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Preface

The notion "fixed point structure" (which we gave in 1986 for singlevalued operators and in 1993 for multivalued operators) is a generalization of some notions such as:

• "topological space with the fixed point property" (L.E.J. Brouwer, J. Schauder, A. Tychonoff, B. Knaster, K. Kuratowski, S. Mazurkiewicz, V.L. Klee, E.H. Connel, E. Fadell, H. Schirmer, S. Kakutani, S. Eilenberg, D. Montgomery, H.F. Bohnenblust, S. Karlin, K. Fan, I.L. Glicksberg, R.F. Brown, J. Dugundji, A. Granas,...);

• "metric space with fixed point property w.r.t. contractions" (S. Banach, R. Caccioppoli, C. Bessaga, P.R. Meyers, E.H. Connel, T.K. Hu, L. Janos, V.I. Opoitsev, P. Amato, L. Leader, W.A. Kirk, S. Park, I.A. Rus, M.C. Anisiu, V. Anisiu, J. Jachymski,...);

• "Menger space with the fixed point property w.r.t. probabilistic contractions" (V.M. Sehgal, A.T. Bharucha-Reid, O. Hadžić, T.L. Hicks, H. Sherwood, Gh. Constantin, V.I. Istrăţescu, E. Pap, V. Radu, R.M. Tardiff, B. Schweizer, D. Miheţ,...);

• "Banach space with the fixed point property w.r.t. nonexpansive operators" (F.E. Browder, D. Göhde, W. Kirk, L.A. Karlovitz, T.C. Lim, M. Edelstein, B. Maurey, J.B. Baillon, R.E. Bruck, K. Goebel, M.A. Khamsi, S. Reich, B. Sims, W. Takahashi, T. Dominguez Benavides, R. Espinola, J. Elton, P.K. Lim, E. Odell, S. Szarek, D.S. Jaggi, G. Kassay,...);

• "ordered set with the fixed point property" (B. Knaster, A. Tarski, G. Birkhoff, S. Ginsburg, L.E. Ward, A.C. Davis, S. Abian, A.B. Brown, A. Abian, I. Rival, A. Pelczar, H. Amann, H. Cohen, D. Duffus, Z. Shmuely,...);

• ”operator with the fixed point property on family of sets” (G.S. Jones, F.S. De Blasi,...);

• ”object with the fixed point property” (F.W. Lawvere, J. Lambek, I.A. Rus, M. Wand, J. Soto-Andrade, F.J. Varela, M. Barr, C. Wells, A. Baranga,...).

The fixed point structure theory offers a solution for the following problem:

If we have a fixed point theorem T and an operator f which does not satisfy the conditions of T, in which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f|_Y$ satisfies the conditions of T.

In the terms of the fixed point structures this problem take the following form:

Let $(X, S(X), M)$ be a fixed point structure on a set X. Let A be a subset of X and $f : A \to A$ an operator. In which conditions there exists $Y \subset A$ such that:

(a) $Y \in S(X)$; (b) $f(Y) \subset Y$; (c) $f|_Y \in M(Y)$.

From the definition of the fixed point structure it follows that if such an Y exists, then the operator f has at least a fixed point.

The aim of this monograph is to present the basic notions, results and today open problems of the fixed point structure theory.

In the construction of the fixed point structure theory we learned much from discussion with C. Avramescu, V. Berinde, A. Petruşel, R. Precup, V. Radu, A. Buică, A. Muntean and M.A. Şerban. So, we would like to thank all of them.

Cluj-Napoca Ioan A. Rus

May, 2006

Part I

Fixed point structures for singlevalued operators

Chapter 1

Sets, operators and fixed points

1.1 Sets and operators

Let X be a set. We denote

$\mathcal{P}(X) := \{Y | Y \subset X\}$ and $P(X) := \{Y \in \mathcal{P}(X) | Y \neq \emptyset\}$.

Let X and Y be two sets. Then we denote by $\mathbb{M}(X, Y)$ the set of all operators $f : X \to Y$. If $Y = X$, then $\mathbb{M}(X) := \mathbb{M}(X, X)$. By $\mathbb{M}^0(X, Y)$ we denote the set of all multivalued operators $T : X \multimap Y$, i.e., $T : X \to \mathcal{P}(Y)$.

Let (X, τ) be a topological space. Then we shall use the following notations:

$P_{cl}(X) := \{Y \in P(X) | Y = \overline{Y}\}$, the set of all nonempty closed subset of X;

$P_{op} := \{Y \in P(X) | Y$ an open subset of $X\}$;

$P_{cp}(X) := \{Y \in P(X) | Y$ is compact$\}$;

$P_{cn}(X) := \{Y \in P(X) | Y$ is connex $\}$.

In the case of a real linear space $(X, +, \mathbb{R})$,

$P_{cv}(X) := \{Y \in P(X) | Y$ is convex$\}$,

$P_{st}(X) := \{Y \in P(X) | Y \text{ is starshaped}\}$,

and in the case of a metric space (X, d),

$P_b(X) := \{Y \in P(X) | \text{diameter of } Y,\ \delta(Y) < +\infty\}$.

1.2 Fixed and periodic points

Let X be a set, $f : X \to X$ a singlevalued operator and $T : X \multimap X$ a multivalued operator. We shall use the following notations:

$F_f := \{x \in X | f(x) = x\}$, the fixed point set of f,

$F_T := \{x \in X | x \in T(x)\}$, the fixed point set of T,

$(SF)_T := \{x \in X | T(x) = \{x\}\}$, the strict fixed point set of T,

$P_f := \bigcup\limits_{n \in \mathbb{N}^*} F_{f^n}$, the periodic point set of f,

$P_T := \bigcup\limits_{n \in \mathbb{N}^*} F_{T^n}$, the periodic point set of T,

$(SP)_T := \bigcup\limits_{n \in \mathbb{N}^*} (SF)_{T^n}$, the strict periodic point set of T.

Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ an operator. Then:

$(UF)_f := \{x \in X | x \geq f(x)\}$, the set of all upper fixed point of f,

$(LF)_f := \{x \in X | x \leq f(x)\}$, the set of all lower fixed point of f.

1.3 Retractible operators

Let X be a set and $Y \subset X$ a subset of X. Then by definition a set retraction of X onto Y is an operator $\rho : X \to Y$ such that the restriction of ρ to Y is the identity operator, $\rho|_Y = 1_Y$. In general:

If $(X, \leq)$ is a partially ordered set and $Y \subset X$, then by definition, $\rho : X \to Y$ is an ordered set retraction if ρ is a set retraction and is increasing.

If (X, τ) is a topological space, $Y \subset X$, then $\rho : X \to Y$ is a topological retraction, if ρ is a set retraction and is continuous.

More general, if X is a structured set and $Y \subset X$, then ρ is a retraction w.r.t. that structure if ρ is a set retraction and ρ is a morphism w.r.t. that structure.

If $\rho : X \to Y$ is a retraction then Y is called a retract of X.

Definition 1.3.1. (R.F. Brown [13]). Let X be a set, $Y \subset X$ and $\rho : X \to Y$ a retraction. Then by definition an operator $f : Y \to X$ is retractible w.r.t. ρ if $F_f = F_{\rho \circ f}$.

Example 1.3.1. Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space and $\overline{B}(0;1) := \{x \in X \mid \|x\| \leq 1\}$. Then the operator $\rho : X \to \overline{B}(0;1)$, defined by

$$\rho(x) := \begin{cases} x & \text{if } \ \|x\| \leq 1, \\ \dfrac{1}{\|x\|}x & \text{if } \ \|x\| \geq 1 \end{cases}$$

is a topological retraction of X onto $\overline{B}(0;1)$.

We name this retraction, the radial retraction.

Let $f : \overline{B}(0;1) \to X$ be an operator. If f satisfies the following condition

$$x \in X,\ \|x\| = 1,\ \lambda \in \mathbb{R}^*_+,\ f(x) = \lambda x \text{ imply } \lambda \leq 1,$$

then f is a retractible operator w.r.t. the radial retraction ρ. The above condition is called the Leray-Schauder boundary condition.

Example 1.3.2. Let $(X, \leq)$ be a partially ordered set with the least element, 0. Let $Y \in P(X)$ be such that:

(i) $0 \in Y$,

(ii) $(Y, \leq)$ is conditionally complete, i.e., $A \in P_b(Y) \Rightarrow \exists \sup A$ and $\inf A$.

Let $\rho : X \to Y$ be defined by

$$\rho(x) := \begin{cases} x & \text{if } \ x \in Y, \\ \sup_Y([0, x] \cap Y) & \text{if } \ x \in X \setminus Y. \end{cases}$$

The operator ρ is an ordered set retraction of X onto Y.

If an operator $f : Y \to X$ is such that

$$f(x) \in X \setminus Y \text{ implies } sup_Y([0, f(x)] \cap Y) \neq x,$$

then f is retractible w.r.t. ρ.

Example 1.3.3. (I.A. Rus [64]). Let $(X, \to)$ be an L-space. An operator $A : X \to X$ is weakly Picard if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A. For a weakly Picard operator A we consider the operator $A^\infty : X \to X$ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

It is clear that $A^\infty(X) = F_A$. So, the operator $A^\infty : X \to F_A$ is a set retraction of X to F_A.

Example 1.3.4. (R.F. Brown [69]). Let $(X, \|\cdot\|)$ be a Banach space, $0 < r < R$ and

$$Y_{r,R} := \{x \in X | r \leq \|x\| \leq R\}.$$

The operator $\rho : X \setminus \{0\} \to Y_{r,R}$ defined by

$$\rho(x) := \begin{cases} \dfrac{r}{\|x\|}x & \text{if} \quad 0 < \|x\| \leq r, \\ \\ x & \text{if} \quad r \leq x \leq R, \\ \\ \dfrac{R}{\|x\|}x & \text{if} \quad \|x\| \geq R. \end{cases}$$

is a topological retraction.

An operator $f : Y_{r,R} \to X \setminus \{0\}$ is retractible w.r.t. ρ if:

(i) $\|x\| = r \;\Rightarrow\; f(x) \neq \lambda x, \; \forall\, \lambda \in]0, 1[$

(ii) $\|x\| = R \;\Rightarrow\; f(x) \neq \lambda x, \; \forall\, \lambda > 1.$

1.4 Closure operators

Definition 1.4.1. Let X be a set. An operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator if

(i) $Y \subset \eta(Y),\ \forall\, Y \in \mathcal{P}(X)$;

(ii) $Y, Z \in \mathcal{P}(X),\ Y \subset Z\ \Rightarrow\ \eta(Y) \subset \eta(Z)$;

(iii) $\eta \circ \eta = \eta$.

Example 1.4.1. Let $(X, +, R)$ be a real linear space. The following operators are closure operators:

$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) :=$ linear hull of Y;

$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) :=$ affine hull of Y;

$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := coY :=$ convex hull of Y.

Example 1.4.2. Let (X, τ) be a topological space. Then, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(Y) = \overline{Y}$ is a closure operator.

Example 1.4.3. Let $(X, +, R, \tau)$ be a linear topological space. Then, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(Y) := \overline{co}Y := \overline{(coY)}$ is a closure operator.

The main property of a closure operator is the following:

Lemma 1.4.1. *If $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator, then*

$$Y_i \in F_\eta,\ i \in I\ \Rightarrow\ \bigcap_{i\in I} Y_i \in F_\eta.$$

1.5 Fractal operators

Let X be a nonempty set and $T : X \to P(X)$ a multivalued operator. We consider the following singlevalued operator generated by T,

$$\widehat{T} : P(X) \to P(X), \quad Y \mapsto T(Y) := \bigcup_{y\in Y} T(y).$$

The operator $\widehat{T}$ is called the fractal operator corresponding to T.

Example 1.5.1. (see [4] or [65], pp. 107-110). Let X be a nonempty set and $f_1, \dots, f_m : X \to X$. The multivalued operator $T_f : X \to P(X)$ defined by

$$T_f(x) := \{f_1(x), \dots, f_m(x)\},$$

is called the Barnsley-Hutchinson operator generated by the operators $f_1, \dots, f_m$. The fractal operator $\widehat{T}_f$ is a basic tool in the fractal theory. For example if (X, d) is a complete metric space and $f_1, \dots, f_m : X \to X$ are α-contractions then the restriction of $\widehat{T}_f$ to $P_{cp}(X)$ is an operator from $P_{cp}(X)$ to $P_{cp}(X)$ which is an α-contraction w.r.t. Pompeiu-Hausdorff metric. In this case the unique fixed point of $\widehat{T}_f$, in $P_{cp}(X)$, is called fractal or attractor.

1.6 Invariant subsets

Let X be a nonempty set and $f : X \to X$ an operator. A subset $Y \subset X$ is an invariant subset for f if $f(Y) \subset Y$. We denote by $I(f)$ the family of all nonempty invariant subsets of f. We have

Lemma 1.6.1. (I.A. Rus (1986), see [65], p. 4). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ and $f : Y \to Y$. Let $A \in P(Y)$. Then there exists $A_0 \subset Y$ such that:*

(i) $A \subset A_0$;

(ii) $A_0 \in F_\eta$;

(iii) $A_0 \in I(f)$;

(iv) $\eta(f(A_0) \cup A) = A_0$.

Proof. Let $\mathcal{B} := \{B \subset Y | B$ satisfies the conditions (i), (ii) and (iii)$\}$. From Lemma 1.4.1 we have that, $\cap\mathcal{B} \in \mathcal{B}$.

This implies that $\cap\mathcal{B}$ is the least element of the partially ordered set $(\mathcal{B}, \subset)$. Let us prove that $A_0 := \cap\mathcal{B}$.

We have $\eta(f(A_0) \cup A) \in \mathcal{B}$ and $\eta(f(A_0) \cup A) \subset A_0$. These imply that

$\eta(f(A_0) \cup A) = A_0$.

Let X be a nonempty set and $T : X \multimap X$ a multivalued operator. A subset $Y \subset X$ is an invariant subset of T if $T(Y) \subset Y$. We denote by $I(T)$ the family of all invariant nonempty subset of T. We have

Lemma 1.6.2. (I.A. Rus (1993); see [65] p. 5). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator, $Y \in F_\eta$ and $T : Y \to P(Y)$ a multivalued operator. Let $A \in P(Y)$. Then there exists $A_0 \subset Y$ such that*

(i) $A \subset A_0$;

(ii) $A_0 \in F_\eta$;

(iii) $A_0 \in I(T)$;

(iv) $\eta(T(A_0) \cup A) = A_0$.

The proof of this lemma is similar with that of Lemma 1.6.1.

Remark 1.6.1. Let $T : X \to P(X)$ be a multivalued operator. In the terms of the fractal operator $\widehat{T}$ we have:

$$Y \in I(T) \;\Leftrightarrow\; Y \in (UF)_{\widehat{T}},$$

$$Y = T(Y) \;\Leftrightarrow\; Y \in F_{\widehat{T}},$$

$$Y \subset T(Y) \;\Leftrightarrow\; Y \in (LF)_{\widehat{T}}.$$

Remark 1.6.2. There exist another type of results on invariant subsets. For example we have

Lemma 1.6.3. (M. Martelli (1973)). *Let X be a compact topological space and $T : X \to P(X)$ a multivalued operator. Then there exists a nonempty closed subset $Y \subset X$ such that $Y = \overline{T(Y)}$. If T is u.s.c. with closed values, then $Y = T(Y)$.*

Lemma 1.6.4. (S. Leader (1982)). *Let X be a compact metric space and $A : X \to X$ be a continuous operator. Then $\bigcap\limits_{n \in \mathbb{N}} A^n(X)$ is a fixed set for A.*

Example 1.6.1. Let (X, τ) be a topological space, $Y \in P_{cl}(X)$ and $f : Y \to Y$ an operator. Let $x \in Y$. Then there exists $A_0 \subset Y$ such that:

(i) $x \in A_0$;

(ii) $A_0 = \overline{A_0}$;

(iii) $A_0 \in I(f)$;

(iv) $\overline{f(A_0) \cup \{x\}} = A_0$.

Indeed, we take in Lemma 1.6.1, $\eta(B) = \overline{B}$ and $A = \{x\}$.

Example 1.6.2. Let $(X, +, R, \tau)$ be a vectorial topological space, $Y \in P_{cl,cv}(X)$ and $f : Y \to Y$ an operator. Let $x \in Y$. Then there exists $A_0 \subset Y$ such that:

(i) $x \in A_0$;

(ii) $\overline{co}A_0 = A_0$;

(iii) $A_0 \in I(f)$;

(iv) $\overline{co}(f(A_0) \cup \{x\}) = A_0$.

Indeed, we take in Lemma 1.6.1, $\eta(B) = \overline{co}B$ and $A = \{x\}$.

1.7 Fixed point theory in categories

By a category $\mathcal{C}$ we understand a class of objects, $Ob\mathcal{C}$, together with the following:

(i) For each ordered pair of objects, (A, B), $A, B \in Ob\mathcal{C}$, a set $Hom(A, B)$ is given. The element of $Hom(A, B)$ are called morphism from A to B. The object A is called the source and B is called the target of $f \in Hom(A, B)$. For $f \in Hom(A, B)$ also, we use the notations, $f : A \to B$ or $A \xrightarrow{f} B$.

(ii) For each ordered triplet of objects, (A, B, C), an operator from $Hom(A, B) \times Hom(B, C)$ to $Hom(A, C)$ is given. We name this operator, the composition operator. If $f \in Hom(A, B)$ and $g \in Hom(B, C)$, then we denote the value of this operator with $g \circ f$. We suppose that the composition operator is associative.

(iii) For each object B a morphism $1_B \in Hom(B, B)$ is given and 1_B is such that if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, then $1_B \circ f = f$ and $g \circ 1_B = g$.

(iv) If $(A, B) \neq (C, D)$, then $Hom(A, B) \cap Hom(C, D) = \emptyset$.

Example 1.7.1. The category Set. The class of objects is the class of all sets. If $A, B \in ObSet$, then $Hom(A, B) := \mathbb{M}(A, B)$. The composition morphism is the composition of operator and the identity morphism is the identity operator.

Example 1.7.2. The category Poset. The class of objects is the class of all partially ordered sets and $Mor(A, B) := \{f : A \to B | f \text{ is increasing}\}$.

Example 1.7.3. The category Top. The class of objects is the class of all topological space and $Mor(A, B) := C(A, B)$.

Example 1.7.4. The category SELF-OP. The objects of this category are the self-operators. Let $f : A \to A$ and $g : B \to B$ be two objects. A morphism from f to g is an operator $h : A \to B$ such that $h \circ f = g \circ h$. The class of self-operators and morphism between them forms a category: the category SELF-OP.

Definition 1.7.1. Let $\mathcal{C}$ be a category and $A \in Ob\mathcal{C}$. A morphism $f \in Hom(A, A)$ has the f.p.p. iff there exist $B \in Ob\mathcal{C}$ and $g \in Hom(B, A)$ such that $f \circ g = g$.

Definition 1.7.2. By the fixed subobject, F_f, of a morphism $f \in Hom(A, A)$ we understand $F_f := Ker(f, 1_A)$. A category in which for each morphism with f.p.p. there exists F_f is by definition a category with fixed subobjects.

Definition 1.7.3. Let $\mathcal{C}$ be a category. An object $A \in Ob\mathcal{C}$ has the f.p.p. if each morphism $f \in Hom(A, A)$ has the f.p.p.

We have

Lemma 1.7.1. *Let $\mathcal{C}$ be a category and A and $B \in Ob\mathcal{C}$. We suppose that:*

(i) the object A has the f.p.p.

(ii) there exists an isomorphism $\varphi \in Hom(A, B)$.

Then B is an object with the f.p.p.

Proof. By definition φ is an isomorphism if there exists $\psi : B \to A$ such that

$$\varphi \circ \psi = 1_B \text{ and } \psi \circ \varphi = 1_A.$$

Let $f : B \to B$. Then $\psi \circ f \circ \varphi \in Hom(A, A)$. From the condition (i) there exists $g \in Hom(C, A)$ such that $\psi \circ f \circ \varphi \circ g = g$. Hence we have $f \circ (\varphi \circ g) = \varphi \circ g$. So, f has the f.p.p.

1.8 References

For the set theory see N. Bourbaki [14], M.A. Khamsi and W.A. Kirk [43], A. Granas and J. Dugundji [36]. See also,

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Chapter 2

Fixed point structures

2.1 Definitions and examples

Definition 2.1.1. A triple $(X, S(X), M)$ is a fixed point structure (briefly, f.p.s.) if

(i) X is a nonempty set, $S(X) \subset P(X)$, $S(X) \neq \emptyset$;

(ii) $M : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}(Y)$, $Y \multimap M(Y) \subset \mathbb{M}(Y)$, is a multivalued operator such that if $Z \subset Y$, $Z \neq \emptyset$, then $M(Z) \supset \{f|_Z|\ f \in M(Y),\ Z \in I(f)\}$;

(iii) Every $Y \in S(X)$ has the fixed point property (f.p.p.) with respect to $M(Y)$.

Definition 2.1.2. A triple $(X, S(X), M)$ which satisfies (i) and (iii) in Definition 2.1.1 and the condition

(ii') $M : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}(Y)$, $Y \multimap M(Y) \subset \mathbb{M}(Y)$ is a multivalued operator;

is called a large fixed point structure (l.f.p.s.).

For a better understanding of the above definitions we suggest to the reader to see pages 2 and 3 in [36].

Example 2.1.1. The trivial f.p.s. X is a nonempty set, $S(X) := \{\{x\}|\ x \in X\}$ and $M(Y) := \mathbb{M}(Y)$.

Example 2.1.2. The Tarski's f.p.s. $(X, \leq)$ is a complete lattice, $S(X) := \{Y \in P(X)|\ (Y, \leq)$ is a complete lattice$\}$ and $M(Y) := \{f : Y \to Y|\ f$ is increasing$\}$.

Condition (iii) follows from Tarski's fixed point theorem.

Example 2.1.3. The f.p.s. of progressive operators. $(X, \leq)$ is a partially ordered set, $S(X) := \{Y \in P(X)|\ (Y, \leq)$ has a maximal element$\}$ and $M(Y) := \{f : Y \to Y|\ x \leq f(x),\ \forall\ x \in Y\}$.

Example 2.1.4. The f.p.s. of contractions. (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is a contraction$\}$.

Example 2.1.5. The f.p.s. of Brouwer-Schauder-Tychonoff. X is a locally convex linear topological space, $S(X) := P_{cp,cv}(X)$ and $M(Y) := C(Y, Y)$.

Example 2.1.6. The f.p.s. of Schauder. X is a Banach space, $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is completely continuous$\}$.

Example 2.1.7. The f.p.s. of Dotson. X is a Banach space, $S(X) := P_{cp,st}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is nonexpansive$\}$.

Example 2.1.8. The l.f.p.s. of Girolo. X is a Banach space, $S(X) := P_{cp,cv}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is connective$\}$.

Example 2.1.9. The f.p.s. of Browder-Ghöde-Kirk. X is a uniformly convex Banach space, $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is nonexpansive$\}$.

Example 2.1.10. The f.p.s. of Nemytzki-Edelstein. (X, d) is a metric space, $S(X) := P_{cp}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is a contractive operator$\}$.

Example 2.1.11. The fixed point structure of Tychonoff. X is a Banach space, $S(X) := P_{wcp,cv}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is weakly continuous operator$\}$.

Example 2.1.12. The fixed point structure of Arino-Gautier-Penot. X is a metrizable locally convex topological vector space, $S(X) := P_{wcp,cv}(X)$ and $M(Y) := \{f : Y \to Y|\ f$ is weakly sequentially continuous$\}$.

It is clear that for any fixed point theorem we have an example of a f.p.s. or of a l.f.p.s.

2.2 Fixed point structures on $P(X)$ generated by fixed point structures on X

The following problem is a tool to generate fixed point structures on $P(X)$ from some fixed point structures on X:

Let $(X, \{X\}, M)$ be a f.p.s. and $S(X) \subset P(X)$ such that

$A \in S(X)$, $f_1, \ldots, f_m \in M(X)$, $m \in \mathbb{N}^*$ imply $f_1(A) \cup \cdots \cup f_m(A) \in S(X)$.

Let $\widetilde{M}(S(X)) := \{\widehat{T}_f|\ f_1, \ldots, f_m \in M(X),\ m \in \mathbb{N}^*\}$.

The problem is in what conditions the triple $(S(X), \{S(X)\}, \widetilde{M})$ is a f.p.s.?

Example 2.2.1. Let (X, d) be a complete metric space and $(X, \{X\}, M)$ the fixed point structure of contractions. Then the triple $(P_{cp}(X), \{P_{cp}(X)\}, \widetilde{M})$ is a f.p.s.

Indeed, $(P_{cp}(X), H_d)$ is a complete metric space and by a theorem of Nadler, the operator $\widehat{T}_f : P_{cp}(X) \to P_{cp}(X)$ is a contraction, if $f_1, \ldots, f_m$ are contractions.

