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THE CONSTRUCTION OF COMPLETED SKEW GROUP RINGS. II

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Abstract. Given a profinite group G acting on a profinite ring R via $\sigma : G \to \mathrm{Aut}R$ (with finite image) we will construct the completition of the skew group ring $R * G$ as a inverse limit of skew group rings of type $R * (G/N)$, where N is a open invariant subgroup of G contained in $\ker \sigma$.

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Key words. Skew group ring, compact ring, profinite group (ring), completion of a topological ring.

1. INTRODUCTION

Let R be a profinite ring, G a profinite group and $\sigma : G \to \mathrm{Aut}R$ be a group homomorphism of G in the group $\mathrm{Aut}R$ of all continuous automorphisms of R such that $\mathrm{Im}\sigma$ is finite. We have proved in [1] that if has a local base consisting of G-invariant ideals, then there exists the completed skew group ring $\widehat{R * G}$. We give in this paper another construction of the completed skew group ring $\widehat{R * G}$. We observe (see Remark 1) that the condition on R to have a local base consisting of G-invariant ideals follows from the condition that $\mathrm{Im}\sigma$ is finite.

2. PRELIMINARIES

By $\mathrm{Aut}R$ is denoted the group of all continuous automorphism of a topological ring R and by $\mathbb{N}^*$ denotes the set of all positive integers.

Let R be a ring with identity, G a group and $\sigma : G \to \mathrm{Aut}R$ be a group homomorphism. If $g \in G$ and $r \in R$ denote by

$$r^g = \sigma(g)(r).$$

The known construction of a *skew group ring* $R * G$ (see, e.g. [4]) is defined to be the free left R-module with G as a free generating set. The multiplication on $R * G$ is defined distributively by using the following rule:

$$(r_1 g_1) \cdot (r_2 g_2) = r_1 r_2^{g_1}\, g_1 g_2\,,$$

for all $r_1, r_2 \in R$ and $g_1, g_2 \in G$.

Evidently, if $\sigma(g) = \mathrm{id}_R$, for all $g \in G$, then the skew group ring $R * G$ coincides with the group ring $R[G]$.

Let R be a profinite ring with identity, let G be a profinite group, and let $\sigma : G \to \mathrm{Aut}R$ be a group homomorphism. If V is an open ideal of R and if

N is an open invariant subgroup of G, consider the subgroup of $R*G$,

$$(V,N) = V*G + (1-N)_l\,,$$

where $(1-N)_l$ is the left ideal of $R*G$ generated by the set $1-N$ and $V*G$ is a right ideal of $R*G$,

$$V*G = \left\{\sum_{i=1}^{k} v_i g_i : k\in\mathbb{N}^*;\ v_1,\ldots,v_k\in V;\ g_1,\ldots,g_k\in G\right\}.$$

If $V^g\subseteq V$, for all $g\in G$, then $V*G$ is a two-sided ideal of $R*G$ and if $N\subseteq\ker\sigma$, then $(1-N)_l$ is a two-sided ideal of $R*G$.

Consider now that $V^g\subseteq V$, for all $g\in G$. We introduce a totally bounded ring topology on $R*G$ as follows: for each left ideal (V,N) of $R*G$ denote by $\widetilde{(V,N)}$ the largest cofinite two-sided ideal of $R*G$ for which

$$V*G\subseteq\widetilde{(V,N)}\subseteq(V,N)\subseteq R*G.$$

Consider the finite intersections of ideals of type $\widetilde{(V,N)}$ as a fundamental system of neighborhoods of zero for a ring topology $\mathfrak{T}$ on $R*G$. Its completion will be called the *completed skew group ring.*

THEOREM 1. *If $V^g\subseteq V$, for all $g\in G$, and for all open ideal V of R and* Imσ *is finite, then $(R,\mathfrak{T}_1)$ is a topological subring of $(R*G,\mathfrak{T})$ and $(G,\mathfrak{T}_2)$ is a topological subgroup of the group of units of the ring $(R*G,\mathfrak{T})$.*

This construction of the completed skew group ring can be found in [1].

REMARK 1. If R is a compact ring with identity and G a finite group of continuous automorphisms of R, then R has a local base consisting of G-invariant ideals.

