Proof. Let $A_i := \partial_i f_i : K_1 \times \cdots \times K_n \to X_i^*$. By the continuity property of $\partial_i f_i$, we have that the operators A_i are generalized w^*-demicontinuous. Now, we apply Theorem 4. □

Corollary 2. *Let X_i, K_i, D_i and f_i as in Theorem 5. In addition, if for all fixed $u_j \in K_j$ $(j \neq i)$,*

$$y_i \mapsto f_i(u_1, \ldots, y_i, \ldots, u_n) \ (y_i \in D_i)$$

is convex, then there exists at least one Nash equilibrium point of functions $f_1, \ldots, f_n$.

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