Example 2.2.2. Let $(X, \{x\}, \mathbb{M})$ be the trivial f.p.s. Let $S(X) \subset P(X)$. Then the triple $(P(X), \{S(X)\}, \widetilde{\mathbb{M}})$ isn't, in general, a f.p.s. For example if $X = \mathbb{R}$, $S(X) := P_b(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, then $F_{\widehat{T}_f} = \emptyset$.

2.3 Maximal fixed point structures

Let $(X, S(X), M)$ be a f.p.s. and $S_1(X) \subset P(X)$ such that $S_1(X) \supset S(X)$.

Definition 2.3.1. (I.A. Rus (1996)). The f.p.s. $(X, S(X), M)$ is maximal in $S_1(X)$ if we have

$$S(X) = \{A \in S_1(X) |\ f \in M(A) \ \Rightarrow\ F_f \neq \emptyset\}.$$

Example 2.3.1. The trivial f.p.s. is maximal in $P(X)$.

Example 2.3.2. The Tarski f.p.s. isn't maximal in $P(X)$ but is maximal in $S_1(X) := \{Y \in P(X) |\ (Y, \leq) \text{ is a lattice}\}$. This follows from a theorem of Davis (1955).

Example 2.3.3. The f.p.s. of contractions isn't maximal in $P(X)$. It is clear that the f.p.s. of contractions is maximal in $P(X)$ if

$$(Y \in P(X),\ f \in M(Y)\ \Rightarrow\ F_f \neq \emptyset)\ \Rightarrow\ Y \in P_{cl}(X).$$

We have

Theorem 2.3.1. (M.C. Anisiu and V. Anisiu (1997), E.H. Connel (1959)). *There exists a complete space and a nonclosed subset with f.p.p. with respect to contractions.*

Theorem 2.3.2. (M.C. Anisiu and V. Anisiu (1997)). *Let X be a Banach space and $Y \in P(X)$ a convex set with $IntY \neq \emptyset$. If each contraction $f : Y \to Y$ has a fixed point, then Y is closed.*

Example 2.3.4. The f.p.s. of Brouwer-Schauder-Tychonoff isn't, in general, maximal in $P(X)$, but if X is a Banach space then it is maximal in $P_{b,cl,cv}(X)$.

This follows from the following theorem of V. Klee (1955):

A closed bounded convex subset of a Banach space has the topological f.p.p. iff it is compact.

In what follow we present some properties of the maximal fixed point structures.

Let $\mathcal{C}$ be a class of structured sets (the class of sets, the class of partially ordered sets, the class of L-spaces, the class of metric spaces,...). By S we denote an operator which attaches to each $X \in \mathcal{C}$ a nonempty set $S(X) \subset P(X)$. Let M be an operator which attaches to each pair (A, B), $A \in P(X)$, $B \in P(Y)$, $X, Y \in \mathcal{C}$, a subset $M(A, B) \subset \mathbb{M}(A, B)$.

From the definition of the maximal fixed point structures, we have

Lemma 2.3.1. *Let $X \in \mathcal{C}$ and $(X, S(X), M)$ be a maximal fixed point structures on $\mathcal{C}$. Let $A \in S(X)$ and B a nonempty subset of A. If there exists a set retraction $\rho \in M(A, B)$, of A onto B such that $f \in M(B)$ implies $f \circ \rho \in M(A)$, then $B \in S(X)$.*

Proof. Let $f \in M(B)$. Then $f \circ \rho \in M(A)$, i.e.,

$$A \xrightarrow{\rho} B \xrightarrow{f} B \xrightarrow{\subseteq} A.$$

But $A \in S(X)$. By definition of a f.p.s. we have that there exists $x^* \in A$ such that $f(\rho(x^*)) = x^*$. Since $x^* \in B$, we have $f(x^*) = x^*$. By the maximality of $(X, S(X), M)$ it follows that $B \in S(X)$.

Lemma 2.3.2. *Let $X, Y \in \mathcal{C}$ and $(X, S(X), M)$ and $(Y, S(Y), M)$ be two fixed point structures on $\mathcal{C}$. Let $A \in S(X)$ and $B \in P(Y)$.*

We suppose that:

(i) $(Y, S(Y), M)$ is a maximal f.p.s. in $P(Y)$;

(ii) there exists a bijection $\varphi \in M(A, B)$ such that $\varphi^{-1} \circ f \circ \varphi \in M(A)$, for all $f \in M(B)$.

Then, $B \in S(Y)$.

Proof. Let $f \in M(B)$. From (ii) it follows that $F_{\varphi^{-1} \circ f \circ \varphi} \neq \emptyset$. If $x^* \in F_{\varphi^{-1} \circ f \circ \varphi}$, then $\varphi(x^*) \in F_f$. By the maximality of $(Y, S(Y), M)$ we have that $B \in S(Y)$.

Remark 2.3.1. The above two results generalize some results given in A. Granas and J. Dugundji [36] (pp.2-3).

Remark 2.3.2. To establish if a given f.p.s. is maximal or not, this is an open problem. For example in some concrete structured sets the problem take the following form:

- Characterize the ordered sets with fixed point property with respect to increasing operators.
- Characterize the topological space with f.p.p. with respect to continuous operators.
- Characterize the metric space with f.p.p. with respect to continuous operators.
- Characterize the metric space with f.p.p. with respect to contractions.
- Characterize the Banach spaces X with the following property:

$$Y \in P_{b,cl,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

- Characterize the Banach spaces X with the following property:

$$Y \in P_{wcp,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

For example we have the following result for the metric space with f.p.p. with respect to contractions.

Let (X, d) be a metric space. We consider the ordered metric space of fractals, $(P_{cp}(X), H_d, \subset)$. We denote:

- $CT(X, X) := \{f : X \to X |\ f \text{ a contraction}\}$.
- For $f \in CT(X, X)$ we denote by $\widehat{f} : P_{cp}(X) \to P_{cp}(X)$ the corresponding fractal operator.

- For $f : X \to X$ we denote by $d(f) := \inf\{d(x, f(x))\}$, the minimal displacement of f (K. Goebel (1973)).

We also need the following notions:

Definition 2.3.2. (F.S. De Blasi and J. Myjak (1989)). Let (X, d) be a metric space an $f : X \to X$ and operator. The fixed point problem for f is well posed iff

(a) $F_f = \{x^*\}$;

(b) if $x_n \in X$, $n \in \mathbb{N}$ and $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.

Definition 2.3.3. A metric space (X, d) has the fixed point property iff

$$f \in CT(X, X) \;\Rightarrow\; F_f \neq \emptyset.$$

Definition 2.3.4. A metric space (X, d) is complete w.r.t. a family of operator $M(X, X)$ iff

$f \in M(X, X), (f^n(x))_{n\in\mathbb{N}}$ is fundamental $\;\Rightarrow\;$ $(f^n(x))_{n\in\mathbb{N}}$ converges.

We have

Theorem 2.3.3. *Let (X, d) be a metric space. Then the following statements are equivalent:*

(i) (X, d) is with f.p.p.

(ii) $f \in CT(X, X) \;\Rightarrow\; f$ is Picard operator.

(iii) (X, d) is complete w.r.t. $CT(X, X)$.

(iv) $\forall\ f \in CT(X, X)\ \exists\ x_f^* \in X :\ d(f) = d(x_f^*, f(x_f^*))$.

(v) $\forall\ f \in CT(X, X)$ the fixed point problem is well posed.

(vi) $\forall\ f \in CT(X, X)\ \exists\ x_0 \in X$ such that

$$\bigcap_{n\in\mathbb{N}} \{x \in X|\ d(x, f(x)) \le L_f^n d(x_0, f(x_0))\} \neq \emptyset.$$

(vii) $F_{\widehat{f}} \neq \emptyset,\ \forall\ f \in CT(X, X)$.

(viii) $(UF)_{\widehat{f}} \neq \emptyset,\ \forall\ f \in CT(X,X)$.

(ix) $(LF)_{\widehat{f}} \neq \emptyset,\ \forall\ f \in CT(X,X)$.

(x) $\forall\ f \in CT(X,X)\ \exists\ x \in X$ *such that* $(f^n(x))_{n\in\mathbb{N}}$ *converges.*

(xi) $\forall\ f \in CT(X,X)\ \exists\ x \in X$ *such that some subsequence of* $(f^n(x))_{n\in\mathbb{N}}$ *converges.*

(xii) $\forall\ f \in CT(X,X)\ \exists\ A \in P_{cp}(X)$ *such that* $(\widehat{f}^n(A))_{n\in\mathbb{N}}$ *converges.*

(xiii) $\forall\ f \in CT(X,X)\ \exists\ A \in P_{cp}(X)$ *such that some subsequence of* $(\widehat{f}^n(A))_{n\in\mathbb{N}}$ *converges.*

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Chapter 3

Functionals with the intersection property

3.1 Diameter functional

Let (X, d) be a metric space. The diameter functional $\delta : P_b(X) \to \mathbb{R}_+$ is defined by $\delta(Y) := \sup\{d(x, y)|\ x, y \in Y\}$.

If $x \in X$ and $Y \in P(X)$, then we denote

$$D(x, Y) := \inf\{d(x, y)|\ y \in Y\}.$$

We have

Lemma 3.1.1. *The functional δ has the following properties:*

(i) $\delta(Y) = 0 \ \Leftrightarrow \ Y = \{y\}$;

(ii) $Y_1, Y_2 \in P_b(X),\ Y_1 \subset Y_2 \ \Rightarrow \ \delta(Y_1) \le \delta(Y_2)$;

(iii) $\delta(V_r(Y)) \le \delta(Y) + 2r,\ \forall\ Y \in P_b(X),\ \forall\ r > 0$;

(iv) $\delta(\overline{Y}) = \delta(Y),\ \forall\ Y \in P_b(X)$.

Proof. (iii) Let $\varepsilon > 0$ and $x, y \in V_r(Y)$. From the definition of $V_r(Y)$,

$V_r(Y) := \{x \in X|\ D(x, Y) < r\}$, there exist $u, v \in Y$ such that

$$d(x, u) < r + \varepsilon,\ d(x, v) < r + \varepsilon.$$

From these we have

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, x) \leq \delta(Y) + 2r + 2\varepsilon$$

for all $x, y \in Y$.

Hence

$$\delta(V_r(Y)) \leq \delta(Y) + 2r + 2\varepsilon,\ \forall\ \varepsilon > 0.$$

So, we have (iii).

(iv) From $Y \subset \overline{Y},\ \delta(Y) \leq \delta(\overline{Y})$.

From $\overline{Y} \subset V_r(Y),\ \forall\ r > 0$ we have $\delta(\overline{Y}) \leq \delta(Y)$.

Lemma 3.1.2. *Let X be a real linear normed space. Then*

(i) $\delta(Y_1 + Y_2) \leq \delta(Y_1) + \sigma(Y_2),\ \forall\ Y_1, Y_2 \in P_b(X)$;

(ii) $\delta(\lambda Y) = |\lambda|\delta(Y),\ \forall\ P_b(X),\ \forall\ \lambda \in \mathbb{R}$;

(iii) $\delta(coY) = \delta(Y),\ \forall\ Y \in P_b(X)$.

Proof. (iii) Let us prove that $\delta(coY) \leq \delta(Y)$. Let $x, y \in coY$. Then there exist $x_i, y_j \in Y$, $\lambda_i, \mu_j \in \mathbb{R}_+$, such that

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad y = \sum_{j=1}^{m} \mu_j y_j, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \sum_{j=1}^{m} \mu_j = 1.$$

From these relations we have

$$\begin{aligned}\|x - y\| &= \left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{m} \mu_j y_j \right\| \\ &= \left\| \left(\sum_{j=1}^{m} \mu_j\right) \sum_{i=1}^{n} \lambda_i x_i - \left(\sum_{i=1}^{n} \lambda_i\right) \sum_{j=1}^{m} \mu_j y_j \right\| \\ &\leq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \|x_i - y_j\| \leq \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j\right) \delta(Y) = \delta(Y).\end{aligned}$$

Lemma 3.1.3. (Cantor's intersection lemma). *Let* (X, d) *be a complete metric space and* $Y_n \in P_{b,cl}(X)$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ *such that* $\delta(Y_n) \to 0$ *as* $n \to \infty$.

Then, $\bigcap_{n\in\mathbb{N}} Y_n = \{x^*\}$.

Proof. First we remark that $card \bigcap_{n\in\mathbb{N}} Y_n \leq 1$. Let $x_n \in Y_n$. Then

$$d(x_n, x_m) \leq \max(\delta(Y_n), \delta(Y_m)) \to 0 \text{ as } n, m \to \infty.$$

Since (X, d) is a complete metric space it follows that $(x_n)_{n\in\mathbb{N}}$ is convergent. Let x^* be its limit. Let $k \in \mathbb{N}$. Then $x_n \in Y_k$ for all $n \geq k$. So, $x^* \in Y_k$, $\forall\, k \in \mathbb{N}$, and

$$\bigcap_{n\in\mathbb{N}} Y_n = \{x^*\}.$$

Remark 3.1.1. $\bigcap_{n\in\mathbb{N}} Y_n = \{x^*\} \;\Leftrightarrow\; \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset$ and $\delta\left(\bigcap_{n\in\mathbb{N}} Y_n\right) = 0$.

3.2 The Kuratowski measure of noncompactness

Let (X, d) be a metric space. The Kuratowski measure of noncompactness is defined as follows

$$\alpha_K : P_b(X) \to \mathbb{R}_+,\ \alpha_K(Y) := \inf\left\{\varepsilon > 0 | \, Y = \bigcup_{i=1}^{n} Y_i,\ \delta(Y_i) \leq \varepsilon,\ n \in \mathbb{N}^*\right\}.$$

We have

Lemma 3.2.1. *Let* (X, d) *be a metric space and* α_K *the Kuratowski measure of noncompactness of* X*. Then*

(i) $\alpha_K(Y) \leq \delta(Y),\ \forall\ Y \in P_b(X)$*;*

(ii) $Y_1, Y_2 \in P_b(X),\ Y_1 \subset Y_2 \;\Rightarrow\; \alpha_K(Y_1) \leq \alpha_K(Y_2)$*;*

(iii) $\alpha_K(Y_1 \cup Y_2) = \max(\alpha_K(Y_1), \alpha_K(Y_2)),\ Y_1, Y_2 \in P_b(X)$*;*

(iv) $\alpha_K(V_r(Y)) \leq \alpha_K(Y) + 2r,\ \forall\ Y \in P_b(X),\ \forall\ r > 0$*;*

(v) $\alpha_K(\overline{Y}) = \alpha_K(Y),\ \forall\ Y \in P_b(X).$

Proof. (iv) Let $\eta > 0$. Then there exists $Y_i \subset Y$ such that $Y = \bigcup_{i=1}^{n} Y_i$ and $\delta(Y_i) \leq \alpha_K(Y) + \eta$. On the other hand $V_r(Y) = \bigcup_{i=1}^{n} V_r(Y_i)$ and $\delta(V_r(Y_i)) \leq \alpha(A) + 2r + \eta$. So, $\alpha_K(V_r(Y)) \leq \alpha_K(Y) + 2r + \eta,\ \forall\ \eta > 0$.

(v) We remark that $\overline{Y} \subset V_r(Y),\ \alpha r > 0$. The proof follows from (iv).

Lemma 3.2.2. *Let (X, d) a complete metric space. Then*

$$\alpha_K(Y) = 0 \ \Leftrightarrow\ \overline{Y} \textit{ is compact.}$$

Proof. The proof follows from the following well known result (see, for example, A. Brown and C. Pearcy [18], pp.198-199):

Lemma 3.2.3. *Let (X, d) be a metric space and $Y \subset X$. The following conditions are equivalent:*

(a) For every $\varepsilon > 0$ there exists a finite ε-net for Y;

(b) For every $\varepsilon > 0$ there exists a finite covering of Y consisting of sets of diameter less than ε;

(c) For every $\varepsilon > 0$ there exists a finite partition of Y into sets of diameter less than ε;

(d) For every $\varepsilon > 0$ there exists a finite ε-net in Y.

By definition a subset $Y \subset X$ is totally bounded if it has any one of the properties (a)-(b).

Let (X, d) be a complete metric space and $Y \subset X$ a closed subset of X. Then Y is compact if and only if it is totally bounded.

Lemma 3.2.4. *Let X be a real linear normed space and α_K the Kuratowski measure of noncompactness of X. Then*

(i) $\alpha_K(Y_1 + Y_2) \leq \alpha_K(Y_1) + \alpha_K(Y_2),\ \forall\ Y_1, Y_2 \in P_b(X)$;

(ii) $\alpha_K(\lambda Y) = |\lambda| \alpha_K(Y),\ \forall\ \lambda \in \mathbb{R},\ \forall\ Y \in P_b(X)$;

(iii) $\alpha_K(coY) = \alpha_K(Y),\ \forall\ Y \in P_b(X)$.

Proof. (iii) Let us prove that $\alpha_K(coY) \leq \alpha_K(Y)$. For this we shall prove the following

$$r > \alpha_K(Y) \ \Rightarrow \ \alpha_K(coY) \leq r.$$

From the definition of α_K, Y can be written as a finite union, $Y = \bigcup_{i=1}^{n} Y_i$, where $\delta(Y_i) \leq r$, $i = \overline{1,m}$. We have

$$Y \subset \bigcup_{i=1}^{n} coY_i \text{ and } \delta(coY_i) \leq r.$$

He (see R. Cristescu, Analiză funcţională, 1978, pp.21-22),

$$coY \subset \bigcup_{\lambda\in\sigma} \sum_{i=1}^{n} \lambda_i coY_i$$

where $\sigma \subset \mathbb{R}^n$ is the standard simplex, i.e.,

$$\sigma := \left\{ \lambda \in \mathbb{R}^n |\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

But σ is a compact set. So, for each $\varepsilon > 0$ there exist $\lambda^1, \ldots, \lambda^m \in \sigma$ such that for all $\lambda \in \sigma$, there exists λ^i such that

$$\|\lambda - \lambda^i\|_{\mathbb{R}^n} \leq \frac{\varepsilon}{\delta\left(0, \bigcup_{i=1}^{n} coY_i\right)}.$$

From this we have that

$$coY \subset \bigcup_{j=1}^{m} \sum_{i=1}^{n} \lambda_i^j coY_i + \varepsilon\overline{B}(0;1).$$

This implies

$$\alpha_K(coY) \leq \max_{i=\overline{1,n}} \alpha_K(coY_i) + \varepsilon \leq r + \varepsilon, \text{ i.e.,}$$

$$\alpha_K(coY) \leq r + \varepsilon,\ \forall\ \varepsilon > 0.$$

So, $\alpha_K(coY) \leq r$.

Lemma 3.2.5. (Kuratowski's intersection lemma). *Let (X,d) be a complete metric space and $Y_n \in P_{b,cl}(X)$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$, be such that $\alpha_K(Y_n) \to 0$ as $n \to \infty$. Then*

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset \text{ and } \alpha_K(Y_\infty) = 0,$$

i.e., Y_∞ is a compact set.

Proof. It is clear that Y_∞ is a closed set and $\alpha_K(Y_\infty) = 0$. Let us prove that $Y_\infty \neq \emptyset$.

Let $x_n \in Y_n$, $n \in \mathbb{N}$. Let $X_n := \{x_n, x_{n+1}, \ldots\}$. We have $X_{n+1} = X_n \cup \{x_{n+1}\}$. Hence

$$\alpha_K(X_0) = \alpha_K(X_1) = \cdots = \alpha_K(X_n) \to 0 \text{ as } n \to \infty.$$

It follows that $\alpha_K(X_0) = 0$. This implies that X_0 is precompact and there exists a convergent subsequence $(x_{n_k})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$.

Let $x^* := \lim\limits_{k\to\infty} x_{n_k}$. We have that $x^* \in Y_\infty$.

3.3 The Hausdorff measure of noncompactness

Let (X,d) be a metric space. The Hausdorff measure of noncompactness of X, $\alpha_H : P_b(X) \to \mathbb{R}_+$ is defined by $\alpha_H := \inf\{\varepsilon > 0 |\ Y$ can be covered by finitely many balls of radius $\leq \varepsilon\}$.

The Hausdorff measure of noncompactness has all the properties of α_K presented in Lemma 3.2.1, 3.2.2 and 3.2.4.

From the definition of α_K and of α_H we have

Lemma 3.3.1. *Let (X,d) be a metric space and α_K and α_H the Kuratowski and Hausdorff measures of noncompactness of X. Then*

$$\alpha_H \leq \alpha_K \leq 2\alpha_H.$$

Lemma 3.3.2. (R.D. Nussbaum (1969), M. Furi and A. Vignoli (1970)) *Let X be a Banach space. Then*

(a) $\alpha_K(\overline{B}(0;1)) = \alpha_H(\overline{B}(0;1)) = 0$ if $\dim X < +\infty$;

(b) $\alpha_K(\overline{B}(0;1)) = 2$ and $\alpha_H(\overline{B}(0;1)) = 1$, if X is infinite dimensional.

Proof. (b) Since $\delta(\overline{B}(0;1)) = 2$, hence $\alpha_K(\overline{B}(0;1)) \leq 2$. Let us suppose that $\alpha_K(\overline{B}(0;1)) < 2$. Then there exists $B_1, \ldots, B_m \in P_{cl}(X)$ such that, $\overline{B}(0;1) \subset \bigcup_{k=1}^{m} B_k$ and $\delta(B_k) < 2$. Let we consider a section of $B(0;1)$ with a n-dimensional subspace $X_m \subset X$. Let $A_k := B_k \cap X_n$. Now, we have a contradiction with the following theorem of antipodes of Borsuk-Lusternik-Schnirelman (see [6] p.23 or [58] p.87):

If $B(0;1)$ is the unit sphere in an m-dimensional Banach space and $A_1, \ldots, A_m$ is a cover of $\partial\overline{B}(0;1)$ by closed subsets, then at least one of the sets $A_1, \ldots, A_m$, contains a pair of antipodes points.

Let us prove that $\alpha_H(\overline{B}(0;1)) = 1$. Suppose that $\alpha_H(\overline{B}(0;1)) < 1$. Let $r := \alpha_H(\overline{B}(0;1))$ and $\varepsilon > 0$ be such that $r + \varepsilon < 1$. By the definition of α_H, there exist $x_1, \ldots, x_m \in X$ such that

$$\overline{B}(0;1) \subset \bigcup_{k=1}^{m} \overline{N}(x_k, r+\varepsilon) = \bigcup_{k=1}^{m} (x_k + (r+\varepsilon)\overline{B}(0;1)).$$

We have

$$r \leq r(r+\varepsilon).$$

This implies that $\alpha_H(\overline{B}(0;1)) = 0$ which is in contradiction with $\dim X = +\infty$.

Lemma 3.3.3. *Let X be a Banach space, $\Omega \subset \mathbb{R}^n$ a compact subset and $Y \subset C(\Omega, X)$ a bounded and equicontinuous subset. Then*

$$\alpha_K(Y) = \sup_{t \in \Omega} \alpha_K(\{x(t) |\ x \in Y\}),$$

with respect to the Chebyshev norm on $C(\Omega, X)$.

Proof. First we shall prove that

$$\mu > \alpha_K(Y) \;\Rightarrow\; \alpha_K(\{x(t)|\ x \in Y\} \le \mu,\ \forall\ t \in \Omega).$$

If $\mu > \alpha_K(Y)$, then there exist $Y_1, \ldots, Y_m$, $Y_k \subset Y$, $k = \overline{1,m}$ such that

$$\delta(Y_k) \le \mu \text{ and } Y \subset \bigcup_{k=1}^{m} Y_k.$$

Now we remark that

$$\delta(\{x(t)|x \in Y_K\}) = \sup_{x,y \in Y_K} \{\|x(t) - y(t)\|\} \le \delta(Y_K) \le \mu.$$

Let now to prove the converse inequality.

Ω a compact subset and Y equicontinuous imply that, given $\varepsilon > 0$ there exist $t_1, \ldots, t_m \in \Omega$ such that

$$\{x(t)|\ x \in Y\} \subset \bigcup_{k=1}^{m} (\{x(t_i)|\ x \in Y\} + \overline{B}(0;\varepsilon))$$

for any $t \in \Omega$.

If $\mu > \sup\limits_{t \in \Omega}(\alpha_K(\{x(t)|\ x \in Y\})$ then there exist $Y_1, \ldots, Y_p$ such that

$$\bigcup_{k=1}^{m} \{x(t_k)|\ x \in Y\} \subset \bigcup_{j=1}^{p} Y_j \text{ and } \delta(Y_j) \le \mu.$$

So, we have

$$\alpha_K(Y) \le \mu + 2\varepsilon,\ \forall\ \varepsilon > 0,\ \text{i.e., } \alpha_K(Y) \le \mu.$$

Remark 3.3.1. The measure α_H has not all the properties of α_K. For example the following definitions are not equivalent.

Let (X, d) be a metric space.