Indeed, if V is an open ideal of R, then there exists an open ideal I of R such that $\alpha(I)\subseteq V$, for all $\alpha\in G$. Consider the open ideal $U=\sum_{\alpha\in G}\alpha(I)$ of R. Obviously, $U\subseteq V$ and $\alpha(U)\subseteq U$, for every $\alpha\in G$.

By Remark 1, the condition that Imσ is finite of Theorem 1 implies the existence of an fundamental system of neighborhoods of 0 consisting of open G-invariant ideals of R.

In this paper we give another construction of the completion of the skew group ring $R*G$ in the case that Imσ is finite, through the inverse limit of skew group rings $R*G/N$, $N\in\mathcal{N}$, where

$$\mathcal{N} = \{N : N \text{ is an open invariant subgroup of } G,\ \text{such that } N\subseteq\ker\sigma\}.$$

This construction is analogous with the construction of the completed group ring $R[[G]]$ given in [3].

3. COMPLETED SKEW GROUP RING

Let R be a compact ring with identity, G a finite group and $\sigma : G \to \mathrm{Aut}R$ be a group homomorphism. For any open ideal V of R, consider the right ideal $\widetilde{V}$ of $R * G$, where

$$\widetilde{V} = V * G = \left\{ \sum_{g \in G} v_g \, g : v_g \in V, \text{ for all } g \in G \right\}.$$

LEMMA 1. *If V is an open ideal of R then*

(1) $\widetilde{V} x \subseteq \widetilde{V}$ *for all* $x \in R * G$;

(2) *there exists an open ideal U of R such that* $(R * G) \cdot \widetilde{U} \subseteq \widetilde{V}$;

(3) $\cap \widetilde{V} = 0$, *where V runs all open ideal of R.*

Proof. The statements 1 and 3 are trivial.

2) Since G is finite and $\sigma(g)$ are continuous for all $g \in G$, there exists an open ideal U of R such that

$$\sigma(g)(U) \subseteq V, \quad \text{for all } g \in G.$$

Therefore $x \cdot \widetilde{U} \subseteq \widetilde{V}$ for all $x \in R * G$ □

We have proved the following

THEOREM 2. *The family $\left\{ \widetilde{V} : V \text{ is a open ideal of } R \right\}$ is a fundamental system of neighborhoods of zero for a compact ring topology on $R * G$.*

PROPOSITION 1. Let G and G' be groups and let $f : G \to G'$ be a group homomorphism. If $\sigma : G \to \mathrm{Aut}R$ and $\sigma' : G' \to \mathrm{Aut}R$ are two group homomorphisms such that the following diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle\sigma} \searrow & & \swarrow {\scriptstyle\sigma'} \\ & \mathrm{Aut}R & \end{array}$$

is commutative (i.e., $\sigma' \circ f = \sigma$), then the mapping

$$\begin{array}{cccc} \overline{f} & : & R * G & \longrightarrow & R * G' \\ & & \sum\limits_{i=1}^{n} r_i \, g_i & \longmapsto & \sum\limits_{i=1}^{n} r_i \, f(g_i) \end{array}$$

is a ring homomorphism which extends f and

$$\ker \overline{f} = (1 - N),$$

where $N = \ker f$ and $(1 - N)$ is the two sided ideal of $R * G$ generated by $1 - N$.

Proof. Obviously, $\overline{f}$ is a group homomorphism which extend f.
If $a, b \in R$ and $x, y \in G$ then

$$\begin{aligned} \overline{f}\left((ax)\cdot(by)\right) &= \overline{f}\left(a\sigma(x)(b)\ xy\right) \\ &= a\sigma(x)(b)\ f(xy), \\ \overline{f}(ax)\cdot\overline{f}(by) &= af(x)\cdot bf(y) \\ &= a\left(\sigma'(f(x))\right)(b)\ f(x)f(y) \\ &= a\sigma(x)(b)\ f(xy). \end{aligned}$$

The statement $\ker\overline{f} = (1-N)$ follows by [1, Lemma 7]. □

Let R be a profinite ring with identity, G a profinite group and $\sigma : G \to \mathrm{Aut}R$ be a group homomorphism such that $\mathrm{Im}\sigma$ is finite. If N is an open invariant subgroup of G such that $N \subseteq \ker\sigma$ and $\varphi_N : G \to G/N$ is the canonical homomorphism, then there exists a group homomorphism

$$\sigma_N : G/N \to \mathrm{Aut}R$$

such that

$$\begin{array}{ccc} G & \xrightarrow{\varphi_N} & G/N \\ {\scriptstyle\sigma}\searrow & & \swarrow{\scriptstyle\sigma_N} \\ & \mathrm{Aut}R & \end{array}$$