Definition 3.3.1. The Hausdorff measure of noncompactness of X is the following functional $\alpha_H : P_b(X) \to \mathbb{R}_+$, where for $A \in P_b(X)$,

$$\alpha_H(A) := \inf\left\{\varepsilon > 0|\ \exists\ x_i \in X,\ 0 < r_i \le \varepsilon :\ A \subset \bigcup_{i=1}^{m} \overline{B}(x_i, r_i)\right\}.$$

Definition 3.3.2. The Hausdorff measure of noncompactness of X is the following functional $\alpha_{iH} : P_b(X) \to \mathbb{R}_+$, where for $A \in P_b(X)$,

$$\alpha_{iH}(A) := \inf\left\{\varepsilon > 0 | \; \exists \; x_i \in A, \; 0 < r_i \leq \varepsilon : \; A \subset \bigcup_{i=1}^{m} \overline{B}(x_i, r_i)\right\}.$$

Example 3.3.1. (M. Furi and A. Vignoli (1970)). Let X be a Hilbert space. Let A be an infinite ortogonal system of X. Then $\alpha_H(A) = 1$ and $\alpha_{iH}(A) = \sqrt{2}$.

3.4 The De Blasi measure of weak noncompactness

Let X be a Banach space and $Y \subset X$ a subset of X. We denote
$\overline{Y}^w$ - the weak closure of Y;
$P_{wcl}(X) := \{Y \subset X | \; Y$ is weakly closed$\}$;
$P_{wcp}(X) := \{Y \subset X | \; Y$ is weakly compact$\}$.
The De Blasi measure of weak noncompactness is defined as follows

$$\omega_D : P_b(X) \to \mathbb{R}_+, \; \omega_D(Y) := \inf\{\varepsilon > 0 | \; \exists \; C \in P_{wcp}(X)$$

$$\text{such that } Y \subset C + \varepsilon\overline{B}(0; 1)\}.$$

We have

Lemma 3.4.1. *The functional ω_D has the following properties:*
(i) $Y_1 \subset Y_2 \;\Rightarrow\; \omega_D(Y_1) \leq \omega_D(Y_2)$;
(ii) $\omega_D(Y_1 \cup Y_2) = \max(\omega_D(Y_1), \omega_D(Y_2)), \; Y_1, Y_2 \in P_b(X)$;
(iii) $\omega_D(Y) = \omega_D(\overline{Y}^w), \; Y \in P_b(X)$;
(iv) $\omega_D(coY) = \omega_D(Y), \; Y \in P_b(X)$;
(v) $\omega_D(Y_1 + Y_2) \leq \omega_D(Y_1) + \omega_D(Y_2), \; Y_1, Y_2 \in P_b(X)$;
(vi) $\omega_D(\lambda Y) = |\lambda|\omega_D(Y), \; \lambda > 0, \; Y \in P_b(X)$;
(vii) $\omega(Y) = 0 \;\Leftrightarrow\; \overline{Y}^w \in P_{wcp}(X)$.

Lemma 3.4.2. (De Blasi's intersection lemma). *Let X be a Banach space and $Y_n \in P_{wcl,b}(X)$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$, be such that*

$$\omega_D(Y_n) \to 0 \text{ as } n \to \infty.$$

Then

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset \text{ and } \omega_D(Y_\infty) = 0.$$

Proof. It is clear that Y_∞ is a weakly closed set and $\omega_D(Y_\infty) = 0$. Let us prove that $Y_\infty \neq \emptyset$.

Let $x_n \in Y_n$, $n \in \mathbb{N}$. Let $X_n := \{x_n, x_{n+1}, \dots\}$. We have $X_{n+1} = X_n \cup \{x_{n+1}\}$. Hence

$$\omega_D(X_0) = \omega_D(X_1) = \cdots = \omega_D(X_n) \to 0 \text{ as } n \to \infty.$$

It follows that $\omega_D(X_0) = 0$. This implies that there exists a weakly convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$.

Let x^* be the limit. We have that $x^* \in Y_\infty$.

3.5 Functional with the intersection property

The above considerations give rise to

Definition 3.5.1. (I.A. Rus [60]-[62]). Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$. A functional $\theta : Z \to \mathbb{R}_+$ has the intersection property if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ and $\theta(Y_n) \to 0$ as $n \to \infty$ imply that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in Z \text{ and } \theta(Y_\infty) = 0.$$

Example 3.5.1. Let (X, d) be a complete metric space. Then the functionals δ, α_K and α_H have the intersection property. In this case $Z = P_{b,cl}(X)$.

Example 3.5.2. Let X be a Banach space. A functional $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ is called Daneš-Pasicki measure of noncompactness if

(i) $\alpha_{DP}(Y) = 0 \Rightarrow \overline{Y} \in P_{cp}(X),\ \forall\, Y \in P_b(X)$;

(ii) $Y_1, Y_2 \in X,\ Y_1 \subset Y_2 \Rightarrow \alpha_{DP}(Y_1) \leq \alpha_{DP}(Y_2)$;

(iii) $\alpha_{DP}(Y \cup \{x\}) = \alpha_{DP}(Y),\ \forall\, Y \in P_b(X),\ \forall\, x \in X$.

If α_{DP} is a Daneš-Pasicki measure of noncompactness then, $\alpha_{DP} : P_{b,cl}(X) \to \mathbb{R}_+$ is a functional with the intersection property. See the proof of Lemma 3.2.5.

Example 3.5.3. Let (X, d) be a metric space. An operator $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X (W. Takahashi (1970)) if $d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1-\lambda)d(z, y)$ for all $x, y, z \in X$ and $\lambda \in [0, 1]$. The triple (X, d, W) is called a convex metric space. A convex metric space is said to have property (c) if every bounded decreasing net of nonempty, closed and convex subsets has nonempty intersection. The Eisenfeld-Lakshmikantham measure of nonconvexity is the following functional

$$\beta_{EL} : P_b(X) \to \mathbb{R}_+,\ \beta_{EL}(Y) := H_d(Y, coY).$$

We have

Lemma 3.5.1. *Let (X, d, W) be a convex metric space with the property (c). Then the functional $\beta_{EL} : P_{b,cl}(X) \to \mathbb{R}_+$ is a functional with intersection property.*

Proof. Let $Y_n \in P_{b,cl}(X)$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ be such that $\beta_{EL}(Y_n) \to 0$ as $n \to \infty$. But $\beta_{EL}(Y_n) = H(Y_n, coY_n) = H(Y_n, \overline{co}Y_n)$ and $\overline{co}Y_n \overset{H_d}{\to} \bigcap\limits_{m\in\mathbb{N}} \overline{co}Y_m \neq \emptyset$. These imply $Y_n \overset{H_d}{\to} \bigcap\limits_{m\in\mathbb{N}} \overline{co}Y_m = \bigcap\limits_{m\in\mathbb{N}} Y_m$.

Example 3.5.4. Let $\theta : Z \to \mathbb{R}_+$ a functional with the intersection property and $Z_1 \subset Z$, $Z_1 \neq \emptyset$. Then $\theta|_{Z_1} : Z_1 \to \mathbb{R}_+$ has the intersection property.

Example 3.5.5. Let $\theta_1, \theta_2 : Z \to \mathbb{R}_+$ be two functionals with the intersection property, and $\lambda_1, \lambda_2 \in \mathbb{R}_+$, $(\lambda_1, \lambda_2) \neq 0$. Then the functional $\lambda_1\theta_1 + \lambda_2\theta_2$

has the intersection property.

Example 3.4.6. If the functionals θ_1, θ_2 have the intersection property, then $\theta := \max(\theta_1, \theta_2)$ has the intersection property.

3.6 Compatible pair with a f.p.s.

Definition 3.6.1. Let $(X, S(X), M)$ be a f.p.s., $\theta : Z \to \mathbb{R}_+$ $(S(X) \subset Z \subset P(X))$ and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$. The pair (θ, η) is compatible with $(X, S(X), M)$ if

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, $\forall\ Y \in Z$;

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 3.6.1. Let $(X, S(X), M)$ be the fixed point structure of Nemytzki-Edelstein, i.e., (X, d) is a complete metric space, $S(X) := P_{cp}(X)$ and $M(Y) := \{f : Y \to Y |\ f$ is contractive$\}$. Let $Z = P_b(X)$, $\theta = \alpha_K$ and $\eta(Y) = \overline{Y}$. Then the pair (θ, η) is compatible with $(X, S(X), M)$.

Example 3.6.2. X is a Banach space, $S(X) := P_{cp,cv}(X)$, $M(Y) := C(Y, Y)$, $Z = P_b(X)$, $\theta = \alpha_K$ and $\eta(Y) = \overline{co}Y$. Then the pair (θ, η) is compatible with $(X, S(X), M)$.

3.7 An abstract measure of noncompactness

The Daneš-Pasiki measure of noncompatness (see Example 3.4.2) is an abstract measure of noncompactness. In what follow we give other examples of abstract measure of noncompactness.

Definition 3.7.1. Let X be a Banach space. A functional $\alpha : P_b(X) \to \mathbb{R}_+$ is called a measure of noncompactness on X iff α satisfies the following conditions:

(i) $\alpha(A) = 0$ implies $\overline{A} \in P_{cp}(X)$;

(ii) $\alpha(A) = \alpha(\overline{A}),\ \forall\ A \in P_b(X)$;

(iii) $A \subset B\ \Rightarrow\ \alpha(A) \leq \alpha(B),\ \forall\ A, B \in P_b(X)$;

(iv) $\alpha(coA) = \alpha(A),\ \forall\ A \in P_b(X)$;

(v) $\alpha|_{P_{b,cl}(X)}$ is a functional with the intersection property.

Example 3.7.1. $\alpha = \delta$ satisfies (i)-(v). We remark that δ isn't a Daneš-Pasicki measure of noncompactness.

Example 3.7.2. α_K and α_H satisfy (i)-(v).

Definition 3.7.2. Let X be a Banach space. A functional $\omega : P_b(X) \to \mathbb{R}_+$ is called a measure of weak noncompactness on X iff it satisfies the following conditions:

(i) $\omega(A) = 0$ implies $\overline{A}^w \in P_{wcp}(X)$;

(ii) $\omega(A) = \omega(\overline{A}^w),\ \forall\ A \in P_b(X)$;

(iii) $A \subset B$ implies $\omega(A) \leq \omega(B),\ \forall\ A, B \in P_b(X)$;

(iv) $\omega(coA) = \omega(A),\ \forall\ A \in P_b(X)$;

(v) $\omega|_{P_{b,wcl}(X)}$ is a functional with the intersection property.

Example 3.7.3. $\omega = \omega_D$ (see §3.4) satisfies (i)-(v).

Definition 3.7.3. Let (X, d) be a metric space. A functional $\alpha : P_b(X) \to \mathbb{R}_+$ is called a measure of noncompactness on the metric space X iff α satisfies the conditions (i), (ii), (iii) and (v) in Definition 3.7.1.

Remark 3.7.1. Let X be a Banach space and $\alpha : P_b(X) \to \mathbb{R}_+$ a functional. The following axioms appear in different definitions of some abstract measure of noncompactness:

(i) $\alpha(A) = 0$ implies $\overline{A} \in P_{cp}(X)$;

(ii) $\alpha(A) = 0$ iff $\overline{A} \in P_{cp}(X)$;

(iii) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$;

(iv) $A \in P_b(X)$, $x \in X$ imply $\alpha(A \cup \{x\}) = \alpha(A)$;

(v) $A \in P_b(X)$, $B \in P_{cp}(X)$ imply $\alpha(A \cup B) = \alpha(A)$;

(vi) $A \in P_b(X)$, $B \in P_b(X)$ imply $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$;

(vii) $\alpha|_{P_{b,cl}}(X)$ is a functional with the intersection property;

(viii) $\alpha(coA) = \alpha(A), \ \forall \ A \in P_b(X)$;

(ix) $\alpha(A + B) \leq \alpha(A) + \alpha(B), \ \forall \ A, B \in P_b(X)$;

(x) $\alpha(\overline{A}) = \alpha(A), \ \forall \ A \in P_b(X)$;

(xi) $\alpha(\lambda A) = |\lambda|\alpha(A), \ \forall \ A \in P_b(X)$ and $\forall \ \lambda \in \mathbb{R}$.

3.8 An abstract measure of nonconvexity

Definition 3.8.1. A triple $(X, \tau, \mathcal{C})$ is a convex topological space if

(i) (X, τ) is a Hausdorff topological space;

(ii) $(X, \mathcal{C})$ is a convex space, i.e., $\mathcal{C} \subset \mathcal{P}(X)$ with the following properties:

(a) $X, \emptyset \in \mathcal{C}$;

(b) $A_i \in \mathcal{C}, \ i \in I \ \Rightarrow \ \bigcap_{i \in I} A_i \in \mathcal{C}$;

(c) $\{x\} \in \mathcal{C}, \ \forall \ x \in X$;

(d) $\{x, y\} \in \mathcal{C} \ \Rightarrow \ x = y$.

(iii) $\mathcal{C}$ is a subbase for τ.

By definition the elements of $\mathcal{C}$ are called convex sets.

Definition 3.8.2. Let $(X, \tau, \mathcal{C})$ be a convex topological space and $Z \subset P(X)$, $Z \neq \emptyset$. A functional $\beta : Z \to \mathbb{R}_+$ is a large measure of nonconvexity if $\beta(A) = 0$ implies that $\overline{A} \in P_{cv}(X)$.

Definition 3.8.3. Let (X, d, W) be a convex metric space. A functional $\beta : P_b(X) \to \mathbb{R}_+$ is a measure of nonconvexity if is a large measure of nonconvexity and satisfies the following conditions:

(i) $\beta(\overline{A}) = \beta(A)$;

(ii) $\beta : (P_{b,cl}(X), H_d) \to \mathbb{R}_+$ is continuous.

Example 3.8.1. Let $(X, \|\cdot\|)$ be a Banach space, $\tau = \tau_{\|\cdot\|}$ and $\mathcal{C} = \mathcal{P}_{cv}(X)$. Then β_{EL} and δ are measures of nonconvexity on X.

Example 3.8.2. The functional β_{EL} defined on a convex metric space, (X, d, W), is a measure of nonconexity.

Example 3.8.3. Let (X, d, W) be a convex metric space and β_1, β_2 two measures of nonconvexity on X. Then $\beta := \max(\beta_1, \beta_2)$ is a measure of nonconvexity.

3.9 Operators with the intersection property

Let $(\mathcal{A}, \to, L)$ be an ordered L-space with the least element, 0.

Definition 3.9.1. Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$. An operator $\theta : Z \to \mathcal{A}$ has the intersection property if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ and $\theta(Y_n) \to 0$ as $n \to \infty$ implies that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in Z \text{ and } \theta(Y_\infty) = 0.$$

Example 3.9.1. Let X be a locally convex space and $(p_i)_{i\in I}$ a family of seminorms which generates the topology on X. Let $Z := P_{b,cl}(X)$ and $\mathcal{A} := \mathbb{M}(I, \mathbb{R}_+)$. We define the operator $\theta : Z \to \mathbb{M}(I, \mathbb{R}_+)$, $\theta(A) :=$ with the function $i \mapsto \alpha_K^i(A)$, where α_K^i is the Kuratowski measure of noncompactness w.r.t. the seminorm p_i. This operator has the intersection property.

Remark 3.9.1. One of the basic problems of the f.p.s. theory is to construct examples of operators with the intersection property. For each such example we have at least a fixed point theorem.

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Chapter 4

(θ, φ)-contractions and θ-condensing operators

4.1 Comparison functions

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Consider relative to φ the following conditions:

(i_φ) φ is increasing.

(ii_φ) $\varphi(t) < t, \ \forall\, t > 0$.

(iii_φ) $\varphi(0) = 0$.

(iv_φ) $\varphi^n(t) \to 0$ as $n \to \infty, \ \forall\, t \in \mathbb{R}_+$.

(v_φ) $t - \varphi(t) \to \infty$ as $t \to \infty$.

(vi_φ) $\sum\limits_{n \in \mathbb{N}} \varphi^n(t) < +\infty$.

Relative to the above conditions we have

Lemma 4.1.1. *The conditions (i_φ) and (ii_φ) imply (iii_φ).*

Proof. From (i_φ), $\varphi(0) \le \varphi(t), \ \forall\, t > 0$. From (ii_φ), $\varphi(0) \le \varphi(t) < t, \ \forall\, t > 0$. So, $\varphi(0) = 0$.

Lemma 4.1.2. *The condition (i_φ) and (iv_φ) imply (ii_φ).*

Proof. Let $t_0 \in \mathbb{R}_+^*$ such that $\varphi(t_0) \geq t_0$. From (i_φ) we have $\varphi^n(t_0) \geq t_0$, $\forall\, n \in \mathbb{N}$, and from (iv_φ) it follows that $t_0 = 0$.

Definition 4.1.1. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if φ satisfies the conditions (i_φ) and (iv_φ).

Definition 4.1.2. A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function if it satisfies (v_φ).

Definition 4.1.3. A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a good comparison function if it satisfies (vi_φ).

Example 4.1.1. Let $\lambda \in [0,1[$. Then $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) := \lambda t$, is a strict and good comparison function.

Example 4.1.2. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) := \dfrac{t}{1+t}$ is a strict comparison function, but isn't a good comparison function.

Example 4.1.3. The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) := \dfrac{1}{2}t$ for $t \in [0,1]$ and $\varphi(t) := t - \dfrac{1}{2}$ for $t > 1$, is a comparison function.

Example 4.1.4. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then each iterate φ^k, $k \geq 1$, is a comparison function.

For more considerations on comparison functions see I.A. Rus [63] and V. Berinde [9].

4.2 (θ,φ)-contractions and θ-condensing operators. Definitions and examples

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$ a functional.

Definition 4.2.1. An operator $f : X \to X$ is a strong (θ,φ)-contraction if

(i) φ is a comparison function;

(ii) $A \in Z \;\Rightarrow\; f(A) \in Z$;

(iii) $\theta(f(A)) \leq \varphi(\theta(A)),\ \forall\, A \in Z$.

Definition 4.2.2. An operator $f : X \to X$ is a (θ, φ)-contraction if satisfies the conditions (i) and (ii) in Definition 4.2.1 and the condition:

(iii') $\theta(f(A)) \leq \varphi(\theta(A)),\ \forall\ A \in Z \cap I(f)$.

Definition 4.2.3. An operator $f : X \to X$ is strong θ-condensing if

(i) $A \in Z \ \Rightarrow\ f(A) \in Z$;

(ii) $A \in Z,\ \theta(A) \neq 0 \ \Rightarrow\ \theta(f(A)) < \theta(A)$.

Definition 4.2.4. An operator $f : X \to X$ is θ-condensing if

(i) $A \in Z \ \Rightarrow\ f(A) \in Z$;

(ii) $A \in Z \cap I(f),\ \theta(A) \neq 0 \ \Rightarrow\ \theta(f(A)) < \theta(A)$.

Example 4.2.1. Let (X, d) be a metric space, $Z := P_b(X)$ and $\theta = \delta$. Then an operator $f : X \to X$ is a strong (δ, φ)-contraction if and only if f is a φ-contraction, i.e.,

$$d(f(x), f(y)) \leq \varphi(d(x, y)),\ \forall\ x, y \in X.$$

Indeed, let f be a φ-contraction. Then

$$d(f(x), f(y)) \leq \varphi(d(x, y)),\ \forall\ x, y \in X.$$

Let $A \in P_b(X)$, and $x, y \in A$. Then

$$d(f(x), f(y)) \leq \varphi(d(x, y)) \leq \varphi(\delta(A)).$$

So,

$$\delta(f(A)) \leq \varphi(\delta(A)),\ \forall\ A \in P_b(X).$$

Let $f : X \to X$ be a strong (δ, φ)-contraction.

Let $A := \{x, y\},\ x, y \in X$. Then

$$\delta(f(A)) = d(f(x), f(y)) \leq \varphi(\delta(A)) = \varphi(d(x, y)).$$

Example 4.2.2. Let (X, d) be a metric space and $f : X \to X$ a Ćirić-Reich-Rus operator, i.e., there exist $a, b \in \mathbb{R}_+$, $a + 2b < 1$, such that

$$d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))],\ \forall\ x, y \in X.$$

Then f is a (δ, φ)-contraction, where $\varphi(t) := (a + 2b)t$, $t \in \mathbb{R}_+$.

Indeed, let $A \in P_b(X) \cap I(f)$. Let $x, y \in A$. Then

$$d(f(x), f(y)) \leq (a + 2b)\delta(A), \ \forall \ x, y \in A.$$

So,

$$\delta(f(A)) \leq (a + 2b)\delta(A).$$

Example 4.2.3. Let (X, d) be a metric space and $f : X \to X$ a compact operator, i.e., $A \in P_b(X)$ implies that $\overline{f(A)} \in P_{cp}(X)$. Then f is a strong $(\alpha_K, 0)$-contraction.

Indeed, let $A \in P_b(X)$. We have

$$\alpha_K(f(A)) = \alpha_K(\overline{f(A)}) = 0 \leq 0\alpha_K(A).$$

Example 4.2.4. Let (X, d) be a metric space and $f : X \to X$ a φ-contraction. Then f is a strong (α_K, φ)-contraction.

Indeed, let $A \in P_b(X)$ and $A = \bigcup_{i=1}^{n} A_i$. Then $f(A) = \bigcup_{i=1}^{n} f(A_i)$ and $\delta(f(A_i)) \leq \varphi(\delta(A_i))$, $i = \overline{1, n}$.

Now the proof follows from the definition of α_K.

Example 4.2.5. Let X be a Banach space, $f : X \to X$ a compact operator and $g : X \to X$ a φ-contraction. Then the operator $h := f + g$ is a strong (α_K, φ)-contraction.

Indeed, for $A \in P_b(X)$ we have

$$\begin{aligned}\alpha_K(h(A)) &= \alpha_K((f + g)(A)) \leq \alpha_K(f(A) + g(A)) \\ &\leq \alpha_K(f(A)) + \alpha_K(g(A)) = \alpha_K(g(A)) \leq \varphi(\alpha_K(A)).\end{aligned}$$

Example 4.2.6. This is an example for the Example 4.2.5. Consider the Banach space $(C[a, b], \|\cdot\|_C)$. Let $K \in C([a, b] \times [a, b] \times \mathbb{R})$ and $H \in C([a, b] \times \mathbb{R})$ such that there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|H(t, \eta_1) - H(t, \eta_2)| \leq \varphi(|\eta_1 - \eta_2|), \ \forall \ t \in [a, b], \ \eta_1, \eta_2 \in \mathbb{R}.$$

We suppose that $(b-a)\varphi$ is a comparison function. Then the operator $h : C[a,b] \to C[a,b]$, defined by

$$h(x)(t) := \int_a^t K(t,s,x(s))ds + H(t,x(t)),\ t \in [a,b],$$

is a strong $(\alpha_K,(b-a)\varphi)$-contraction.

Indeed, the operator $f : C[a,b] \to C[a,b]$ defined by

$$f(x)(t) := \int_a^t K(t,s,x(s))ds$$

is compact and the operator $g : C[a,b] \to C[a,b]$ defined by

$$g(x)(t) := H(t,x(t))$$

is a $(b-a)\varphi$-contraction.

Example 4.2.7. Let X be a Banach space and α_K the Kuratowski measure of noncompactness of X. Let $\Omega \subset X$ and $l \in \mathbb{R}_+$. An operator $f : \Omega \to X$ is strong (α_K,l)-Lipschitz if

$$\alpha_K(f(A)) \le l\alpha_K(A),\ \forall\ A \in P_b(\Omega).$$

Let $\Omega \subset X$ be an open subset and $f : \Omega \to X$ be such that

(i) f is differentiable;

(ii) f is strong (α_K,l)-Lipschitz.

Then the differential of f at x, $\partial f(x) : X \to X$ is strong (α_K,l)-Lipschitz for all $x \in X$.

Proof. From the definition of Fréchet differential, we have $\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon)$ and $\omega(x,h)$ such that

$$\partial f(x)(h) = f(x+h) - f(x) + \omega(x,h),\ \forall\ h \in X,\ \|h\| \le \delta(\varepsilon)$$

and $\|\omega(x,h)\| \le \varepsilon\|h\|$.

Let $A \in P_b(X)$. Then

$$\begin{aligned}\alpha_K(\partial f(x)(A)) &\leq \alpha_K(f(x+A)) + \alpha_K(\omega(x,A)) \\ &\leq l\alpha_K(A) + \varepsilon\delta(0,A), \ \forall\, \varepsilon > 0.\end{aligned}$$

So,

$$\alpha_K(\partial f(x)(A)) \leq l\alpha_K(A), \ \forall\, A \in P_b(X).$$

Example 4.2.8. The radial retraction ρ on a Banach space X to $\overline{B}(0;1)$ (see Example 1.3.1) is strong α_K-nonexpansive. It is known that ρ is $l(X)$-Lipschitz with $1 \leq l(X) \leq 2$ (see D.G. Defigueiredo and L.A. Karlovitz (1967), Chp.1). Now we prove that ρ is strong $(\alpha_K, 1)$-Lipschitz, i.e., ρ is strong α_K-nonexpansive. First we remark that

$$\rho(A) \subset \overline{co}(A \cup \{0\}), \ \forall\, A \in P_b(X).$$

So,

$$\alpha_K(\rho(A)) \leq \alpha_K(\overline{co}(A \cup \{0\})) = \alpha_K(A).$$

Example 4.2.9. Let X_i, $i = 1,2,3$, be Banach spaces. Let $f_1 : X_1 \to X_2$ be a strong (α_K, l_1)-Lipschitz operator and $f_2 : X_2 \to X_3$ be a strong (α_K, l_2)-Lipschitz. Then the operator $f_2 \circ f_1 : X_1 \to X_3$ is strong $(\alpha_K, l_1 l_2)$-Lipschitz.

Example 4.2.10. Let X_1, X_2 be two Banach spaces and $f_i : X_1 \to X_2$ an operator strong (α_K, l_i)-Lipschitz, $i = 1,2$. Then the operator $f_1 + f_2 : X_1 \to X_2$ is strong $(\alpha_K, l_1 + l_2)$-Lipschitz.

Example 4.2.11. ([6], p.40). Let X be an infinite dimensional Banach space and $\overline{B}(0;1) \subset X$. Consider the operator $f : \overline{B}(0;1) \to \overline{B}(0;1)$ defined by $f(x) := (1 - \|x\|)x$. This operator is α_K-condensing and is not (α_K, l)-contraction for any $l \in (0,1)$.

4.3 References

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Chapter 5

First general fixed point principle and applications

5.1 First general fixed point principle

One of the main results in the f.p.s. theory is the following

Theorem 5.1.1. (First general fixed point principle).

Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) f is a (θ, φ)-contraction.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. (a)+(b). $Y \in \eta(Z)$ implies $Y \in Z$, Condition (ii) implies $f(Y) \in Z$. Let $Y_1 := \eta(f(Y))$, $Y_2 := \eta(f(Y_1)), \ldots, Y_n := \eta(f(Y_{n-1})), \ldots$. We remark that

$Y_{n+1} \subset Y_n$, $Y_n \in F_\eta$ and $Y_n \in I(f)$. Let we denote $Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n$. From the condition (ii) we have that

$$\theta(Y_n) = \theta(\eta(f(Y_{n-1}))) = \theta(f(Y_{n-1}))$$

$$\leq \varphi(\theta(Y_{n-1})) \leq \cdots \leq \varphi^n(\theta(Y)) \to 0 \text{ as } n \to \infty.$$

Since $\theta : \eta(Z) \to \mathbb{R}_+$ is a functional with the intersection property, we have that $Y_\infty \neq \emptyset$, $Y_\infty \in \eta(Z)$, $Y_\infty \in I(f)$ and $\theta(Y_\infty) = 0$. Since the pair (θ, η) is a compatible pair with $(X, S(X), M)$ we have that $Y_\infty \in I(f) \cap S(X)$ and $f|_{Y_\infty} \in M(Y_\infty)$. So, $F_f \neq \emptyset$.

(c). Let $F_f \in Z$. From $f(F_f) = F_f$ and the condition (ii) we have that

$$\theta(F_f) = \theta(f(F_f)) \leq \varphi(\theta(F_f)).$$

Remark 5.1.1. In Theorem 5.1.1 is not necessarily that $M(Y)$ be defined for all $Y \in P(X)$. It is sufficiently that $M(Y)$ be defined for $Y \in \eta(Z)$.

From the proof of Theorem 5.1.1 we have

Theorem 5.1.2. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$ and $f \in M(Y)$ be such that $f(Y) \in Z$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) f is a (θ, φ)-contraction.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. First we remark that $\eta(f(Y)) \in I(f)$. After that we apply Theorem 5.1.1 for the operator $f|_{\eta(f(Y))} : \eta(f(Y)) \to \eta(f(Y))$.

Remark 5.1.2. In Theorem 5.1.2 is not necessarily that $M(Y)$ be defined for all $Y \in P(X)$. It is sufficiently that $M(Y)$ be defined for $Y \in F_\eta$.

Remark 5.1.3. In the above results θ take values in $\mathbb{R}_+$ and φ is from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let us consider instead of $\mathbb{R}_+$ an ordered L-space $(\mathcal{A}, \leq, \to)$ with the least element 0.

Definition 5.1.1. An operator $\varphi : \mathcal{A} \to \mathcal{A}$ is a comparison operator iff

(i) φ is increasing;

(ii) $\varphi(0) = 0$ and $\alpha \leq \varphi(\alpha)$ implies $\alpha = 0$, $\forall\ \alpha \in \mathcal{A}$;

(iii) $\varphi^n(\alpha) \to 0$ as $n \to \infty$, $\forall\ \alpha \in \mathcal{A}$.

In terms of comparison operators and of operator with intersection property (Definition 3.9.1), the Theorem 5.1.1 and 5.1.2 take the following form:

Theorem 5.1.1'. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathcal{A})$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) f is a (θ, φ)-contraction, where $\varphi : \mathcal{A} \to \mathcal{A}$.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Theorem 5.1.2'. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathcal{A})$ a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$ and $f \in M(Y)$ be such that $f(Y) \in Z$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) f is a (θ, φ)-contraction, where $\varphi : \mathcal{A} \to \mathcal{A}$.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Remark 5.1.4. All terms in the above results are set-theoretic. So, Theorem 5.1.1, Theorem 5.1.2, Theorem 5.1.1' and Theorem 5.1.2' are on an arbitrary set.

In what follow we shall present some consequences of the above theorems.

5.2 (δ, φ)-contraction principle

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function, (X, d) a metric space and $\delta : P_b(X) \to \mathbb{R}_+$ the diameter functional. We have

Theorem 5.2.1. *Let (X, d) be a bounded and a complete metric space and $f : X \to X$ a (δ, φ)-contraction. Then $F_f = \{x^*\}$.*

Proof. Consider the trivial f.p.s. on X. Let $Z = P(X)$, $\theta = \delta$, $\eta(A) = \overline{A}$, $Y = X$. It is clear that we are in the conditions of the Theorem 5.1.1. So, $F_f \neq \emptyset$ and $\delta(F_f) = 0$, i.e., $F_f = \{x^*\}$.

Theorem 5.2.1'. *Let (X, d) be a bounded and complete metric space, $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ a comparison function and $f : X \to X$ an operator. We suppose that*

$$d(f(x), f(y)) \leq \varphi(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)))$$

for all $x, y \in X$.

Then, $F_f = \{x^*\}$.

Proof. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\psi(t) := \varphi(t, t, t, t, t)$. The function ψ is a comparison function. We remark that the operator f is a (δ, ψ)-contraction. The proof follows from Theorem 5.2.1.

Theorem 5.2.2. *Let (X, d) be a complete metric space and $f : X \to X$ a (δ, φ)-contraction such that $f(X) \in P_b(X)$. Then, $F_f = \{x^*\}$.*

Proof. Follows from Theorem 5.1.2.

Theorem 5.2.2'. *Let (X, d) be a complete metric space, $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a complete metric space, $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ a comparison function and $f : X \to X$ an operator. We suppose that* *(i)* $d(f(x), f(y)) \leq \varphi(d(x,y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)))$, *for all* $x, y \in X$.

(ii) $f(X) \in P_b(X)$.

Then, $F_f = \{x^*\}$.

Proof. See the proof of Theorem 5.2.1' and considers the operator $f : \overline{f(X)} \to \overline{f(X)}$.

Theorem 5.2.3. *Let* (X, d) *be a bounded and complete metric space and* $f : X \to X$ *an operator. We suppose that:*

(i) f *is an* (α_k, φ)*-contraction;*

(ii) f *is a contractive operator.*

Then, $F_f = \{x^*\}$.

Proof. Consider on X the f.p.s. of Nemytzki-Edelstein. If we take $Z = P(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{A}$, then we are in the conditions of Theorem 5.1.1. From this theorem we have that $F_f \neq \emptyset$. From the condition (ii) we have that $Card F_f \leq 1$. So, $F_f = \{x^*\}$.

Theorem 5.2.4. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *an operator. We suppose that:*

(i) f *is an* (α_K, φ)*-contraction;*

(ii) f *is a contractive operator;*

(iii) $f(X) \in P_b(X)$.

Then, $F_f = \{x^*\}$.

Proof. Consider on X the f.p.s. of Nemytzki-Edelstein. If we take $Z = P_b(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{A}$, then we are in the conditions of Theorem 5.1.2. From this theorem we have that $F_f \neq \emptyset$. From the condition (ii) we have that $card F_f \leq 1$. So, $F_f = \{x^*\}$.

5.3 (α, φ)-contraction principle

In this section $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha : P_b(X) \to \mathbb{R}_+$, where X is a Banach space.

We have:

Theorem 5.3.1. *Let X be a Banach space, α an abstract measure of noncompactness, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) $f \in C(Y, Y)$;

(ii) f is an (α, φ)-contraction.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Proof. Let $(X, P_{cp,cv}(X), M)$ be the f.p.s. of Schauder ($M(Y) := C(Y, Y)$). Let $Z = P_b(X)$, $\theta = \alpha$ and $\eta(A) = \overline{co}A$. The proof follows from Theorem 5.1.1.

Theorem 5.3.2. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$, α an abstract measure of noncompactness on X and $f : Y \to Y$ an operator. We suppose that:*

(i) $f \in C(Y, Y)$;

(ii) f is an (α, φ)-contraction;

(iii) $f(Y) \in P_b(X)$.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Proof. The proof follows from Theorem 5.1.2, where instead of Y we take $\overline{co}f(Y)$.

From the Theorem 5.3.1 we have

Theorem 5.3.3. (Darbo (1955)). *Let X be a Banach space, $l \in [0, 1[$,*

$Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:

(i) $f \in C(Y, Y)$;

(ii) f is an (α_K, l)-contraction.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Proof. We take in the theorem 5.3.1, $\alpha := \alpha_K$ and $\varphi(t) := lt$, $t \in \mathbb{R}_+$.

From Theorem 5.3.2 we have:

Theorem 5.3.4. *Let X be a Banach space, $l \in [0, 1[$, $Y \in P_{cl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) $f \in C(Y, Y)$;

(ii) f is an (α_K, l)-contraction;

(iii) $f(Y) \in P_b(X)$.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Remark 5.3.3 Darbo works with strong (α_K, l)-contractions.

From Darbo's fixed point theorem we have

Theorem 5.3.5. (Krasnoselskii (1958)). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$ two operators. We suppose that:*

(i) f is completete continuous;

(ii) g is a contraction;

(iii) $f(x) + g(x) \in Y$, $\forall\ x \in Y$.

Then the operator $f + g$ has at least a fixed point.

Proof. Let g be an l-contraction. Then $f + g$ is an (α_K, l)-contraction (see Example 4.2.5). Now the proof follows from the fixed point theorem of Darbo.

Remark 5.3.4. In the Theorem 5.3.5 instead of condition (ii) we can put the condition (ii') g is a φ-contraction.

5.4 (ω, φ)-contraction principle

In this section $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega : P_b(X) \to \mathbb{R}_+$ where X is a Banach space.

We have

Theorem 5.4.1. *Let X be a Banach space, ω an abstract measure of weak noncompactness on X, $Y \in P_{b,wcl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is weakly continuous;

(ii) f is an (ω, φ)-contraction.

Then:

(a) $F_f \neq \emptyset$

(b) F_f is a weak compact subset of Y.

Proof. Let $(X, P_{wcp,cv}(X), M)$ be the fixed point structure of Tychonoff ($M(Y) := \{f : Y \to Y | f$ is weakly continuous operator$\}$). Let $Z = P_b(X)$, $\theta = \omega$ and $\eta(A) = \overline{co}A$. The proof follows from Theorem 5.1.1.

Theorem 5.4.2. (G. Emmanuele (1981)). *Let X be a Banach space, ω_D the De Blasi weak measure of noncompactness on X, $Y \in P_{b,wcl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is weakly continuous;

(ii) f is an (ω_D, l)-contraction.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a weak compact subset of Y.

Proof. One takes $\omega = \omega_D$, $\varphi(t) = lt$ in the Theorem 5.4.1.

Remark 5.4.1. In the Theorem 5.4.1 and 5.4.2 we can take $Y \in P_{wcl,cv}(X)$ with $f(Y) \in P_b(X)$.

5.5 (β,φ)-contraction principle

We begin our considerations in this section with

Lemma 5.5.1. *Let $(X,\tau,\mathcal{C})$ be a convex topological space and $\beta : Z \to \mathbb{R}_+$ a large measure of nonconvexity on X. Let we suppose that:*

(i) $Z \subset P(X)$ and if $A \in P(X)$ with $cardA < +\infty$ imply that $A \in Z$;

(ii) f is a (β,φ)-contraction.

Then, $cardF_f \leq 1$.

Proof. Let $x, y \in F_f$. Then $\{x, y\} \in Z$. We have

$$\beta(\{x,y\}) = \beta(f(\{x,y\})) \leq \varphi(\beta(\{x,y\})).$$

This implies that $\beta(\{x,y\}) = 0$. So, $x = y$.

Theorem 5.5.1. *Let X be a Banach space, β a large measure of nonconvexity on X and $Y \in P_{cp}(X)$. If $f : Y \to Y$ is a continuous (β,φ)-contraction, then $F_f = \{x^*\}$.*

Proof. We remark that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} f^n(Y) \neq \emptyset \text{ and } f(Y_\infty) = Y_\infty.$$

Since f is a (β,φ)-contraction, it follows that $\beta(A_\infty) = 0$. This implies that $A_\infty \in P_{cp,cv}(X)$. The proof follows from the fixed point theorem of Schauder.

In order to present another result we consider (X, d, W) Takahashi's convex metric space. Moreover we suppose that if $x, y \in X$ and $\{x, y\}$ is a convex set, then $x = y$. We recall that a convex metric space X is with the property (C) iff every bounded decreasing net of nonempty, closed convex subsets of X has nonempty intersection.

Theorem 5.5.2. *Let (X, d, W) be a strictly convex metric space with property (C), $Y \in P_{b,cl}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is a nonexpansive operator;

(ii) f is a (β_{EL}, φ)-contraction.

Then, $F_f = \{x^*\}$.

Proof. We consider on X the f.p.s. of Takahashi, i.e., $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y | f$ a nonexpansive operator$\}$. Let $Z = P_b(X)$, $\theta := \beta_{EL}$ and $\eta(A) := \overline{A}$. Now, the proof follows from Theorem 5.1.1.

5.6 References

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Chapter 6

Second general fixed point principle and applications

6.1 Second general fixed point principle

The second main result in the f.p.s. theory is the following

Theorem 6.1.1. (Second general fixed point principle).

Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:

(i) $A \in Z$, $x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) f is a θ-condensing operator.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) if $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. (a)+(b). Let $x \in Y$ and $A = \{x\}$. By Lemma 1.6.1 there exists $A_0 \in F_\eta \cap I(f)$ such that $\eta(f(A_0) \cup \{x\}) = A_0$. We have that

$$\theta(\eta(f(A_0) \cup \{x\}) = \theta(f(A_0) \cup \{x\}) = \theta(f(A_0)) = \theta(A_0).$$

From the condition (ii) we have that $A_0 \in Z_\theta$. Hence $A_0 \in F_\eta \cap I(f)$. So, $A_0 \in S(X)$ and $f|_{A_0} \in M(A_0)$. Since $(X, S(X), M)$ is a f.p.s. we have that $F_f \neq \emptyset$.

(c) From the condition (ii) and from $f(F_f) = F_f$, we have that $\theta(F_f) = 0$.

From the proof of the above theorem we have

Theorem 6.1.2. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$ and $f \in M(Y)$ with $f(Y) \in Z$. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) f is a θ-condensing operator.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Remark 6.1.1. In the above results θ take values in $\mathbb{R}_+$. If instead of $\mathbb{R}_+$ we take an ordered set, $(\mathcal{A}, \leq)$, with the least element 0, then Theorem 6.1.1 and Theorem 6.1.2 take the following form:

Theorem 6.1.1'. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathcal{A})$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$.

(ii) $\theta(f(A)) = \theta(A)$ implies $\theta(A) = 0$.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Theorem 6.1.2'. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathcal{A})$ a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$ and $f \in M(Y)$ with $f(Y) \in Z$. We suppose that:*

(i) $A \in Z,\ x \in Y$ *imply that* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$*;*

(ii) $\theta(f(A)) = \theta(A)$ *implies* $\theta(A) = 0$.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$*;*

(b) $F_f \neq \emptyset$*;*

(c) If $F_f \in Z$*, then* $\theta(F_f) = 0$.

Remark 6.1.2. All terms in the above results are set-theoretic. So, Theorem 6.1.1, 6.1.2, 6.1.1' and 6.1.2' are on an arbitrary set.

Remark 6.1.3. See Remark 5.1.1 and Remark 5.1.2.

In the following sections we shall present some consequences of these abstract theorems.

6.2 α_{DP}-condensing operator principle

In this section X is a Banach space and $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ is the abstract measure of noncompactness of Daneš-Pasicki.

We have

Theorem 6.2.1. *Let* X *be a Banach space,* $Y \in P_{b,cl,cv}(X)$ *and* $f : Y \to Y$ *a continuous* α_{DP}*-condensing operator. Then,* $F_f \neq \emptyset$ *and* $\alpha_{DP}(F_f) = 0$.

Proof. If we consider $S(X) := P_{cp,cv}(X)$, $M(Y) := C(Y,Y)$, $\theta := \alpha_{DP}$ and $\eta(A) := \overline{co}A$, then we are in the conditions of the Theorem 6.1.1.

Theorem 6.2.2. *Let* X *be a Banach space,* $Y \in P_{cl,cv}(X)$ *and* $f : Y \to Y$ *a continuous* α_{DP}*-condensing operator with* $f(Y) \in P_b(X)$*. Then,* $F_f \neq \emptyset$ *and* $\alpha_{DP}(F_f) = 0$.

Proof. Let $S(X) := P_{cp,cv}(X)$, $M(Y) := C(Y,Y)$, $\theta := \alpha_{DP}$ and $\eta(A) := \overline{co}A$. Then we are in the conditions of Theorem 6.1.2.

Theorem 6.2.3. (Sadovskij (1967)). *Let* X *be a Banach space,* $Y \in P_{b,cl,cv}(X)$ *and* $f : Y \to Y$ *a continuous* α_H*-condensing operator. Then,* $F_f \neq \emptyset$

and F_f is a compact subset of Y.

Proof. We take $\alpha_{DP} = \alpha_H$ in Theorem 6.2.1.

Remark 6.2.1. Sadovskij works with strong α_H-condensing operators.

Let X be a topological vector space $S(X) := P_{cp,cv}(X)$ and $M(Y) := \{f : Y \to Y \mid f \text{ continuous affine operator}\}$, $Y \in P_{cv}(X)$. Let L be a lattice and $\alpha : P(X) \to L$ a set-argument operator which satisfies the following conditions:

(i) $\alpha(A) = 0 \Rightarrow \overline{A} \in P_{cp}(X)$;

(ii) $\alpha(A \cup \{x\}) = \alpha(A),\ A \in P(X),\ x \in X$

(iii) $\alpha(\overline{co}A) = \alpha(A)$.

If we take, in Theorem 6.1.1, the above fixed point structure, $\theta := \alpha$, $\eta = \overline{co}$, then we have

Theorem 6.2.4. (H.S. Chon - W. Lee (2000)). *Let X be a Hausdorff topological vector space, $Y \in P_{cl,cv}(X)$ and $f : Y \to Y$ a continuous affine operator. If f is α-condensing, then $F_f \neq \emptyset$.*

6.3 ω-condensing operator principle

In this section X is a Banach space and $\omega : P_b(X) \to \mathbb{R}_+$ is a weak noncompactness measure on X.

We have

Theorem 6.3.1. *Let X be a Banach space, ω an abstract measure of weak noncompactness on X, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is weakly continuous;

(ii) f is ω-condensing operator.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a weak compact subset of Y.

Proof. Let (X, P_{wcp}, M) be the fixed point structure of Tychonoff, $Z := P_b(X)$, $\theta := \omega$ and $\eta(A) = \overline{co}^w A$. The proof follows from Theorem 6.1.1.

Theorem 6.3.2. *Let X be a Banach space, ω an abstract measure of weak noncompactness on $X, Y \in P_{wcl,cv}(X)$ and $f : Y \to Y$ an operator with $f(Y) \in P_b(X)$. We suppose that:*

(i) f is weakly continuous;

(ii) f is ω-condensing operator.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a weak compact subset of Y.

Proof. We apply Theorem 6.1.2.

Theorem 6.3.3. (G. Emmanuele (1981)). *Let X be a Banach space, ω_D the De Blasi weak measure of noncompactness on X, $Y \in P_{b,wcl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is weakly continuous;

(ii) f is ω_D-condensing operator.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a weak compact subset of Y.

6.4 β-condensing operator principle

We have

Lemma 6.4.1. *Let $(X, \tau, \mathcal{C})$ be a convex topological space and $\beta : Z \to \mathbb{R}_+$ a large measure of nonconvexity on X. Let $Y \in P(X)$ and $f : Y \to Y$ be an operator. We suppose that:*

(i) $Z \subset P(X)$ and if $A \in P(X)$ with $cardA < +\infty$ imply that $A \in Z$;

(ii) f is β-condensing.

Then, $cardF_f \leq 1$.

Proof. Let $x, y \in F_f$. Then $\{x, y\} \in Z$. We have

$$\beta(\{x,y\}) = \beta(f(\{x,y\})).$$

From the condition (ii) it follows that $\beta(\{x,y\}) = 0$. So, $x = y$.

Theorem 6.4.1. *Let X be a Banach space, $\beta : P_b(X) \to \mathbb{R}_+$ a large measure of nonconvexity on X and $Y \in P_{cp}(X)$. If $f : Y \to Y$ is a continuous β-condensing operator, then $F_f = \{x^*\}$.*

Proof. Let $Y_\infty := \bigcap\limits_{n\in\mathbb{N}} f^n(Y)$. We remark that $Y_\infty \neq \emptyset$, $Y_\infty \in P_{cp}(X)$ and $f(Y_\infty) = Y_\infty$. Since f is β-condensing it follows that $\beta(Y_\infty) = 0$. This implies that $Y_\infty \in P_{cp,cv}(X)$. Now, the proof follows from the fixed point theorem of Schauder.

Theorem 6.4.2. *Let X be a Banach space, β a large measure of nonconvexity on X and α_{DP} a Daneš-Pasicki measure of noncompactness on X. Let $Y \in P_{b,cl}(X)$ and $f : Y \to Y$ a continuous operator. We suppose that:*

(i) f is α_{DP}-condensing;

(ii) f is β-condensing.

Then, $F_f = \{x^*\}$.

Proof. From Theorem 6.2.1 we have that $F_f \neq \emptyset$. From Lemma 6.4.1, $F_f = \{x^*\}$.

6.5 δ-condensing operator principle

Theorem 6.5.1. *Let (X,d) be a compact metric space and $f : X \to X$ a continuous δ-condensing operator. Then $F_f = \{x^*\}$.*

Proof. We remark that $X_\infty := \bigcap\limits_{n\in\mathbb{N}} f^n(X) \neq \emptyset$ and $f(X_\infty) = X_\infty$. From the δ-condensing condition it follows that $\delta(X_\infty) = 0$, i.e., $X_\infty = \{x^*\}$ and

$F_f = \{x^*\}$.

Theorem 6.5.2. *Let* (X, d) *be a bounded and complete metric space,* α *a measure of noncompactness space and* $f : X \to X$ *an operator. We suppose that:*

(i) f *is a* (α, φ)*-contraction;*

(ii) f *is* δ*-condensing operator.*

Then, $F_f = \{x^*\}$.

Proof. From the condition (i) it follow that there exists a compact set X_∞ with the following properties:

- X_∞ is an invariant subset for f;
- X_∞ is compact;
- $F_f \subset X_\infty$.

Now, the proof follows from the condition (ii) and Theorem 6.5.2.

6.6 References

For the second general fixed point principle and its application see I.A. Rus [62]. See also, I.A. Rus [60], [63] and I.A. Rus, A. Petruşel and G. Petruşel [65]. For other considerations on second fixed point principle see

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For other fixed point theorems in terms of strong θ-condensing operator (θ, $\theta = \alpha$, $\theta = \beta$, $\theta = \omega, \dots$) see R.R. Akhmerov, M.I. Kamenskij, A.S. Potapov, A.E. Rodkina and B.N. Sadovskij [2], H. Amann [3], J. Appell [5], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo [6], J. Banas and K. Goebel [8]. S. Czerwik [26], K. Deimling [27], B.N. Sadovskij [66], E. Zeidler [74].

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Chapter 7

Fixed point structures with the common fixed point property

7.1 Commuting operators, common fixed points and common invariant subsets

Let X be a nonempty set and $f, g : X \to X$ two operators. We have

Lemma 7.1.1. *If $f \circ g = g \circ f$, then*

(a) $F_f, F_g \in I(f) \cap I(g)$;

(b) $f(X), g(X) \in I(f) \cap I(g)$.

Proof. (a) Let, for example, $x \in F_g$. Then

$$f(x) = f(g(x)) = g(f(x)).$$

So, $f(x) \in F_g$.

(b) It is clear that $f(X) \in I(f)$. We have

$$g(f(X)) = f(g(X)) \subset f(X).$$

So, $f(X) \in I(f) \cap I(g)$.