$\sigma_N \circ \varphi_N = \sigma$. According to Proposition 1 the mapping

$$\begin{array}{cccc} \overline{\varphi}_N & : & R * G & \longrightarrow & R*(G/N) \\ & & \sum\limits_{i=1}^{n} r_i\, g_i & \longmapsto & \sum\limits_{i=1}^{n} r_i\ (g_i N) \end{array}$$

is a ring homomorphism which extends φ_N.

Consider now the family $\mathcal{N}$ of all open invariant subgroups of G which are contained in $\ker\sigma$. Let M, N be open invariant subgroups of G such that $N \subseteq M \subseteq \ker\sigma$. Since

$$\phi_{MN} : G/N \to G/M, \quad gN \longmapsto gM$$

is a group homomorphism for which the following diagram

$$\begin{array}{ccc} G/N & \xrightarrow{\phi_{MN}} & G/M \\ {\scriptstyle\sigma_N}\searrow & & \swarrow{\scriptstyle\sigma_M} \\ & \mathrm{Aut}R & \end{array}$$

is commutative, i.e. $\sigma_M \circ \phi_{MN} = \sigma_N$, by Proposition 1 we can extend ϕ_{MN} to a ring homomorphism

$$\overline{\phi}_{MN} : R*(G/N) \to R*(G/M).$$

Moreover, since if V is an open ideal of R, $\overline{\phi}_{MN}(V*(G/N)) \subseteq V*(G/M)$, $\overline{\phi}_{MN}$ is continuous.

Hence, we obtained an inverse system $\left\{R*(G/N), \overline{\phi}_{MN}\right\}_{M,N\in\mathcal{N}}$ of compact rings. Consider the inverse limit of this system and denote

$$\widehat{R*G} = \varprojlim \left(R*(G/N)\right).$$

Since $R*(G/N)$ are compact rings, $\widehat{R*G}$ is complete.

Lemma 2. *The mapping*

$$f : R*G \longrightarrow \widehat{R*G}, \quad \sum_{i=1}^{n} r_i\, g_i \longmapsto \left(\sum_{i=1}^{n} r_i\, g_i N\right)_{N\in\mathcal{N}}$$

is a injective ring homomorphism,

$$\begin{array}{ccc} R*G & \xrightarrow{f} & \widehat{R*G} \\ \overline{\varphi}_N \searrow & & \swarrow \overline{\phi}_N \\ & R*G/N & \end{array}$$

$\overline{\phi}_N \circ f = \overline{\varphi}_N$ *and* $\mathrm{Im} f$ *is a dense subring of* $\widehat{R*G}$.

Proof. Let $\sum_{i=1}^{n} r_i\, g_i \in \ker f$. There exists $N_0 \in \mathcal{N}$ such that the cosets g_iN_0, $i = 1, \ldots, n$ are pairwise different. Since $\sum_{i=1}^{n} r_i\,(g_iN_0) = 0$, we obtain that $r_1 = \cdots = r_n = 0$ and so $\sum_{i=1}^{n} r_i\, g_i = 0$. Therefore f is injective.

Consider $y = (y_N)_{N\in\mathcal{N}} \in \widehat{R*G}$ and let $\widehat{V}$ an neighborhood of 0 in $\widehat{R*G}$. There exists an open ideal U_0 of R and an open invariant subgroup N_0 of G with $N_0 \subseteq \ker\sigma$ such that $\overline{\phi}_{N_0}^{-1}(U_0 * G/N_0) \subseteq \widehat{V}$. Let $x \in \overline{\varphi}_{N_0}^{-1}(y_{N_0})$. Then

$$\overline{\phi}_{N_0}(f(x) - y) = \left(\overline{\phi}_{N_0} \circ f\right)(x) - \overline{\phi}_{N_0}(y) = 0.$$

Thus $f(x) - y \in \overline{\phi}_{N_0}^{-1}(U_0 * G/N_0) \subseteq \widehat{V}$ i.e. $f(x) \in y + \widehat{V}$. □

Theorem 3. $\widehat{R*G}$ *is the completion of the skew group ring* $R*G$.

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