Lemma 7.1.2. *If $f \circ g = g \circ f$, then*

$$f(X) \cup g(X) \subset Z \subset X \;\Rightarrow\; Z \in I(f) \cap I(g).$$

Lemma 7.1.3. *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ and $f, g : Y \to Y$ such that $f \circ g = g \circ f$. Let $A_1 \in P(Y)$. Then there exists $A_0 \subset Y$ such that:*

(a) $A_1 \subset A_0$;

(b) $A_0 \in F_\eta$;

(c) $A_0 \in I(f) \cap I(g)$;

(d) $\eta(f(A_0) \cup g(A_0) \cup A_1) = A_0$.

Proof. Let $\mathcal{B} := \{B \subset Y |\ B$ satisfies the conditions (a), (b) and (c)$\}$. From Lemma 1.4.1 we have that $\cap\mathcal{B} \in \mathcal{B}$. This implies that $\cap\mathcal{B}$ is the least element of the partially ordered set $(\mathcal{B}, \subset)$. Let us prove that $A_0 = \cap\mathcal{B}$.

We remark that $\eta(f(A_0) \cup g(A_0) \cup A_1) \in \mathcal{B}$ and $\eta(f(A_0) \cup g(A_0)) \subset A_0$. These imply that $\eta(f(A_0) \cup g(A_0) \cup A_1) = A_0$.

7.2 Fixed point structures with the common fixed point property

Definition 7.2.1. A fixed point structure $(X, S(X), M)$ is with the common fixed point property iff

$$Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \;\Rightarrow\; F_f \cap F_g \neq \emptyset.$$

Example 7.2.1. The Tarski f.p.s. is with the common fixed point property. Indeed, let $(X, \leq)$ be an ordered set and $Y \subset X$ a complete lattice. Let $f, g : Y \to Y$ be increasing operators such that $f \circ g = g \circ f$. By the Tarski's fixed point theorem we have that $F_f \neq \emptyset$ and $(F_f, \leq)$ is a complete lattice. By Lemma 7.1.1, $F_f \in I_g$. By Tarski's fixed point theorem the operator $g|_{F_f} : F_f \to F_f$ has at least a fixed point. So, $F_f \cap F_g \neq \emptyset$.

More general we have

Example 7.2.2. Let $(X, S(X), M)$ be a fixed point structure such that

$$Y \in S(X),\ f \in M(Y) \ \Rightarrow\ F_f \in S(X).$$

Then, $(X, S(X), M)$ is with common fixed point property.

Example 7.2.3. A continuous function $f : [0,1] \to [0,1]$ is full if the interval $[0,1]$ may be subdivided into a finite number $I_1, \dots, I_m$ of subintervals on each of which $f|_{I_k} : I_k \to [0,1]$ is a topological isomorphism. Let $X = [0,1]$, $S(X) := \{X\}$ and $M(X) := \{f \in C([0,1],[0,1]) |\ f \text{ is full}\}$. Then the triple $(X, S(X), M)$ is a large f.p.s. with the common fixed point property. This follows from the following result

Theorem 7.2.1. (H. Cohen (1964)). *Let $f, g \in C([0,1],[0,1])$ be commuting full functions. Then $F_f \cap F_g \neq \emptyset$.*

Remark 7.2.1. The following result is given by T. Suzuki in 2002:

Theorem 7.2.2. *Let X be a Banach space, $Y \in P_{cp,cv}(X)$ and $f, g : Y \to Y$ be nonexpansive operators with $f \circ g = g \circ f$. Then for $y \in Y$, the following statements are equivalent:*

(i) $y \in F_f \cap F_g$;

(ii) $\liminf_{n\to\infty} \left\| \dfrac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} f^i g^j(y) - y \right\| = 0.$

7.3 (θ, φ)-contraction pair

Let X be a nonempty set, $Y \subset X$ and $\theta : Z \to \mathbb{R}_+$, where $Z \subset P(X)$, $Z \neq \emptyset$.

Definition 7.3.1. A pair of operators $f, g : Y \to Y$ is a (θ, φ)-contraction pair if

(i) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function;

(ii) $A \in P(Y) \cap Z \;\Rightarrow\; f(A) \cup g(A) \in Z$;

(iii) $\theta(f(A) \cup g(A)) \le \varphi(\theta(A)),\ \forall\ A \in I(f) \cap I(g) \cap Z$.

If $l \in [0,1[$ and $\varphi(t) = lt$, then by (θ, l)-contraction pair we understand a $(\theta, l(\cdot))$-contraction pair.

Example 7.3.1. Let (X, d) be a metric space and α_K the Kuratowski measure of noncompactness on X. If $f_i : X \to X$ is an (α_K, l_i)-contraction, $i = 1, 2$, then the pair f_1, f_2 is an $(\alpha_K, \max(l_1, l_2))$-contraction pair.

We have

Theorem 7.3.1. *Let $(X, S(X), M)$ be a f.p.s. with the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$. We suppose that*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $f \circ g = g \circ f$;

(iii) the pair (f, g) is a (θ, φ)-contraction pair.

Then:

(a) $F_f \cap F_g \neq \emptyset$;

(b) If $F_f \cap F_g \in Z$, then $\theta(F_f \cap F_g) = 0$.

Proof. (a) Let $Y_1 := \eta(f(Y) \cup g(Y)), \ldots, Y_{n+1} := \eta(f(Y_n) \cup g(Y_n))$, $n \in \mathbb{N}$. From Lemma 7.1.2 we have that $Y_n \in I(f) \cap I(g)$, $n \in \mathbb{N}$. From the conditions (ii) and (iii) we have

$$\theta(Y_{n+1}) = \theta(\eta(f(Y_n) \cup g(Y_n))) = \theta(f(Y_n) \cup g(Y_n))$$

$$\le \varphi(\theta(Y_n)) \le \cdots \le \varphi^{n+1}(\theta(Y)) \to 0 \text{ as } n \to \infty.$$

From the condition (i) it follows that $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$ and $\theta(Y_\infty) = 0$. It is clear that $\eta(Y_\infty) = Y_\infty$ and $Y_\infty \in I(f) \cap I(g)$. These imply that $Y_\infty \in S(X)$

and so, $F_f \cap F_g \neq \emptyset$.

(b) This follows from

$$f(F_f \cap F_g) \cup g(F_f \cap F_g) = F_f \cap F_g.$$

Theorem 7.3.2. *Let $(X, S(X), M)$ be a f.p.s. with common fixed point property and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$, $f, g \in M(Y)$ and $f(Y) \cup g(Y) \in Z$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $f \circ g = g \circ f$;

(iii) the pair (f, g) is a (θ, φ)-contraction pair.

Then:

(a) $F_f \cap F_g \neq \emptyset$;

(b) If $F_f \cap F_g \in Z$, then $\theta(F_f \cap F_g) = 0$.

Proof. We apply Theorem 7.3.1 to the operators

$$f, g : \eta(f(Y) \cup g(Y)) \to \eta(f(Y) \cup g(Y)).$$

Theorem 7.3.3. *Let X be a strictly convex Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$, two nonexpansive operators. We suppose that:*

(i) $f \circ g = g \circ f$;

(ii) the pair (f, g) is an (α_K, φ)-contraction pair.

Then:

(a) $F_f \cap F_g \neq \emptyset$;

(b) $\alpha_K(F_f \cap F_g) = 0$.

Proof. We consider in the Theorem 7.3.1 the following f.p.s. on X, $S(X) := P_{cp,cv}(X)$ and $M(Y) := \{h : Y \to Y |\ h$ is a nonexpansive operator$\}$. This f.p.s. is as that in Example 7.2.2.

7.4 θ-condensing pair

Let X be a nonempty set, $Y \subset X$ and $\theta : Z \to \mathbb{R}_+$, $Z \subset P(X)$, $Z \neq \emptyset$.

Definition 7.4.1. A pair of operators $f, g : Y \to Y$ is a θ-condensing pair iff:

(i) $A_i \in Z,\ i \in I,\ \bigcap\limits_{i \in I} A_i \neq \emptyset \ \Rightarrow \ \bigcap\limits_{i \in I} A_i \in Z$;

(ii) $A \in P(Y) \cap Z \ \Rightarrow \ f(A) \cup g(A) \in Z$;

(iii) $\theta(f(A) \cup g(A)) < \theta(A)$, for all $A \in I(f) \cap I(g) \cap Z$ such that $\theta(A) \neq 0$.

We have

Theorem 7.4.1. *Let $(X, S(X), M)$ be a f.p.s. with common fixed point property and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$. We suppose that:*

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $f \circ g = g \circ f$;

(iii) the pair (f, g) is θ-condensing pair.

Then:

(a) $F_f \cap F_g \neq \emptyset$;

(b) if $F_f \cap F_g \in Z$, then $\theta(F_f \cap F_g) = 0$.

Proof. (a) Let $x_0 \in Y$. By Lemma 7.1.3 there exists $A_0 \subset Y$ such that $x_0 \in A_0$, $A_0 \in F_\eta \cap I(f) \cap I(g) \cap Z$ and $\eta(f(A_0) \cup g(A_0) \cup \{x_0\}) = A_0$. From the condition (iii) we have that $\theta(A_0) = 0$. This implies that $A_0 \in S(X)$. So, $F_f \cap F_g \neq \emptyset$.

(b) From $f(F_f \cap F_g) \cup g(F_f \cap F_g) = F_f \cap F_g$, we have that $\theta(F_f \cap F_g) = 0$.

From the proof of Theorem 7.4.1 we have

Theorem 7.4.2. *Let $(X, S(X), M)$ be a f.p.s. with the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$, $f, g \in M(Y)$ and $f(Y) \cup g(Y) \in Z$. We suppose that*

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $f \circ g = g \circ f$;

(iii) the pair (f, g) *is* θ*-condensing pair.*

Then:

(a) $F_f \cap F_g \neq \emptyset$;

(b) if $F_f \cap F_g \in Z$, *then* $\theta(F_f \cap F_g) = 0$.

Theorem 7.4.3. *Let* X *be a strictly convex Banach space,* $Y \in P_{b,cl,cv}(X)$ *and* $f, g : Y \to Y$ *two nonexpansive operators. We suppose that:*

(i) $f \circ g = g \circ f$;

(ii) the pair (f, g) *is an* α_K*-condensing pair.*

Then, $F_f \cap F_g \neq \emptyset$ *and* $\alpha_K(F_f \cap F_g) = 0$.

Proof. We consider in Theorem 7.4.1 the fixed point structure $S(X) := P_{cp,cv}(X)$, $M(Y) := \{f : Y \to Y |\ f \text{ is nonexpansive}\}$, $Z = P_b(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{co}A$.

Remark 7.4.1. We can put instead of α_K, in Theorem 7.4.3, an α_{DP} measure of noncompactness.

7.5 References

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Chapter 8

Fixed point property and coincidence property

8.1 Set-theoretic aspects of coincidence theory

The following remarks are useful in order to apply the technique of the fixed point theory to the coincidence theory.

Remarks 8.1.1. Let U and V be two nonempty sets and $f, g : U \to V$ two operators. If the operator g is injective and $f(U) \subset g(U)$ then for a left-inverse $g_l^{-1} : g(U) \to V$ of the operator g, we consider the operator

$$g_l^{-1} \circ f : U \to U.$$

Let $u_0 \in C(f, g) := \{u \in U |\ f(u) = g(u)\}$. Then we have that $g_l^{-1}(f(u_0)) = u_0$.

Let $u_0 \in F_{g_l^{-1} \circ f}$. Then $g_l^{-1}(f(u_0)) = u_0$. But $g : U \to g(u)$ is a bijection. Hence $f(u_0) = g(u_0)$.

So, in the above conditions we have that

$$C(f, g) = F_{g_l^{-1} \circ f}.$$

Remark 8.1.2. Let $f, g : U \to V$ be two operators. We suppose that g is surjective. Let g_r^{-1} be a right-inverse of g. Then

$$g_r^{-1}(F_{f \circ g_r^{-1}}) \subset C(f, g).$$

8.2 Applications of the first general fixed point principle

We have

Theorem 8.2.1. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $U \in \eta(Z)$ and $f, g : U \to U$ be two operators. We suppose that:*

(i) $A \in Z$ implies $P(A) \subset Z$;

(ii) $\theta|_{\eta(Z)}$ has the intersection property;

(iii) g has a left-inverse, g_l^{-1}, $f(U) \subset g(U)$ and $g_l^{-1} \circ f \in M(U)$;

(iv) there exist $\alpha, \beta \in \mathbb{R}_+^*$, $\alpha \cdot \beta < 1$, such that

(a) $\alpha\theta(g(A)) \geq \theta(A)$, for all $A \in P(U)$ with $f(A) \subset g(A)$;

(b) $\theta(f(A)) \leq \beta\theta(A)$, for all $A \in P(U)$ with $f(A) \subset g(A)$.

Then $C(f, g) \neq \emptyset$ and $\theta(C(f, g)) = 0$.

Proof. Let $A \in P(U)$ such that $f(A) \subset g(A)$. From (iii) there exists $B \subset A$ such that $f(A) = g(B)$, and

$$\theta(g_l^{-1}(f(A))) = \theta(B) \leq \alpha\theta(g(B)) = \alpha\theta(f(A)) \leq \alpha\beta\theta(A).$$

Now the proof follows from Remark 8.1.1 and the first general fixed point principle.

Theorem 8.2.2. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) a compatible pair with $(X, S(X), M)$. Let U be a set, $V \in \eta(Z)$, $f, g : U \to V$ and $\alpha \in [0, 1[$. We suppose that*

(i) $A \in Z$ implies $P(A) \subset Z$;

(ii) $\theta|_{\eta(Z)}$ *has the intersection property;*

(iii) g *has a right inverse,* g_r^{-1}*, such that* $f \circ g_r^{-1} \in M(V)$*;*

(iv) $\theta(f(A)) \leq \alpha(\theta(g(A)))$*, for all* $A \in P(U)$*, such that* $f(A) \subset g(A)$*.*

Then, $C(f,g) \neq \emptyset$*.*

Proof. Let $B \in P(V)$. From (iv) we have that

$$\theta(f(g_r^{-1}(B))) \leq \alpha\theta(B), \text{ for all } B \in P(V),$$

such that $f(g_r^{-1}(B)) \subset B$. Now the proof follows from Remark 8.1.2 and the first general fixed point principle.

8.3 Applications of the second general fixed point principle

Theorem 8.3.1. *Let* $(X, S(X), M)$ *be a f.p.s. and* (θ, η) *a compatible pair with* $(X, S(X), M)$*. Let* $U \in \eta(Z)$ *and* $f, g : U \to U$ *be two operators. We suppose that:*

(i) $A \in Z$ *implies* $P(A) \in Z$*;*

(ii) $\theta(A \cup \{x\}) = \theta(A)$*, for all* $A \in P(U)$*,* $x \in U$*;*

(iii) $f(U) \subset g(U)$ *and* g *has a left-inverse,* g_l^{-1}*, such that* $g_l^{-1} \circ f \in M(U)$*;*

(iv) $\theta(g(A)) \geq \theta(A)$*, for all* $A \in P(U)$ *such that* $f(A) \subset g(A)$*;*

(v) $\theta(f(A)) < \theta(A)$*, for all* $A \in P(U)$ *such that* $f(A) \subset g(A)$*,* $\theta(A) \neq 0$*.*

Then, $C(f,g) \neq \emptyset$ *and* $\theta(C(fd,g)) = 0$*.*

Proof. Let $A \in P(U)$ such that $f(A) \subset g(A)$ and $\theta(A) \neq 0$. From the condition (iii) there exists $B \subset A$ such that $f(A) = f(B)$. We have $f(B) \subset g(B)$ and

$$\theta(g_l^{-1}(f(A))) = \theta(B) \leq \theta(g(B)) = \theta(f(A)) < \theta(A).$$

Now the proof follows from the Remark 8.1.1 and the second general fixed point principle.

Theorem 8.3.2. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) a compatible pair with $(X, S(X), M)$. Let U be a set, $V \in \eta(Z)$ and $f, g : U \to V$ be two operators. We suppose that*

(i) $A \in Z$ implies $P(A) \subset Z$;

(ii) g has a right inverse, g_r^{-1}, such that $f \circ g_r^{-1} \in M(V)$;

(iii) $\theta(A \cup \{x\}) = \theta(A),\ \forall\ A \in P(V),\ x \in V$;

(iv) $A \in P(U),\ f(A) \subset g(A)$ and $\theta(g(A)) \neq 0$, imply that $\theta(f(A)) < \theta(g(A))$.

Then, $C(f, g) \neq \emptyset$.

Proof. Let $B \in P(V)$. From (iv) we have that $\theta(f(g_r^{-1}(B))) < \theta(B)$, for all B such that $\theta(B) \neq 0$ and $f(g_r^{-1}(B)) \subset B$. Now the proof follows from Remark 8.1.2 and the second general fixed point principle.

8.4 Fixed point structures with the coincidence property

Definition 8.4.1. A f.p.s. $(X, S(X), M)$ is with the coincidence property iff $Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \ \Rightarrow\ C(f, g) \neq \emptyset$.

Example 8.4.1. Each f.p.s. with the common fixed point property is a f.p.s. with the coincidence property.

Example 8.4.2. (W.A. Horn (1970)). Let $X = \mathbb{R}$, $S(X) := P_{cp,cv}(\mathbb{R})$ and $M(Y) := C(Y, Y)$. Then $(\mathbb{R}, P_{cp,cv}(\mathbb{R}), M)$ is a f.p.s. with the coincidence property.

Proof. Let $I \subset \mathbb{R}$ be a compact interval and $f, g \in C(I, I)$ such that $f \circ g = g \circ f$. If f is surjective then we consider the sets $A := \{x \in I |\ f(x) \leq g(x)\}$ and $B := \{x \in I |\ f(x) \geq g(x)\}$. Suppose $C(f, g) = \emptyset$. It is clear that, A and B are nonempty closed-open subsets of I and, $I = A \cup B$. So, in this case, $C(f,g) \neq \emptyset$. In the general case we consider the subset $I_\infty := \bigcap\limits_{n\in\mathbb{N}} f^n(I)$. It is clear that $f(I_\infty) = I_\infty$ and if there exists $J \subset I$ such that $f(J) = J$, then $J \subset I_\infty$. On the other hand we have

$$g(I_\infty) = g(f(I_\infty)) = f(g(I_\infty)).$$

This implies that $g(I_\infty) \subset I_\infty$. Now we consider the functions

$$f|_{I_\infty}, g|_{I_\infty} : I_\infty \to I_\infty.$$

In this case I_∞ is a compact interval of $\mathbb{R}$ and $f|_{I_\infty}$ is a surjective function.

Problem 8.4.1. Which are the f.p.s. with the coincidence property?

The above problem has the following well known particular cases.

Horn's conjecture. The Schauder f.p.s., $(X, P_{cp,cv}(X), M)$, is with the coincidence property.

Schauder's conjecture. Let X be a Banach space and $Y \in P_{cl,cv}(X)$. If $f : Y \to Y$ is a continuous operator such that f^n is compact for some $n \in \mathbb{N}^*$, then f has at least a fixed point.

Remark 8.4.1. If the Horn conjecture is a theorem, then the Schauder conjecture is a theorem.

Indeed, let f as in Schauder's conjecture. Then the pair f^n, f^{n+1} is as in the Horn conjecture $(f^n|_{\overline{co}f^n(Y)}, f^{n+1}|_{\overline{co}f^n(Y)})$.

We have

Theorem 8.4.1. *Let $(X, S(X), M)$ be a f.p.s. with the coincidence property, (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$ such that $f \circ g = g \circ f$.*

We suppose that the pair (f,g) is a θ-condensing pair. Then, $C(f,g) \neq \emptyset$.

Proof. Let $A_1 = F_f$. From Lemma 7.1.3, there exists $A_0 \subset Y$ such that

$$\eta(f(A_0) \cup g(A_0) \cup F_f) = A_0.$$

This relation implies that, $\theta(A_0) = 0$.

But $A_0 \in F_\eta \cap Z_\theta \subset S(X)$. So, $C(f,g) \neq \emptyset$.

Theorem 8.4.2. *The following statements are equivalent:*

(i) (Horn's conjecture). Let Y be a compact convex subset of a Banach space X and let $f,g : Y \to Y$ be commuting continuous operators. Then $C(f,g) \neq \emptyset$.

(ii) Let Y be a bounded closed convex subset of a Banach space X and let $f,g : Y \to Y$ be commuting continuous operators. If the pair (f,g) is α_K-condensing, then $C(f,g) \neq \emptyset$.

Proof. It is clear that (ii) $\Rightarrow$ (i). Let us prove that (i) $\Rightarrow$ (ii). Consider the fixed point structure of Schauder. From (i) it follows that this f.p.s. is with the coincidence property. We are in the conditions of the Theorem 8.4.1 where we take $\theta = \alpha_K$ and $\eta(A) = \overline{co}A$.

8.5 Coincidence structures

Let X be a nonempty set.

Definition 8.5.1. A triple $(X, S(X), M)$ is a coincidence structure iff

(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;

(ii) $M : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}(Y),\ Y \mapsto\!\circ\, M(Y) \subset \mathbb{M}(Y)$, is an operator such that if $Y_1 \subset Y$, $Y_1 \neq \emptyset$, then $M(Y_1) \supset \{f|_{Y_1} : f \in M(Y),\ f(Y_1) \subset Y_1\}$;

(iii) $Y \in S(X),\ f,g \in M(Y),\ f \circ g = g \circ f$ imply $C(f,g) \neq \emptyset$.

Definition 8.5.2. Let $(X, S(X), M)$ be a coincidence structure. A pair (θ, η) is compatible with $(X, S(X), M)$ iff

(i) $\theta : Z \to \mathbb{R}_+,\ S(X) \subset Z \subset P(X)$;

(ii) $\eta : P(X) \to P(X)$ is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;

(iii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 8.5.1. A fixed point structure with the common fixed point property is a coincidence structure.

Example 8.5.2. A fixed point structure with the coincidence property is a coincidence structure.

We have

Theorem 8.5.1. *Let $(X, S(X), M)$ be a coincidence structure and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$ such that $f \circ g = g \circ f$. We suppose that*

(i) $\theta(f(A) \cup g(A)) < \theta(A)$, for all $A \in I(f, g)$, $\theta(A) \neq 0$;

(ii) $F_f \neq \emptyset$.

Then, $C(f, g) \neq \emptyset$.

Proof. Let $A_1 = F_f$. From Lemma 7.1.3 there exists $A_0 \subset Y$ such that

$$\eta(f(A_0) \cup g(A_0) \cup F_f) = A_0.$$

Since (θ, η) is a compatible pair with $(X, S(X), M)$, it follows

$$\theta(\eta(f(A_0) \cup g(A_0) \cup F_f)) = \theta(A_0).$$

This implies $\theta(A_0) = 0$. So, $A_0 \in F_\eta \cap Z_\theta$, which imply that $C(f, g) \neq \emptyset$.

8.6 References

For the main results of this Chapter see:

- I.A. Rus, *On a conjecture of Horn in coincidence theory*, Studia Univ. Babeş-Bolyai, Math., 36(1991), No.2, 71-75.

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For the basic results in the coincidence point theory see: R.F. Brown [20], A. Buică [22], J. Mawhin [49], I.A. Rus [57], [58], T. Van der Walt [73]. See also:

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For the coincidence producing operators see Chapter 17.

Chapter 9

Fixed point theory for retractible operators

9.1 Set-theoretic aspects for non self-operators

Let X be a nonempty set and $Y \subset X$ a nonempty subset of X. An operator $\rho : X \to Y$ is a set-retraction if $\rho|_Y = 1_Y$. An operator $f : Y \to X$ is retractible w.r.t. the retraction $\rho : X \to Y$ iff $F_{\rho \circ f} = F_f$.

Lemma 9.1.1. *Let $(X, S(X), M)$ be a fixed point structure. Let $Y \in S(X)$, $\rho : X \to Y$ a retraction and $f : Y \to X$ an operator. We suppose that:*

(i) $\rho \circ f \in M(Y)$; *(ii) f is retractible w.r.t. ρ.*

Then, $F_f \neq \emptyset$.

Proof. From (i) we have that $F_{\rho \circ f} \neq \emptyset$. From (ii) it follows that $F_f \neq \emptyset$.

Remark 9.1.1. In the terminology of R.F. Brown, Lemma 9.1.1 is the general retraction operator principle.

Theorem 9.1.1. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$, $f : Y \to X$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ *is with the intersection property;*

(ii) f *is retractible w.r.t.* ρ *and* $\rho \circ f \in M(Y)$*;*

(iii) ρ *is* (θ, l)*-Lipschitz* $(l \in \mathbb{R}_+)$*;*

(iv) f *is a strong* (θ, φ)*-contraction;*

(v) the function $l\varphi$ *is a comparison function.*

Then $F_f \neq \emptyset$ *and if* $F_f \in Z$*, then* $\theta(F_f) = 0$.

Proof. From the conditions (iii), (iv) and (v), the operator $\rho \circ f : Y \to Y$ is a strong $(\theta, l\varphi)$-contraction. By Theorem 5.1.1, $F_{\rho\circ f} \neq \emptyset$. From the condition (ii) it follows that $F_f \neq \emptyset$. From $f(F_f) = F_f$ we have that $\theta(F_f) = 0$.

Theorem 9.1.2. *Let* $(X, S(X), M)$ *be a f.p.s. and* (θ, η) $(\theta : Z \to \mathbb{R}_+)$ *a compatible pair with* $(X, S(X), M)$*. Let* $Y \in \eta(Z)$*,* $f : Y \to X$ *an operator and* $\rho : X \to Y$ *a retraction. We suppose that:*

(i) $A \in Z$*,* $x \in Y$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$*;*

(ii) f *is retractible w.r.t.* ρ *and* $\rho \circ f \in M(Y)$*;*

(iii) ρ *is* $(\theta, 1)$*-Lipschitz;*

(iv) f *is strong* θ*-condensing.*

Then, $F_f \neq \emptyset$ *and if* $F_f \in Z$*, then* $\theta(F_f) = 0$.

Proof. From the conditions (iii) and (iv) the operator $\rho \circ f : Y \to Y$ is strong θ-condensing. By the Theorem 6.1.1, $F_{\rho\circ f} \neq \emptyset$. From the condition (ii) it follows that $F_f \neq \emptyset$. From $F_f \in Z$, $f(F_f) = F_f$ and the condition (iv) we have that $\theta(F_f) = 0$.

9.2 Retractible operators on ordered sets

In this section we shall give some fixed point theorem for retractible operators on ordered sets.

Theorem 9.2.1. *Let* $(X, \leq)$ *be an ordered set with the least element* 0*. Let* $Y \in P(X)$ *and* $f : Y \to X$ *be such that* *(i)* $0 \in Y$*;*

(ii) $(Y, \leq)$ *is a complete lattice;*

(iii) f *is increasing;*

(iv) $f(x) \in X \setminus Y$ *implies* $\sup_Y([0, f(x)] \cap Y) \neq x$*.*

Then, $F_f \neq \emptyset$*.*

Proof. Let $(X, S(X), M)$ be the fixed point structure of Tarski, i.e., $S(X) := \{A \in P(X)|\ (A, \leq)$ is a complete lattice and $M(A) := \{f : A \to A|\ f$ is increasing$\}$. We remark that $Y \in S(X)$ and the $\rho : X \to Y$, defined by

$$\rho(x) := \begin{cases} x, & \text{if} \quad x \in Y \\ \sup_Y([0, x] \cap Y), & \text{if} \quad x \in X \setminus Y \end{cases}$$

is an ordered-set retraction. From (iv) it follows that f is a retractible operator with respect to ρ. From (iii) we have that $\rho \circ f \in M(Y)$. Now the proof follows from Lemma 9.1.1.

Theorem 9.2.2. *Let* $(X, \leq)$ *be an ordered set with the least element* 0*. Let* $Y \in P(X)$ *and* $f : Y \to X$ *an operator. We suppose that:*

(i) $0 \in Y$*;*

(ii) $(Y, \leq)$ *is a right inductive ordered set;*

(iii) f *is a progressive operator;*

(iv) $f(x) \in X \setminus Y$ *implies* $x < \sup_Y([0, f(x)] \cap Y)$*.*

Then, $F_f \neq \emptyset$*.*

Proof. Consider the fixed point structure $(X, S(X), M)$, where $S(X) := \{A \in P(X)|\ (A, \leq)$ is a right inductive ordered set$\}$ and $M(A) := \{f : A \to A|\ f$ is progressive$\}$. Let $\rho : X \to Y$ be the retraction from the proof of Theorem 9.2.1. It is clear that $f(x) \in Y$ implies $\rho(f(x)) = f(x) \geq x$. If $f(x) \in X \setminus Y$, then by the condition (iv) we have that $x < \rho(f(x))$. Hence $\rho \circ f \in M(Y)$ and f is retractible w.r.t. ρ. Now the proof follows from Lemma 9.1.1.

9.3 Retractible operators on metric spaces

In this section we consider fixed point structures on metric spaces.

Theorem 9.3.1. *Let (X, d) be a complete metric space and α a measure of noncompactness on X. Let $(X, S(X), M)$ be a fixed point structure and (α, η) a compatible pair with $(X, S(X), M)$. Let $Y \in P_{b,cl}(X)$, $f : Y \to X$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) f is a strong (α, φ)-contraction;

(ii) f is retractible w.r.t. the retraction ρ and $\rho \circ f \in M(Y)$;

(iii) ρ is (α, l)-Lipschitz;

(iv) the function $l\varphi$ is a comparison function.

Then, $F_f \neq \emptyset$ and $\alpha(F_f) = 0$.

Proof. We take in the Theorem 9.1.1, $Z := P_b(X)$, $\theta = \alpha$ and $\eta(A) = \overline{A}$.

Theorem 9.3.2. *Let (X, d) be a complete metric space and α_{DP} a Daneš-Pasicki measure of noncompactness on X. Let $(X, S(X), M)$ be a fixed point structure and (α_{DP}, η) a compatible pair with $(X, S(X), M)$. Let $Y \in P_{b,cl}(X)$, $f : Y \to X$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) f is strong α_{DP} condensing;

(ii) f is retractible w.r.t. the retraction ρ and $\rho \circ f \in M(Y)$;

(iii) ρ is $(\alpha_{DP}, 1)$-Lipschitz.

Then, $F_f \neq \emptyset$ and $\alpha_{DP}(F_f) = 0$.

Proof. We take in the Theorem 9.1.2, $Z := P_b(X)$, $\theta = \alpha_{DP}$ and $\eta(A) = \overline{A}$.

Theorem 9.3.3. *Let (X, d) be a complete metric space, α_K the Kuratowski measure of noncompactness on X, $Y \in P_{b,cl}(X)$, $\rho : X \to Y$ a retraction and $f : Y \to X$ an operator. We suppose that:*

(i) f is a strong (α_K, φ)-contraction;

(ii) f is retractible w.r.t. the retraction ρ;

(iii) ρ is (α, l)-Lipschitz;

(iv) f is a contractive operator;

(v) the function $l\varphi$ is a comparison function.

Then, $F_f = \{x^*\}$.

Proof. We take in the Theorem 9.3.1, the fixed point structure of Nemytzki-Edelstein, $\alpha = \alpha_K$ and $\eta(A) = \overline{A}$.

9.4 Retractible operators on Banach spaces

In this section we consider f.p.s. on Banach spaces.

Theorem 9.4.1. *Let X be a Banach space, $\alpha_K : P_b(X) \to \mathbb{R}_+$ Kuratowski measure of noncompactnes on X and $f : \overline{B}(0; R) \to X$ a continuous operator. We suppose that:*

(i) f is a strong (α_K, φ)-contraction;

(ii) f is retractible w.r.t. the radial retraction.

Then, $F_f \neq \emptyset$ and $\alpha_K(F_f) = 0$.

Proof. We consider in the Theorem 9.1.1, the fixed point structure of Schauder, $Z = P_b(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{co}A$ and ρ the radial retraction. We remark that the radial retraction is $(\alpha_K, 1)$-Lipschitz.

Theorem 9.4.2. *Let X be a Banach space and $f : \overline{B}(0; R) \to X$ a continuous operator. We suppose that:*

(i) f is α_K-condensing;

(ii) f is retractible w.r.t. the radial retraction.

Then, $F_f \neq \emptyset$ and $\alpha_K(F_f) = 0$.

Proof. We consider in the Theorem 9.1.2 the fixed point structure of Schauder, $Z = P_b(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{co}\overline{A}$.

Remark 9.4.1. Each of the following conditions implies the condition (ii) in the Theorem 9.4.1 and 9.4.2:

(a) (Leray-Schauder). $x \in \partial\overline{B}(0; R)$, $f(x) = \lambda x$ imply $\lambda \leq 1$.

(b) (E. Rothe). $f(\partial\overline{B}(0;R)) \subset \overline{B}(0;R)$.

(c) (M. Altman). $\|f(x)-x\|^2 \geq \|f(x)\|^2 - \|x\|^2,\ \forall\, x \in \partial\overline{B}(0;R)$.

Remark 9.4.2. There are many other boundary conditions which appear in fixed point theorems for non self-operator. The problem is if each of these conditions imply the retractibility w.r.t. a suitable retraction.

Theorem 9.4.3. *Let X be a Banach space and $f : X \to X$ a continuous operator. We suppose that:*

(i) f is α_K-condensator;

(ii) f is quasibounded with $|f| < 1$.

Then, $F_f \neq \emptyset$.

Proof. We consider the operator $f|_{\overline{B}(0;R)}\overline{B}(0;R) \to X$. Condition (ii) implies that there exist $a, b \in \mathbb{R}_+$, $a < 1$ such that

$$\|f(x)\| \leq a\|x\| + b,\ \forall\, x \in X.$$

This condition implies that there exists $R > 0$ such that the operator $f|_{\overline{B}(0;R)}$ is retractible w.r.t. the radial retraction $\rho : X \to \overline{B}(0;R)$. So, we are in the conditions of Theorem 9.4.2.

Theorem 9.4.4. *Let X be a Banach space and $f : X \to X$ a continuous operator. We suppose that:*

(i) f is α_K-condensing;

(ii) f is quasibounded with $|f| < 1$.

Then $1_X - f : X \to X$ is a surjective operator.

Proof. Let $y \in X$. We consider the operator $g_y : X \to X$ defined by $g_y(x) := f(x) + y$. It is clear that g_y is quasibounded with $|g_y| < 1$ and there exist $a \in [|f|, 1[$ and $b > 0$ such that

$$\|g_y(x)\| \leq a\|x\| + b,\ \forall\, x \in X.$$

This implies that, for $R \geq \dfrac{b}{1-a}$, $\overline{B}(0;R) \in I(g_y)$. So, the proof follows from Theorem 9.4.2.

9.5 References

For the main results of this Chapter see:

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For the retraction principle see

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Part II

Fixed point structures for multivalued operators

Chapter 10

Background in multivalued operator theory

10.1 Sets and multivalued operators

Let X and Y be nonempty sets. By definition an operator $T : X \to \mathcal{P}(Y)$ is a multivalued (or set-valued) operator from X to Y and we shall use the notation $T : X \multimap Y$ for such an operator. Let $T : X \multimap Y$ be a multivalued operator, $A \subset X$ and $B \subset Y$. Then:
$G(T) := \{(x, y) |\ x \in X,\ y \in T(x)\}$ denotes the graph of the operator T;
$T(A) := \bigcup\limits_{a \in A} T(a)$ denotes the image of A under the operator T;
$T^{-1}(B) := \{a \in A |\ T(a) \cap B \neq \emptyset\}$ the counter image of B under the operator T.

From the above definitions we have

Lemma 10.1.1. *Let X and Y be two nonempty sets, $(A_i)_{i \in I}$ a family of subsets of X and $(B_i)_{i \in I}$ a family of subsets of Y. If $T : X \multimap Y$ is a multivalued operator, then:*

(i) $T\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} T(A_i)$;

(ii) $T\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} T(A_i)$;

(iii) $T^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} T^{-1}(B_i)$;

(iv) $T^{-1}\left(\bigcap_{i\in I} B_i\right) \subset \bigcap_{i\in I} T^{-1}(B_i)$.

If $T : X \multimap X$ is a multivalued operator, then:

$T^1 := T,\ T^2 := T \circ T, \ldots, T^{n+1} := T \circ T^n,\ n \in \mathbb{N}^*$;

$F_T := \{x \in X|\ x \in T(x)\}$ denotes the fixed point set of T;

$(SF)_T := \{x \in X|\ T(x) = \{x\}\}$ denotes the strict fixed point set of T.

Let $T : X \to P(Y)$ and $s : X \to Y$. The singlevalued operator s is a selection of the multivalued operator T iff $s(x) \in T(x)$, for all $x \in X$. It is clear that if s is a selection of T, then $F_s \subset F_T$.

Remark 10.1.1. For the invariant subsets of multivalued operators see section 1.6.

10.2 Functionals on $P(X) \times P(X)$

Let (X, d) be a metric space. We consider the following functionals on $P(X) \times P(X)$:

1) $D : P(X) \times P(X) \to \mathbb{R}_+$ defined by

$$A, B \in P(X),\ D(A, B) := \inf\{d(a, b)|\ a \in A,\ b \in B\}.$$

This functional is called the gap functional.

2) $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$A, B \in P(X),\ \delta(A, B) := \sup\{d(a, b)|\ a \in A,\ b \in B\}.$$

This functional is called the diameter functional with two arguments.

3) $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$A, B \in P(X),\ \ \rho(A,B) := \sup\{D(a,B) |\ a \in A\}.$$

This functional is called the excess functional.

4) $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$A, B \in P(X),\ \ H(A,B) := \max(\rho(A,B), \rho(B,A)).$$

This functional is called the Pompeiu-Hausdorff functional.

From the definitions of these functionals we have:

Lemma 10.2.1. *Let (X,d) be a metric space and $D : P(X) \times P(X) \to \mathbb{R}_+$ the gap functional on (X,d). Then:*

(a) the functional $D(\cdot, A) : X \to \mathbb{R}_+$ is nonexpansive for each $A \in P(X)$;

(b) the functional $D(x, \cdot) : (P_{b,cl}(X), H) \to \mathbb{R}_+$ is nonexpansive for each $x \in X$;

(c) if $x \in X$ and $A \in P(X)$, then

$$D(x,A) = 0 \text{ if and only if } x \in \overline{A}.$$

Lemma 10.2.2. *Let (X,d) be a metric space, $A, B, C \in P_b(X)$ and $\delta : P(X) \times P(X) \to \mathbb{R}_+$ the diameter functional on (X,d). Then:*

(a) $\delta(A,B) = 0 \ \Leftrightarrow\ A = B = \{a\}$;

(b) $\delta(A,B) = \delta(B,A)$;

(c) $\delta(A,B) \leq \delta(B,C) + \delta(C,B)$;

(d) for every $x \in X$ and $0 < q < 1$, there exists $a \in A$ such that $q\delta(x,A) \leq d(x,a)$.

Lemma 10.2.3. *Let (X,d) be a metric space and $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ the Pompeiu-Hausdorff functional on (X,d). Then:*

(a) $(P_{b,cl}(X), H)$ is a metric space;

(b) $H(A,B) = \inf\{r > 0 |\ V_r(A) \supset B, V_r(B) \supset A\}$;

(c) $H(A,B) = \sup\{|D(x,A) - D(x,B)| \mid x \in X\}$*;*

(d) for $\varepsilon > 0$ *and* $a \in A$ *there exists* $b(\varepsilon, a) \in B$ *such that* $d(a,b) \leq H(A,B) + \varepsilon$*;*

(e) for $q > 1$ *and* $a \in A$ *there exists* $b(q,a) \in B$ *such that* $d(a,b) \leq qH(A,B)$*;*

(f) if $\eta > 0$ *is such that:*

(1) for each $a \in A$ *there exists* $b \in B$ *such that* $d(a,b) \leq \eta$*,*

(2) for each $b \in B$ *there exists* $a \in A$ *such that* $d(a,b) \leq \eta$*,*

then $H(A,B) \leq \eta$*;*

(g) if (X,d) *is a complete metric space, then* $(P_{b,cl}(X), H)$ *is a complete metric space;*

(h) if (X,d) *is a complete metric space, then* $(P_{cp}(X), H)$ *is a complete metric space;*

(i) if (X,d) *is a compact metric space, then* $(P_{cp}(X), H)$ *is a compact metric space;*

(j) if (X,d) *is a complete metric space, then* $(P_{cl}(X), H)$ *is a complete generalized metric space.*

10.3 Continuity

Let X and Y be two Hausdorff topological spaces and $T : X \to P(Y)$ be a multivalued operator.

Definition 10.3.1. By definition T is upper semicontinuous (u.s.c.) iff for each closed set $A \subset Y$, $T^{-1}(A)$ is a closed subset of X.

Definition 10.3.2. By definition T is lower semicontinuous (l.s.c.) iff for each open set $A \subset Y$, $T^{-1}(A)$ is an open subset of X.

Definition 10.3.3. By definition T is continuous iff it is both l.s.c. and u.s.c.

Definition 10.3.4. By definition T is closed iff the graph of T, $G(T) \subset X \times Y$ is closed.

From the above definitions we have:

Theorem 10.3.1. *If T is u.s.c. with closed values then T is closed.*

Theorem 10.3.2. *If T is closed and Y is compact then T is u.s.c.*

Theorem 10.3.3. *(i) If T is u.s.c. with compact values and X is compact, then $T(X)$ is compact.*

(ii) If T is u.s.c. or l.s.c., with connected values, and $C \in P_{cn}(X)$, then $T(C)$ is connected.

Let (X, d) and (Y, ρ) be two metric spaces. An operator $T : X \to P_{b,cl}(Y)$ is called Lipschitz (respective, contraction, contractive, nonexpansive, expansive, dilatation,...) if the singlevalued operator $T : (X, d) \to (P_{b,cl}(X), H)$ is Lipschitz (respective, contraction, contractive, nonexpansive, expansive, dilatation,...).

Theorem 10.3.4. *If an operator $T : X \to P_{b,cl}(X)$ is Lipschitz, then T is closed.*

Let X be a Hausdorff topological space and (Y, d) a metric space.

Definition 10.3.5. An operator $T : X \to P(Y)$ is H-u.s.c. iff the functional $\rho(F(\cdot), F(x_0)) : X \to \mathbb{R}_+$, $x \mapsto \rho(F(x), F(x_0))$ is continuous at x_0, for all $x_0 \in X$.

Definition 10.3.6. An operator $T : X \to P(Y)$ is H-l.s.c. iff the functional $\rho(F(x_0), F(\cdot)) : X \to \mathbb{R}_+$, $x \mapsto \rho(F(x_0), F(x))$ is continuous at x_0, for all $x_0 \in X$.

Definition 10.3.7. An operator $T : X \to P(Y)$ is H-continuous iff T is H-u.s.c. and H-l.s.c.

We have:

Theorem 10.3.5. *Let (X, τ) be a Hausdorff topological space, (Y, d) a metric space and $T : X \to P(Y)$. Then:*

(a) If T is u.s.c., then T is H-u.s.c.

(b) If T is H-l.s.c., then T is l.s.c.

(c) If $T(x) \in P_{cp}(Y)$, $\forall\ x \in X$, then the converse of (a) and (b) hold.

(d) $T : X \to P_{cp}(Y)$ is continuous if and only if T is H-continuous.

For the algebraic operations on multivalued operators we have

Theorem 10.3.6. *Let X_1, X_2 and X_3 be Hausdorff topological spaces and $T : X_1 \to P(X_2)$, $S : X_2 \to P(X_3)$.*

Then:

(a) T and S u.s.c. imply $S \circ T$ u.s.c.

(b) T and S l.s.c. imply $S \circ T$ l.s.c.

Theorem 10.3.7. *Let X be a Hausdorff topological space, Y a Hausdorff linear topological space and $T, S : X \to P(Y)$. Then:*

(a) $T(x) \in P_{cp}(Y)$, $S(x) \in P_{cp}(Y)$, $\forall\ x \in X$ and T, S u.s.c. imply $T + S$ u.s.c.

(b) T and S l.s.c. imply $T + S$ l.s.c.

Theorem 10.3.8. *Let X and Y be Hausdorff topological spaces and $T, S : X \to P(Y)$. Then:*

(a) T and S u.s.c. imply $T \cup S$ u.s.c.

(b) T and S l.s.c. imply $T \cup S$ l.s.c.

(c) Y normal, $T(x) \cap S(x) \neq \emptyset$, $\forall\ x \in X$ and T and S u.s.c. imply $T \cap S$ u.s.c.

(d) T and S closed imply $T \cap S$ closed.

10.4 References

For the basic notions and results in the multivalued operator theory see: J.P. Aubin and H. Frankowska [5], T. Bânzaru and B. Rendi [6], G. Beer [7], C. Berge [8], Yu. G. Borisovitch, B.D. Gelman, A.D. Myshkis and V.V.

Obukhovskii [10]-[12], L. Górniewicz [20], S. Hu and N.S. Papageorgiou [22], M. Mureşan [27], A. Petruşel [29]-[30], A. Petruşel and Gh. Moţ [31], R.E. Smithson [39] and R. Wegrzyk [40].

For the continuity of multivalued operators see also M.C. Anisiu [2], K. Deimling [19], M. Kamenskii, V. Obukhovskii and P. Zecca [24], and

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Chapter 11

Fixed point structures for multivalued operators

11.1 Definitions

Let X be a nonempty set.

Definition 11.1.1. A triple $(X, S(X), M^0)$ is a fixed point structure on X (f.p.s.) iff:

(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}^0(Y),\ Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator such that if $Z \subset Y$, $Z \neq \emptyset$, then $M^0(Z) \supset \{T|_Z : T \in M^0(Y) \text{ and } Z \in I(T)\}$;

(iii) every $Y \in S(X)$ has the fixed point property with respect to $M^0(Y)$, i.e., $Y \in S(X)$, $T \in M^0(Y)$ imply $F_T \neq \emptyset$.

Definition 11.1.2. A triple $(X, S(X), M^0)$ which satisfies (i) and (iii) in Definition 11.1.1 and the condition

(ii') $M : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}^0(Y)$, $Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator;

is called large fixed point structure (l.f.p.s.).

Remark 11.1.1. In 1973 T.B. Muenzenberger and R.E. Smithson have introduced another type of f.p.s. By Muenzenberger and Smithson a triple $(X, S(X), M^0)$ is a fixed point structure iff:

(a) For all $x, y \in X$, there exists $A \in S(X)$ such that $x, y \in A$.

(b) If $Z \subset S(X)$, $Z \neq \emptyset$, then $\cap Z = \emptyset$ or $\cap Z \in S(X)$.

(c) For all $A \in S(X)$ there exist $x, y \in X$ such that $A = [x, y]$. Here $[x, y]$ is the minimal element in $(S(X), \subset)$ containing x, y.

(d) The union of two chainable sets with nonempty intersection is chainable. By definition a subset $A \subset X$ is chainable if and only if, for all $x, y \in A$, $[x, y] \subset A$.

(e) If $Z \subset S(X)$ is nested (i.e. totally ordered), then there exists $A \in S(X)$ such that $\cup Z \subset A$.

(f) If $x \neq y$, then $[x, y]$ contains at least three points.

(g) Fix a point $e \in X$ and define a relation $\leq$ by: $x \leq y$ iff $x \in [e, y]$. This axiom states that if $x \leq y$, then $T([x, y])$ is chainable for all $T \in M^0(X)$.

(h) A subset $A \subset X$ is closed iff for all $y, z \in X$ with $y \leq z$, $\inf(A \cap [y, z]) \in A$ and $\sup(A \cap [y, z]) \in A$ whenever $A \cap [y, z] \neq \emptyset$. This axiom states that for all $T \in M^0(X)$, $T(x)$ is closed for all $x \in X$.

(i) For all $T \in M^0(X)$ either $T^{-1}(x)$ is chainable or $T^{-1}(x)$ is closed for all $x \in X$.

(j) $T \in M^0(X)$ implies $F_T \neq \emptyset$.

In this f.p.s. the authors only consider the multivalued operators from X to X.

11.2 Examples

Example 11.2.1. Trivial f.p.s. X is a nonempty set,

$$S(X) := \{\{x\} |\ x \in X\} \quad \text{şi} \quad M^0(Y) := \mathbb{M}^0(Y).$$

Example 11.2.2. (I.A. Rus (2005)). **Fixed point structure of progressive operators.** Let $(X, \leq)$ be a partially ordered set and $A, B \in P(X)$. We denote $A \leq B$ if $a \in A$, $b \in B$ imply $a \leq b$. Let $Y \in P(X)$. By definition an operator $T : Y \to P(Y)$ is progressive if $y \leq T(y)$, $\forall\ y \in Y$. Let $S(X) := \{Y \in P(X) |\ (Y, \leq)$ is such that every chain in Y has an upper bound in $Y\}$, and $M^0(Y) := \{T : Y \to P(Y) |\ T$ is a progressive operator$\}$. Then the triple $(X, S(X), M^0)$ is a f.p.s.

Example 11.2.3. The fixed point structure of contractions (Avramescu-Markin-Nadler). (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ is contraction, i.e. $H(T(x), T(y)) \leq ld(x, y)$, $\forall\ x, y \in Y$, with $0 < l < 1\}$.

Example 11.2.4. The fixed point structure of S. Reich (1972). (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ is such that there exist $a, b, c \in \mathbb{R}_+$, $a + b + c < 1$ and $H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y))$, $\forall\ x, y \in X\}$.

Example 11.2.5. The fixed point structure of T. Wang (1989). (X, d) is a complete metric space, $S(X) := P_{cl}(X)$, $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ is such that there exists $a, b, c \in \mathbb{R}_+$, $a+b+c < 1$ and $\rho(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y))$, $\forall\ x, y \in Y\}$.

Example 11.2.6. The f.p.s. of graphic contraction (I.A. Rus (1975)). (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cp}(Y) |$ there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha+\beta < 1$, such that $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$, and T is closed$\}$.

Example 11.2.7. The f.p.s. of nonexpansive operators (T.C. Lim (1974)). X is a uniformly convex Banach space, $S(X) := P_{b,cl,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp}(Y) |\ T$ is nonexpansive$\}$.

Example 11.2.8. The f.p.s. of contractive operators (R.E. Smithson (1971)). (X, d) is a complete metric space, $S(X) := P_{cp}(X)$ and $M^0(Y) :=$

$\{T : Y \to P_{cl}(Y) |\ T$ is a contractive operator$\}$.

Example 11.2.9. The f.p.s. of M. Frigon (2002). Let $(X, (d_i)_{i\in I})$ be a sequentially complete Hausdorff gauge space and $Y \in P_{cl}(X)$. An operator $T : Y \to P_{cl}(Y)$ is called admissible l_i-contraction, $i \in I$, if $l_i \in]0, 1[$, $\forall\ i \in I$ and the following conditions are satisfied:

(a) $H_{d_i}(T(x), T(y)) \leq l_i d_i(x, y),\ \forall\ x, y \in Y$ and $\forall\ i \in I$;

(b) for every $x \in Y$ and every $q \in]1, +\infty[$ there exists $y \in T(x)$ such that $d_i(x, y) \leq q_i D_i(x, T(x))$, for each $i \in I$.

If we take $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ is an admissible contraction$\}$, then $(X, P_{cl}(X), M^0)$ is a f.p.s.

Example 11.2.10. The f.p.s. of S. Kakutani (1941). $X = \mathbb{R}^n$, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp,cv}(Y) | T$ is u.s.c.$\}$.

Example 11.2.11. The f.p.s. of Bohnenblust-Karlin (1950). X is a Banach space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp,cv}(Y) |\ T$ is u.s.c.$\}$.

Example 11.2.12. The f.p.s. of Fan-Glicksberg (1952). X is a Hausdorff locally convex topological space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp,cv}(Y) |\ T$ is u.s.c.$\}$.

Example 11.2.13. The f.p.s. of F.E. Browder (1968). X is a Hausdorff topological vector space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cv}(Y) |\ T^{-1}(y)$ is an open subset in Y, $\forall\ y \in Y\}$.

It is clear that for any fixed point theorem we have an example of a f.p.s. or of a l.f.p.s.

11.3 Compatible pair with a fixed point structure

The following notion is fundamental in the f.p.s. theory.

Definition 11.3.1. Let $(X, S(X), M^0)$ be a f.p.s., $S(X) \subset Z \subset P(X)$, $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. The pair (θ, η) is a compatible pair with $(X, S(X), M^0)$ iff:

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 11.3.1. Let $(X, S(X), M^0)$ be the f.p.s. in Example 11.2.7, $Z = P_b(X)$, $\theta = \alpha_K$ or α_H and $\eta(A) = \overline{A}$. Then the pairs (α_K, η) and (α_H, η) are compatible with $(X, S(X), M^0)$.

Example 11.3.2. Let $(X, S(X), M^0)$ be the f.p.s. in Example 11.2.10, $Z = P_b(X)$, $\theta = \alpha_K$ or α_H and $\eta(A) = \overline{co}A$. Then the pairs (α_K, η), (α_H, η) are compatible with $(X, S(X), M^0)$.

11.4 Maximal fixed point structures

Let $(X, S(X), M^0)$ be a f.p.s. and $S_1(X) \subset P(X)$ such that $S(X) \subset S_1(X)$.

Definition 11.4.1. The f.p.s. $(X, S(X), M^0)$ is maximal in $S_1(X)$ if we have $S(X) = \{A \in S_1(X) | \ T \in M^0(A) \ \Rightarrow \ F_T \neq \emptyset\}$.

Example 11.4.1. (Generic example). Let $(X, S(X), M)$ be a f.p.s. for singlevalued operators. Let $(X, S(X), M^0)$ be a f.p.s. for multivalued operators. Let $f \in M(Y)$. We consider the multivalued operator $\widetilde{f} : Y \multimap Y$ defined by $\widetilde{f}(x) = \{f(x)\}$, $\forall\, x \in Y$. We denote by $\widetilde{M}(Y)$ the set of multivalued operators generated, in the above way, by the singlevalued operators from $M(Y)$. We suppose that $\widetilde{M}(Y) \subset M^0(Y)$ for all $Y \in S_1(X)$. Then the maximality of the f.p.s. $(X, S(X), \widetilde{M})$, in $S_1(X)$, implies the maximality of $(X, S(X), M^0)$ in $S_1(X)$.

From this example and for the examples given in section 2.3, we have

Example 11.4.2. The trivial f.p.s., $(X, S(X), M^0)$ is maximal in $P(X)$.

Example 11.4.3. Let X be a Banach space and $(X, P_{cp,cv}(X), M^0)$ the Bohnenblust-Karlin f.p.s. on X. This f.p.s. is maximal in $P_{b,cl,cv}(X)$.

In spite of the above remarks, to establish if a given f.p.s. is maximal or not, this is an open problem.

11.5 References

For the notion of the f.p.s. for multivalued operators see I.A. Rus [32].

See also: T.B. Muenzenberger and R.E. Smithson, *Fixed point structures*, Trans. A.M.S., 184(1973), 153-173.

For the basic fixed point theorems for multivalued operators see J. Andres and L. Górniewicz [1], C. Berge [8], K.C. Border [9], Yu. G. Borisovitch, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii [10]-[12], F.E. Browder [13]-[14], B. Cong Cuong [16], K. Deimling [18], Górniewicz [20], S. Hu and N.S. Papageorgiou [22], W.A. Kirk and B. Sims [25], S. Park [28], A. Petruşel [29]-[31], I.A. Rus [34], [35] and [37], R. Wegrzyk [40].

For the fixed point theorems on ordered set, see also:

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For topological fixed point theorems see:

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Chapter 12

Strict fixed point structures

12.1 Definitions and examples

Let X be a nonempty set.

Definition 12.1.1. A triple $(X, S(X), M^0)$ is a strict fixed point structure on X (s.f.p.s.) iff:

(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \to \bigcup\limits_{Y \in P(X)} \mathbb{M}^0(Y),\ Y \mapsto\!\circ\ M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator such that if $Z \subset Y$, $Z \neq \emptyset$, then

$$M^0(Z) \supset \{T|_Z :\ T \in M^0(Y) \text{ and } Z \in I(T)\};$$

(iii) every $Y \in S(X)$ has the strict fixed point property w.r.t. $M^0(Y)$.

Definition 12.1.2. A triple $(X, S(X), M)$ which satisfies (i) and (iii) in Definition 12.1.1 and the condition

(ii') $M^0 : P(X) \multimap \bigcup\limits_{Y \in P(X)} \mathbb{M}^0(Y)$, $Y \mapsto\!\circ\ M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator;

is called large strict fixed point structure (l.s.f.p.s.).

Example 12.1.1. The trivial f.p.s. is a s.f.p.s.

Example 12.1.2. The s.f.p.s. of S. Reich (1972). (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_b(X) |$ there exists $a, b, c \in \mathbb{R}_+$, $a + b + c < 1$ such that $\delta(T(x), T(y)) \leq ad(x, y) + b\delta(x, T(x)) + c\delta(y, T(y)),\ \forall\ x, y \in Y\}$.

Example 12.1.3. The s.f.p.s. of reflexive operators (S. Dancs, M. Hegedüs and P. Medvegyev (1983)). Let (X, d) be a complete metric space and $Y \in P_{cl}(X)$. For an operator $T : Y \to P(Y)$ we consider the following conditions:

(i) $x \in T(x),\ \forall\ x \in X$;

(ii) $T(x) \in P_{cl}(X)$;

(iii) $x_2 \in T(x_1) \ \Rightarrow\ T(x_2) \subset T(x_1),\ \forall\ x_1, x_2 \in X$;

(iv) if $(x_n)_{n\in\mathbb{N}}$ is an orbit of T, i.e., $x_{n+1} \in T(x_n),\ \forall\ n \in \mathbb{N}$, then $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

If we take $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ satisfies conditions (i)-(iv)$\}$, then the triple $(X, S(X), M^0)$ is a s.f.p.s.

Example 12.1.4. The s.f.p.s. of H.W. Corley (1986). (X, d) is a metric space, $S(X) := P_{cp}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) |\ T$ is reflexive, antisymmetric and transitive$\}$.

Example 12.1.5. The s.f.p.s. of reductible operators (I.A. Rus (1990)). Let X be a nonempty set. A family $U \subset P(X)$, $U \neq \emptyset$, has the intersection property if for any totally ordered subset $V \subset U$ (U is partial ordered by the set inclusion) we have $\cap V \in U$. By definition a multivalued operator $T : X \to P(X)$ is said to be reductible on U if for any $A \in U$, $A \in I(T)$ and $cardA > 1$, there exists $B \in I(T) \cap U$, proper subset of A.

Let X be a set, $U \subset P(X)$, $U \neq \emptyset$, a family with the intersection property. If we take $S(X) := \{x\}$, $M^0(X) := \{T : X \to P(X) |\ I(Y) \cap U \neq \emptyset$ and T is reducible on $U\}$, then $(X, S(X), M^0)$ is a s.f.p.s.

Example 12.1.6. (X, d) is a complete metric space, $S(X) := P_{b,cl}(X)$ and $M^0(Y) := \{T : Y \to P(Y) |\ T$ is a (δ, φ)-contraction$\}$. By definition,

$T : Y \to P(Y)$ is a (δ, φ)-contraction iff $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and $\delta(T(A)) \le \varphi(\delta(A)),\ \forall\ A \in I(T)$.

12.2 Compatible pair with a fixed point structures

Definition 12.2.1. Let $(X, S(X), M^0)$ be a s.f.p.s., $S(X) \subset Z \subset P(X)$, $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. The pair (θ, η) is a compatible pair with $(X, S(X), M^0)$ iff:

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 12.2.1. Let $(X, S(X), M^0)$ be the trivial s.f.p.s. on a metric space (X, d), $Z := P_b(X)$, $\theta := \delta$ and $\eta(A) = \overline{A}$. Then the pair (δ, η) is a compatible pair with $(X, S(X), M^0)$.

12.3 Maximal strict fixed point structures

Let $(X, S(X), M^0)$ be a s.f.p.s. and $S_1(X) \subset P(X)$ such that $S(X) \subset S_1(X)$.

Definition 12.3.1. The s.f.p.s. $(X, S(X), M^0)$ is maximal in $S_1(X)$ if we have

$$S(X) := \{A \in S_1(X) |\ T \in M^0(A)\ \Rightarrow\ (SF)_T \neq \emptyset\}.$$

Example 12.3.1. (Generic example). Let $(X, S(X), M)$ be a f.p.s. for singlevalued operators. Let $(X, S(X), M^0)$ be a s.f.p.s. for multivalued operators. Let $f \in M(Y)$. We consider the multivalued operator, $\widetilde{f} : Y \multimap Y$ defined by $\widetilde{f}(x) = \{f(x)\},\ \forall\ x \in Y$. It is clear that $(SF)_{\widetilde{f}} = F_{\widetilde{f}} = F_f$. We denote by $\widetilde{M}(Y)$ the set of multivalued operator generated, in the above way, by the singlevalued operators from $M(Y)$. We suppose that $\widetilde{M}(Y) \subset M^0(Y)$ for all $Y \in S_1(X)$. Then the maximality of the s.f.p.s., $(X, S(X), \widetilde{M})$, in $S_1(X)$ implies the maximality of the s.f.p.s., $(X, S(X), M^0)$ in $S_1(X)$.

Remark 12.3.1. To establish if a given s.f.p.s. is maximal or not, this is an open problem.

12.4 References

For the notion of the s.f.p.s. see I.A. Rus [35].

For strict fixed point theorems see:

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Chapter 13

(θ, φ)-contractions and θ-condensing operators

13.1 (θ, φ)-contractions

Let X be a nonempty set, $Y \in P(X)$ and $Z \in P(P(X))$.

Definition 13.1.1. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function and $\theta : Z \to \mathbb{R}_+$ a functional. An operator $T : Y \multimap X$ is said to be strong (θ, φ)-contraction iff:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii) $\theta(T(A)) \le \varphi(\theta(A))$, for all $A \in P(Y) \cap Z$.

Definition 13.1.2. An operator $T : Y \to Y$ is said to be a (θ, φ)-contraction iff:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii') $\theta(T(A)) \le \varphi(\theta(A))$ for all $A \in P(Y) \cap Z \cap I(T)$.

Example 13.1.1. Let (X, d) be a metric space, δ the diameter functional on X, $Z = P_b(X)$, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function. Then an operator

$T : Y \to P(Y)$, $Y \in P_b(X)$, is a (δ, φ)-contraction iff

$$\delta(T(A)) \le \varphi(\delta(A)),\ \forall\ A \in I(T).$$

The operator T is a strong (δ, φ)-contraction iff

$$\delta(T(A)) \le \varphi(\delta(A)),\ \forall\ A \in P(Y).$$

We have

Lemma 13.1.1. *Let (X, d) be a metric space, $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ a comparison function, $Y \in P_b(X)$ and $T : Y \to P(Y)$. If*

$$\delta(T(x), T(y)) \le \varphi(d(x, y), \delta(x, T(x)), \delta(y, T(y)), \delta(x, T(y)), \delta(y, T(x)))$$

for all $x, y \in Y$, then T is a (δ, ψ)-contraction, where $\delta : P_b(X) \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := \varphi(t, t, t, t, t)$.

Proof. Let $A \in I(T)$, and $x, y \in A$. Then

$$\delta(T(x), T(y)) \le \varphi(\delta(A), \delta(A), \delta(A), \delta(A), \delta(A)),$$

for all $x, y \in A$. This implies that

$$\delta(T(A)) \le \psi(\delta(A)).$$

Lemma 13.1.2. *Let $T : Y \to P(Y)$ be a strong (δ, φ)-contraction. Then T is a singlevalued φ-contraction.*

Proof. Since T is a strong (δ, φ)-contraction we have that

$$\delta(T(A)) \le \varphi(\delta(A)),\ \forall\ A \in P(Y).$$

If we take $A = \{x\}$, $x \in Y$, then $\delta(A) = 0$ and we have $\delta(T(x)) = 0$, i.e. $card T(x) = 1$.

Now we take $A = \{x, y\}$, $x, y \in X$. Then

$$d(T(x), T(y)) \le \varphi(d(x, y)),\ \forall\ x, y \in Y,$$

i.e. T is a singlevalued φ-contraction.

Example 13.1.2. Let (X, d) a metric space, $Z = P_b(X)$, $\alpha : P_b(X) \to \mathbb{R}_+$ an abstract measure of noncompactness on the metric space (X, d) and $Y \in P_b(X)$. Then $T : Y \to P(Y)$ is a strong (α, φ)-contraction iff

$$\alpha(T(A)) \leq \varphi(\alpha(A)), \ \forall \ A \in P_b(X),$$

and T is an (α, φ)-contraction iff

$$\alpha(T(A)) \leq \varphi(\alpha(A)), \ \forall \ A \in I(T).$$

For example, if T is a compact operator, then T is a strong $(\alpha, 0)$-contraction.

Example 13.1.3. Let X be a Banach space, $\omega_D : P_b(X) \to \mathbb{R}_+$ the De Blasi measure of weak noncompactness on X and $Y \in P_b(X)$.

Then T is a (ω_D, φ)-contraction iff

$$\omega_D(T(A)) \leq \varphi(\omega_D(A)), \ \forall \ A \in I(T)$$

and T is a strong (ω_D, φ)-contraction iff

$$\omega_D(T(A)) \leq \varphi(\omega_D(A)), \ \forall \ A \in P(Y).$$

For example, if T is a weak compact operator, then T is a strong $(\omega_D, 0)$-contraction.

Example 13.1.4. Let (X, d, W) be a convex metric space and $\beta : P_b(X) \to \mathbb{R}_+$ an abstract measure of nonconvexity on X. Then an operator $T : Y \to P(Y)$, $Y \in P_b(X)$ is a (β, φ)-contraction iff

$$\beta(T(A)) \leq \varphi(\beta(A)), \ \forall \ A \in I(T)$$

and T is a strong (β, φ)-contraction iff

$$\beta(T(A)) \leq \varphi(\beta(A)), \ \forall \ A \in P(Y).$$

Remark 13.1.1. if T is a strong (β, φ)-contraction then

$$A \in P_{cv}(Y) \;\Rightarrow\; \overline{T(A)} \in P_{cv}(Y).$$

Example 13.1.5. Let X be a Banach space, $Y \in P_b(X)$ and $T : X \to P_{cp,cv}(X)$ an l-contraction. Then T is a (α_H, l) strong contraction on Y.

Indeed, let $A \in P(Y)$ and $\alpha_H(A) = r$. For $\varepsilon > 0$ we choose $\{a_1, \ldots, a_m\} \subset X$ such that $A \subset \bigcup_{i=1}^{m} B(a_i, r + \varepsilon)$. For each $i \in \{1, \ldots, m\}$ we choose b_j^i, $j = \overline{1, m(i)}$ such that $T(x_i) \subset \bigcup_{j=1}^{m(i)} B(b_j^i, \varepsilon)$. From this we have that

$$T(A) \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{m(i)} B(b_j^i, l(r + \varepsilon) + \varepsilon).$$

This implies that $\alpha_H(T(A)) \leq l(r+\varepsilon)+\varepsilon$, $\forall\, \varepsilon > 0$. So, $\alpha_H(T(A)) \leq l\alpha_H(A)$, $\forall\, A \in P(Y)$.

Example 13.1.6. Let X be a Banach space, $Y \in P_b(X)$, $T : X \to P_{cp,cv}(X)$ and $S : Y \to P_{cp,cv}(X)$. We suppose that

(a) T is an l-contraction;

(b) S is a compact operator.

Then $T + S$ is an (α_H, l)-contraction.

13.2 θ-condensing operators

Let X be a nonempty set, $Y \in P(X)$ and $Z \in P(P(X))$. Let $\theta : Z \to \mathbb{R}_+$ a setargument functional.

Definition 13.2.1. An operator $T : Y \multimap X$ is said to be strong θ-condensing iff

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii) $A \in P(Y) \cap Z$, $\theta(A) \neq 0$ implies $\theta(T(A)) < \theta(A)$.

Definition 13.2.2. An operator $T : Y \multimap Y$ is said to be θ-condensing iff

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii) $A \in I(T) \cap Z$, $\theta(A) \neq 0$ implies $\theta(T(A)) < \theta(A)$.

Example 13.2.1. Let (X,d) be a metric space, $Y \in P_b(X)$ and $T : Y \to P(Y)$. Then T is strong δ-condensing iff

$$\delta(T(A)) < \delta(A),\ \forall\ A \in P(Y), \quad \delta(A) \neq 0,$$

and T is δ-condensing iff

$$\delta(T(A)) < \delta(A),\ \forall\ A \in I(T), \quad \delta(A) \neq 0.$$

Lemma 13.2.1. *If $T : Y \to P(Y)$ is a strong δ-condensing, then*

$$\delta(T(x) \cup T(y)) < d(x,y),\ \forall\ x,y \in Y,\ x \neq y.$$

Proof. We take $A = \{x,y\}$ in Definition 13.2.1.

Example 13.2.2. Let (X,d) be a metric space and $\alpha_K : P_b(X) \to \mathbb{R}_+$ the Kuratowski measure of noncompactness on X and $Y \in P_b(X)$. Then $T : Y \to P(Y)$ is a strong α_K-condensing iff

$$\alpha_K(T(A)) < \alpha_K(A),\ \forall\ A \in P(Y) \text{ with } \alpha_K(A) \neq 0,$$

and T is α_K-condensing iff

$$\alpha_K(T(A)) < \alpha_K(A),\ \forall\ A \in I(T), \quad \alpha_K(A) \neq 0.$$

For example, if T is compact, then T is strong α_K-condensing.

Remark 13.2.1. Let $(\mathcal{A},\leq)$ be an ordered set with the least element, 0. If in Definition 13.2.1 and 13.2.2, instead of $\theta : Z \to \mathbb{R}_+$ we put $\theta : Z \to \mathcal{A}$ then we shall have different classes of condensing operators if we put instead of condition $\theta(T(A)) < \theta(A)$ for all $A \in I(T)$ $(A \in P(Y))$, with $\theta(A) \neq 0$ each of the following conditions:

(ii') $A \in I(T)(P(Y))$, $\theta(T(A)) \geq \theta(A)$ implies $\theta(A) = 0$.

(ii") $A \in I(T)(P(Y))$, $\theta(T(A)) = \theta(A)$ implies $\theta(A) = 0$.

For example, W.V. Petryshyn and P.M. Fitzpatrick (1974) work with the following setargument operator.

Let $(\mathcal{A}, \leq)$ be a lattice with the first element, 0. Let X be a Hausdorff locally convex topological vector space. An operator $\alpha : P(X) \to \mathcal{A}$ is called a measure of noncompactness if it satisfies the following conditions:

(i) $\alpha(coA) = \alpha(A),\ \forall\ A \in P(X)$;

(ii) $\alpha(A) = 0$ iff A is precompact;

(iii) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B)),\ \forall\ A, B \in P(X)$.

Let $Y \subset X$. An operator $T : Y \to P_{cl,cv}(X)$ is said to be α-condensing iff:

$$\alpha(T(A)) \not\geq \alpha(A),\ \forall\ A \in P(Y),$$

such that A is not precompact.

So, Petryshyn and Fitzpatrick (1974) work with strong α-condensing operators. E. Tarafdar and R. Výborný (1975) use α-condensing operators w.r.t. a setargument operator.

13.3 References

For the Definitions 13.1.1, 13.1.2, 13.2.1 and 13.2.2 see I.A. Rus [36].

For examples of (θ, φ)-contractions and θ-condensing operators see I.A. Rus [36] and

- E. Tarafdar and R. Výborný, *Fixed point theorems for condensing multivalued mappings on a locally convex topological space*, Bull. Austral. Math. Soc., 12(1975), 161-170.

For examples of strong (θ,φ)-contractions and strong θ-condensing operators see: J. Appell, E. De Pascale, H.T. Nguyen and P.P. Zabreiko [3], B. Cong Cuong [16], Yu. G. Borisovitch, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii [10]-[12], S. Czerwik [17], K. Deimling [18] and [19], L. Górniewicz [20], O. Hadžić [21], M. Kamenskii, V. Obukhovskii and P. Zecca [24], S. Hu and N.S. Papageorgiou [22], I.A. Rus [37], I.A. Rus, A. Petruşel and G. Petruşel [38]. See also,

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Chapter 14

First general fixed point principle for multivalued operators

14.1 First general fixed point principle

Let X be a nonempty set.

Theorem 14.1.1. (First general fixed point principle).

Let $(X, S(X), M^0)$ be a f.p.s. on the set X and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) T is a (θ, φ)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$;

(c) if:

(1) T is a strong (θ,φ)-contraction;

(2) $A, B \in Z$, $A \subset B \Rightarrow \theta(A) \le \theta(B)$;

(3) $F_T \in Z$;

then, $\theta(F_T) = 0$.

Proof. (a) From $Y \in \eta(Z)$ we have that $T(Y) \in Z$. Let $Y_1 := \eta(T(Y)), \ldots, Y_n := \eta(T(Y_{n-1}))$. From the definition of Y_n it follows that $Y_{n+1} \subset Y_n$, $Y_n \in I(T)$ and $Y_n \in F_\eta$, $n \in N$ $(Y_0 := Y)$.

Let $Y_\infty := \bigcap\limits_{n\in\mathbb{N}} Y_n$. From the condition (ii) we have

$$\theta(Y_n) \le \varphi(\theta(Y_{n-1})) \le \cdots \le \varphi^n(\theta(Y)) \to 0 \text{ as } n \to \infty.$$

Since $\theta : \eta(Z) \to \mathbb{R}_+$ is a functional with the intersection property, it follows that $Y_\infty \neq \emptyset$, $Y_\infty \in \eta(Z)$, $Y_\infty \in I(T)$ and $\theta(Y_\infty) = 0$. These imply that $Y_\infty \in S(X)$ and $T|_{Y_\infty} \in M^0(Y_\infty)$, i.e., $F_T \neq \emptyset$.

(b) From $T(F_T) = F_T$, we have that

$$\theta(F_T) = \theta(T(F_T)) \le \varphi(\theta(F_T)).$$

But φ is a comparison function, so, $\theta(F_T) = 0$.

(c) From (1), (2) and (3) we have

$$\theta(F_T) \le \theta(T(F_T)) \le \varphi(\theta(F_T)).$$

Hence $\theta(F_T) = 0$.

Remark 14.1.1. In Theorem 14.1.1 is not necessarily that $M(A)$ be defined for all $A \in P(X)$. It is sufficiently that $M(A)$ be defined for $A \in \eta(Z)$.

Theorem 14.1.2. *Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) a compatible pair with $(X, S(X), M^0)$. Let $Y \in F_\eta$ and $T \in M^0(Y)$ be such that $T(Y) \in Z$. In the conditions (i) and (ii) in Theorem 14.1.1 we have (a), (b) and (c) in that theorem.*

Proof. First we remark that $\eta(T(Y)) \in I(T)$ and $T|_{\eta(T(Y))} : \eta(T(Y)) \to \eta(T(Y))$ is in the conditions of Theorem 14.1.1.

The proof follows from this theorem.

Remark 14.1.2. In Theorem 14.1.2 it is sufficiently that $M(A)$ be defined for $A \in F_\eta$.

Remark 14.1.3. In terms of comparison operator $\varphi : \mathcal{A} \to \mathcal{A}$ (Definition 5.1.1) and of operator with intersection property $\theta : Z \to \mathcal{A}$ (Definition 3.9.1), Theorem 14.1.1 and 14.1.2 take the following form:

Theorem 14.1.1'. *Let $(X, S(X), M^0)$ be a f.p.s., on the set X, and (θ, η) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:*

(i) the operator $\theta|_{\eta(Z)} : \eta(Z) \to \mathcal{A}$ has the intersection property;

(ii) T is a (θ, φ)-contraction, where $\varphi : \mathcal{A} \to \mathcal{A}$.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$.

Theorem 14.1.2'. *Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathcal{A})$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in F_\eta$ and $T \in M^0(Y)$ be such that $T(Y) \in Z$. We suppose that:*

(i) the operator $\theta|_{\eta(Z)} : \eta(Z) \to \mathcal{A}$ has the intersection property;

(ii) T is a (θ, φ)-contraction, where $\varphi : \mathcal{A} \to \mathcal{A}$.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$.

Remark 14.1.4. If in addition, in Theorem 14.1.1' and 14.1.2' T is a strong (θ, φ)-contraction and $F_T \in Z$, then $\theta(F_T) = 0$. So, in this case for to have $\theta(F_T) = 0$ it is not necessarily that $T(F_T) = F_T$.

Remark 14.1.5. The Theorems 14.1.1, 14.1.2, 14.1.1' and 14.1.2' are settheoretical.

In what follow we shall present some consequences of these general results.

14.2 (α, φ)-contraction principle

Let (X, d) be a metric space and $\alpha : P_b(X) \to \mathbb{R}_+$ an abstract measure of noncompactness on the metric space X (Definition 3.7.3).

Theorem 14.2.1. *Let (X, d) be a bounded complete metric space and $T : X \to P_{cl}(X)$. We suppose that:*

(i) T is an (α, φ)-contraction;

(ii) T is contractive, i.e.,

$$H(T(x), T(y)) < d(x, y), \ \forall\ x, y \in X, \ x \neq y.$$

Then:

(a) $F_T \neq \emptyset$;

(b) if $T(F_T) = F_T$, then $\alpha(F_T) = 0$;

(c) if T is a strong (α, φ)-contraction, then $\alpha(F_T) = 0$.

Proof. We consider, in Theorem 14.1.1, the f.p.s. of R.E. Smithson (see Example 11.2.8), $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) = \overline{A}$.

Theorem 14.2.2. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$. We suppose that:*

(i) T is an (α, φ)-contraction;

(ii) T is contractive;

(iii) $T(X) \in P_b(X)$.

Then:

(a) $F_T \neq \emptyset$;

(b) if $T(F_T) = F_T$, then $\alpha(F_T) = 0$;

(c) if T is a strong (α, φ)-contraction, then $\alpha(F_T) = 0$.

Proof. We consider, in Theorem 14.1.2, the f.p.s. of R.E. Smithson, $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) := \overline{A}$.

Theorem 14.2.3. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and α an abstract measure of noncompactness on the Banach space X (Definition 3.7.1). If $T : Y \to P_{cp,cv}(Y)$ is u.s.c. and (α, φ)-contraction, then $F_T \neq \emptyset$. If T is a strong (α, φ)-contraction, then $\alpha(F_T) = 0$.*

Proof. We consider, in Theorem 14.1.1, the f.p.s. of Bohnenblust-Karlin (see Example 11.2.11), $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) := \overline{co}A$.

Theorem 14.2.4. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and α an abstract measure of noncompactness on the Banach space X. We suppose that:*

(i) $T : Y \to P_{cp,cv}(Y)$ is u.s.c.;

(ii) T is an (α, φ)-contraction;

(iii) $T(Y) \in P_b(X)$.

Then, $F_T \neq \emptyset$. If T is a strong (α, φ)-contraction, then $\theta(F_T) = 0$.

Proof. We take, in Theorem 14.1.2, the f.p.s. of Bohnenblust-Karlin, $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) := \overline{co}A$.

Theorem 14.2.5. (S. Czerwik (1980)). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $T : Y \to P_{cp,cv}(Y)$ an operator. We suppose that:*

(i) T is u.s.c.;

(ii) T is an (α_H, φ)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if $T(F_T) = F_T$, then $\alpha_H(F_T) = 0$;

(c) if T is a strong (α_H, φ)-contraction, then $\alpha_H(F_T) = 0$.

Proof. We take $\alpha := \alpha_H$ in Theorem 14.2.3.

Remark 14.2.1. We can take, in Theorem 14.2.5, $Y \in P_{cl,cv}(X)$ and $T : Y \to P_{cp,cv}(Y)$ with $T(Y) \in P_b(X)$.

Theorem 14.2.6. (W.V. Petryshyn and P.M. Fitzpatrick (1974)). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$, $T : X \to P_{cp,cv}(X)$ and $S : Y \to P_{cp,cv}(Y)$. We suppose that:*

(i) T is a strong (α_H, l)-contraction;

(ii) $Y \in I(T)$;

(iii) T and S are u.s.c.;

(iv) S is compact;

(v) $T(x) + S(x) \in Y, \ \forall \ x \in Y$.

Then, $F_{T+S} \neq \emptyset$.

Proof. We remark that $T + S$ is a strong (α_H, l)-contraction. The proof follows from Theorem 14.2.3.

Remark 14.2.2. We can take in Theorem 14.2.6 instead of condition (i) the condition

(i') $H(T(x), T(y)) \leq ld(x, y), \ \forall \ x, y \in X$.

14.3 (ω, φ)-contraction principle

Let X be a Banach space and $\omega : P_b(X) \to \mathbb{R}_+$ an abstract measure of weak noncompactness on X. We have

Theorem 14.3.1. *Let X be a Banach space, $Y \in P_{b,wcl,cv}(X)$ and $T : Y \to P_{wcp,cv}(Y)$. We suppose that:*

(i) T is weakly u.s.c., i.e., $T : (X, \tau_w) \to (X, \tau_w)$ is u.s.c.;

(ii) T is an (ω, φ)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in I(T)$, then $\omega(F_T) = 0$;

(c) if T is strong (ω, φ)-contraction, then $\omega(F_T) = 0$.

Proof. Let $S(X) := P_{wcp,cv}(X)$ and $M^0(A) := \{T : A \to P_{wcp,cv}(A) \mid T$ is weakly u.s.c.$\}$. Then $(X, S(X), M^0)$ is a f.p.s. on X (see J. Ewert (1986)). Now, we take $Z := P_b(X)$, $\theta := \omega$ and $\eta(A) := \overline{co}^w A$, in Theorem 14.1.1.

Theorem 14.3.2. *Let X be a Banach space, $Y \in P_{wcl,cv}(X)$ and $T : Y \to P_{wcp,cv}(Y)$. We suppose that:*

(i) T is weakly u.s.c.;

(ii) T is an (ω, φ)-contraction;

(iii) $T(Y) \in P_b(X)$.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in I(T)$, then $\omega(F_T) = 0$;

(c) if T is a strong (ω, φ)-contraction then, $\omega(F_T) = 0$.

Proof. We remark that $\overline{co}^w(T(Y)) \in I(T)$. The proof follows from Theorem 14.3.1.

Theorem 14.3.3. (J. Ewert (1986)). *Let X be a Banach space, $Y \in P_{b,wcl,cv}(X)$ and $T : Y \to P_{wcp,cv}(Y)$. We suppose that:*

(i) T is weakly u.s.c.;

(ii) T is an (ω_D, l)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in I(T)$, then $\omega_D(F_T) = 0$;

(c) if T is a strong (ω_D, l)-contraction, then $\omega_D(F_T) = 0$.

Proof. We take $\omega := \omega_D$ and $\varphi(t) := lt$, in Theorem 14.3.1.

14.4 First general strict fixed point principle

Let X be a nonempty set.

Theorem 14.4.1. *Let $(X, S(X), M^0)$ be a s.f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) T is a (θ, φ)-contraction.

Then:

(a) $(SF)_T \neq \emptyset$;

(b) if $(SF)_T \in Z$, then $\theta((SF)_T) = 0$.

Proof. See the proof of Theorem 14.1.1.

Theorem 14.4.2. *Let $(X, S(X), M^0)$ be a s.f.p.s. and (θ, η) a compatible pair with $(X, S(X), M^0)$. Let $Y \in F_\eta$ and $T \in M^0(Y)$ be such that $T(Y) \in Z$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) T is a (θ, φ)-contraction.

Then:

(a) $(SF)_T \neq \emptyset$;

(b) if $(SF)_T \in Z$, then $\theta((SF)_T) = 0$.

Proof. See the proof of Theorem 14.1.2.

Theorem 14.4.3. *Let (X, d) be a bounded complete metric space and $T : X \to P(X)$ a (δ, φ)-contraction. Then:*

(a) $(SF)_T = \{x^*\}$;

(b) $F_T = (SF)_T$.

Proof. (a) We take in Theorem 14.4.1 the trivial s.f.p.s. on X, $Z := P_b(X)$, $\theta := \delta$ and $\eta(A) := \overline{A}$.

(b) We remark that $F_T \subset X_\infty$ (see the proof of Theorem 14.1.1) and $\delta(X_\infty) = 0$.

Theorem 14.4.4. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ a (δ, φ)-contraction. If $T(X) \in P_b(X)$, then:*

(a) $(SF)_T = \{x^*\}$*;*

(b) $F_T = (SF)_T$*.*

Proof. We take in Theorem 14.4.2 the trivial s.f.p.s., $Z := P_b(X)$, $\theta := \delta$ and $\eta(A) := \overline{A}$. See also the proof of Theorem 14.4.3.

14.5 References

For the first general fixed point principle and the first general strict fixed point principle see I.A. Rus [36].

For the (α, φ)-contraction principle see S. Czerwik [17], K. Deimling [18] and [19], S. Hu and N.S. Papageorgiou [22], O. Hadžić [21], M. Kamenskii, V. Obukhovskii and P. Zecca [24], A. Petruşel [30], I.A. Rus [36],... See also:

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For (δ, φ)-contractions see I.A. Rus [36] and [37].
For the multivalued version of the Krasnoselskii fixed point theorem see:

- W.V. Petryshyn and P.M. Fitzpatrick, *A degree theory, fixed point theorems and mapping theorems for multivalued noncompact mappings*, Trans. A.M.S., 194(1974), 1-25.

- A. Petruşel, *Multivalued operators and fixed points*, Pure Math. Appl., 11(2000), No.2, 361-368.

- A. Petruşel, *Fixed points and selection for multivalued operators*, Seminar on Fixed Point Theory Cluj-Napoca, 2(2001), 3-22.
- C. Avramescu and C. Vladimirescu, *Fixed point theorems of Krasnoselskii type in a space of continuous functions*, Fixed Point Theory, 5(2004), No.2, 181-195.

See also M. Kamenskii, V. Obukhovskii and P. Zecca [24], p.47.

Chapter 15

Second general fixed point principle for multivalued operators

15.1 Second general fixed point principle

Let X be a nonempty set, $Z \in P(P(X))$ and $(O, \leq)$ an ordered set with the first element, denoted by 0. Let $\theta : Z \to O$ be an operator.

Theorem 15.1.1. (Second general fixed point principle).

Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) a compatible pair with $(X, S(X), M^0)$. let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is θ-condensing, i.e.,

$$A \in Z \;\Rightarrow\; T(A) \in Z, \text{ and}$$

$$A \in Z \cap I(T),\; \theta(T(A)) \geq \theta(A) \;\Rightarrow\; \theta(A) = 0.$$

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$.

Proof. (a) Let $y_0 \in Y$ and $A = \{y_0\}$. By Lemma 1.6.2 there exists $A_0 \in F_\eta \cap I(T)$ such that $\eta(T(A_0) \cup \{y_0\}) = A_0$. From the condition (i) we have

$$\theta(\eta(T(A_0) \cup \{y_0\}) = \theta(T(A_0) \cup \{y_0\}) = \theta(T(A_0)) = \theta(A_0).$$

From the condition (ii) it follows that $A_0 \in Z_\theta$. Thus, $A_0 \in F_\eta \cap Z_\theta$ and $T|_{A_0} \in M^0(A_0)$. Since $(X, S(X), M^0)$ is a f.p.s. and (θ, η) is a compatible pair with this f.p.s., we have $F_T \neq \emptyset$.

(b) From $T(F_T) = F_T$ and the condition (ii) it follows that $\theta(F_T) = 0$.

Remark 15.1.1. In the above theorem is not necessarily that $M(A)$ be defined for all $A \in P(X)$. It is sufficiently that $M(A)$ be defined for $A \in \eta(Z)$.

Theorem 15.1.2. *Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) $(\theta : Z \to O)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in F_\eta$ and $T \in M^0(Y)$ be such that $T(Y) \in Z$. We suppose that:*

(i) $A \in Z,\ x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is θ-condensing.

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$.

Proof. The operator $T|_{\eta(T(Y))} : \eta(T(Y)) \multimap \eta(T(Y))$ is in the conditions of Theorem 15.1.1.

Remark 15.1.2. In Theorem 15.1.2 is not necessarily that $M(A)$ be defined for all $A \in P(X)$. It is sufficiently that $M(A)$ be defined for $A \in F_\eta$.

Remark 15.1.2. If $O = \mathbb{R}_+$, then the condition (ii) in the above results take the following form:

(ii') $A \in Z \ \Rightarrow\ T(A) \in Z$,

and

$$A \in Z \cap I(T),\ \theta(A) \neq 0 \ \Rightarrow\ \theta(T(A)) < \theta(A).$$

Remark 15.1.3. All terms in the above results are set-theoretic.

In what follow we shall present some consequences of these general results.

15.2 α-condensing operators principle

Let X be a locally convex space and $(p_i)_{i\in I}$ a family of seminorms which generate the topology on X. Let $\alpha_K : P_b(X) \to \mathbb{M}(I, \mathbb{R}_+)$ be the operator defined in Example 3.9.1, i.e., $\alpha(A)(i) = \alpha_K^i(A)$, $A \in P_b(X)$, where α_K^i is the Kuratowski measure of noncompactness w.r.t. the seminorm p_i. We call this operator the Kuratowski measure of noncompactness on X. In a similar way we define the Hausdorff measure of noncompactness on X, α_H.

Theorem 15.2.1. *Let X be a Hausdorff locally convex linear topological space, $Y \in P_{b,cl,cv}(X)$ and $T : Y \to P_{cl,cv}(Y)$. We suppose that:*

(i) T is u.s.c.;

(ii) T is α-condensing, where $\alpha = \alpha_K$ or α_H.

Then, F_T is nonempty, and F_T is compact, if $T(F_T) = F_T$.

Proof. We take in Theorem 15.1.1, the Glicksberg-Fan f.p.s., $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) = \overline{co}A$. From this theorem we have that $F_T \neq \emptyset$ and $\alpha(F_T) = 0$. But $F_T = \overline{F}_T$, so F_T is a nonempty compact subset of Y.

Theorem 15.2.2. *Let X be a Hausdorff locally convex linear topological space, $Y \in P_{cl,cv}(X)$ and $T : Y \to P_{cl,cv}(Y)$. We suppose that:*

(i) T is u.s.c.;

(ii) T is α-condensing, where $\alpha = \alpha_K$ or α_H;

(iii) $T(Y) \in P_b(X)$.

Then F_T is a nonempty, and F_T is compact subset of Y if $T(F_T) = F_T$.

Proof. We take in the Theorem 15.1.2 the Glicksberg-Fan f.p.s.

Theorem 15.2.3. *Let X be a Banach space, $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ the Daneš-Pasicki measure of noncompactness on X, $Y \in P_{b,cl,cv}(X)$ and T :*

$Y \to P_{cl,cv}(Y)$. *We suppose that:*

(i) T is u.s.c.;

(ii) T is α_{DP}-condensing.

Then F_T is nonempty, and if $T(F_T) = F_T$, then $\alpha_{DP}(F_T) = 0$.

Proof. We take in Theorem 15.1.1 the Bohnenblust-Karlin f.p.s., $Z := P_b(X)$, $\theta := \alpha_{DP}$ and $\eta(A) := \overline{co}A$. From this theorem we have that $F_T \neq \emptyset$ and $\alpha_{DP}(F_T) = 0$, if $T(F_T) = F_T$.

Theorem 15.2.4. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $T : Y \to P_{cl,cv}(Y)$. We suppose that:*

(i) T is u.s.c.;

(ii) T is α_{DP}-condensing;

(iii) $T(Y) \in P_b(X)$.

Then F_T is a nonempty subset of Y.

Proof. We take in Theorem 15.1.2 the Bohnenblust-Karlin f.p.s.

15.3 ω-condensing operators principle

Let X be a Banach and $\omega_D : P_b(X) \to \mathbb{R}_+$ the De Blasi weak measure of noncompactness on X.

Theorem 15.3.1. *Let X be a Banach space, $Y \in P_{b,wcl,cv}(X)$ and $T : Y \to P_{wcp,cv}(X)$. We suppose that:*

(i) T is weakly u.s.c.;

(ii) T is ω_D-condensing.

Then $F_T \neq \emptyset$ and if $T(F_T) = F_T$, then $\omega_D(F_T) = 0$.

Proof. Let $S(X) := P_{wcp,cv}(X)$ and $M^0(A) := \{T : A \to P_{wcp,cv}(A) \mid T$ is weakly u.s.c.$\}$. Then $(X, S(X), M^0)$ is a f.p.s. Now we take in Theorem 15.1.1, the above f.p.s., $Z = P_b(X)$, $\theta := \omega_D$ and $\eta(A) := \overline{co}^w(A)$.

Theorem 15.3.2. *Let X be a Banach space, $Y \in P_{wcl,cv}(X)$ and $T : Y \to P_{wcp,cv}(X)$. We suppose that:*

(i) T is weakly u.s.c.;

(ii) T is ω_D-condensing;

(iii) $T(Y) \in P_b(X)$.

Then $F_T \neq \emptyset$ and if $T(F_T) = F_T$, then $\omega_D(F_T) = 0$.

Proof. We take in Theorem 15.1.2, $S(X) := P_{wcp,cv}(X)$, $M_0(A) := \{T : A \to P_{wcp,cv}(A) \mid T$ is weakly u.s.c.$\}$, $Z = P_b(X)$, $\theta = \omega_D$ and $\eta(A) = \overline{co}^w(A)$.

15.4 Second general strict fixed point principle

Let X be a nonempty set.

Theorem 15.4.1. (Second general strict fixed point principle). *Let $(X, S(X), M^0)$ be a s.f.p.s., and (θ, η) $(\theta : Z \to O)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is θ-condensing.

Then:

(a) $(SF)_T \neq \emptyset$;

(b) if $(SF)_T \in Z$, then $\theta((SF)_T) = 0$.

Proof. (a) See the proof of Theorem 15.1.1.

(b) We remark that $T((SF)_T) = (SF)_T$. From (ii) we have that $\theta((SF)_T) = 0$.

Theorem 15.4.2. *Let $(X, S(X), M^0)$ be a s.f.p.s., and (θ, η) $(\theta : Z \to O)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in F_\eta$ and $T \in M^0(Y)$. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is θ-condensing;

(iii) $T(Y) \in Z$.

Then:

(a) $(SF)_T \neq \emptyset$,

(b) if $(SF)_T \in Z$, *then* $\theta((SF)_T) = 0$.

Proof. The proof is similar with that of Theorem 15.1.2.

Theorem 15.4.3. *Let* (X, d) *be a bounded complete metric space and* $T : X \to P(X)$ *a* (δ, φ)*-contraction. Then:*

(a) $(SF)_T = \{x^*\}$;

(b) $F_T = (SF)_T$.

Proof. (a) We consider in Theorem 14.1.1 the trivial f.p.s., $Z := P(X)$, $\theta := \delta$ and $\eta(A) := \overline{A}$. The proof follows from Theorem 14.1.1.

(b) We remark that $F_T \subset X_\infty$ (see the proof of the Theorem 14.1.1) and $\delta(X_\infty) = 0$.

Theorem 15.4.4. *Let* (X, d) *a complete metric space and* $T : X \to P(X)$ *a* (δ, φ)*-contraction with* $T(X) \in P_b(X)$. *Then:*

(a) $(SF)_T = \{x^*\}$; *(b)* $F_T = (SF)_T$.

Proof. We apply Theorem 15.4.3 to the operator, $T|_{\overline{T(X)}} : \overline{T(X)} \to \overline{T(X)}$.

Theorem 15.4.5. *Let* (X, d) *a compact metric space and* $T : X \to P(X)$ *a* δ*-condensing operator. Then,* $F_T = (SF)_T = \{x^*\}$.

Proof. (a) By Martelli's lemma (see Lemma 1.6.3) there exists a nonempty closed subset $Y \subset X$ such that $Y = \overline{T(Y)}$. Since $Y \in I(T)$ and $\delta(Y) = \delta(T(Y))$ it follows that $\delta(Y) = 0$. So, $x^* \in (SF)_T$. Since $T((SF)_T) = (SF)_T$ we have that $(SF)_T = \{x^*\}$.

(b) Now we shall prove that $F_T = (SF)_T$. Let $x_0 \in F_T$ and consider the following family of subsets of X, $\mathcal{B}(x_0) := \{B \in P_{cl}(X) \mid B \in I(T),\ x_0 \in B\}$. Let $B(x_0) := \cap\mathcal{B}(x_0)$. $B(x_0)$ is the first element of the ordered set $(\mathcal{B}(x_0), \subset)$. But, $\overline{T(B(x_0))} \in \mathcal{B}(x_0)$ so, $\overline{T(B(x_0))} = B(x_0)$. Hence $\text{card}B(x_0) = 1$. Since $T(x_0) \subset B(x_0)$, hence $T(x_0) = \{x_0\}$.

15.5 References

For the second general fixed point principle and strict fixed point principles see I.A. Rus [36].

For α-condensing operators principles see I.A. Rus [36] and

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Chapter 16

Common fixed point property for a pair of multivalued operators

16.1 Commuting operators, common fixed points and common invariant subsets

Let X be a nonempty set and $T, S : X \to P(X)$ two multivalued operators. we have:

Lemma 16.1.1. $F_T \cap F_S = F_{T \cap S}$.

Remark 16.1.1. In general, $\{x \in X \mid T(x) \cap S(x) \neq \emptyset\} \neq X$.

Lemma 16.1.2. *If $T \circ S = S \circ T$, then:*

(a) $T(X), S(X) \in I(T) \cap I(S)$;

(b) $T(X) \cup S(X) \subset Z \subset X \;\Rightarrow\; Z \in I(T) \cap I(S)$;

(c) $S(F_T) \subset T(S(F_T))$ *and* $T(F_S) \subset S(T(F_S))$;

(d) If $(SF)_T \neq \emptyset$, *then* $(SF)_T \in F_{\widehat{T}}$;

(e) $F_{\widehat{T}} \in I(\widehat{S})$ *and* $F_{\widehat{S}} \in I(\widehat{T})$.

Proof. (c) From $F_T \subset T(F_T)$ it follows

$$S(F_T) \subset S \circ T(F_T) = T(S(F_T)).$$

(e) We remark that $\widehat{T} \circ \widehat{S} = \widehat{S} \circ \widehat{T}$. The proof follows from Lemma 7.1.1.

Lemma 16.1.3. *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ and $T, S : Y \to P(Y)$ such that $T \circ S = S \circ T$. Let $A_1 \in P(Y)$. Then there exists $A_0 \in Y$ such that:*

(a) $A_1 \subset A_0$;

(b) $A_0 \in F_\eta$;

(c) $A_0 \in I(T) \cap I(S)$;

(d) $\eta(T(A_0) \cup S(A_0) \cup A_1) = A_0$.

Proof. Let $\mathcal{B} := \{B \subset Y \mid B$ satisfies the conditions (a), (b) and (c)$\}$. From Lemma 1.4.1 we have that $\cap\mathcal{B} \in \mathcal{B}$. This implies that $\cap\mathcal{B}$ is the least element of the partially ordered set $(\mathcal{B}, \subset)$. We shall prove that $A_0 = \cap\mathcal{B}$. We remark that

$$\eta(T(A_0) \cup S(A_0) \cup A_1) \in \mathcal{B} \text{ and } \eta(T(A_0) \cup S(A_0) \cup A_1) \subset A_0.$$

These imply that $\eta(T(A_0) \cup S(A_0) \cup A_1) = A_0$.

Example 16.1.1. Let (X, τ) be a topological space, $Y \in P_{cl}(X)$ and $T, S : Y \to P(Y)$ such that $T \circ S = S \circ T$. Let $A_1 \in P(Y)$. Then there exists $A_0 \subset Y$ such that:

(a) $A_1 \subset A_0$;

(b) $A_0 = \overline{A}_0$;

(c) $A_0 \in I(T) \cap I(S)$;

(d) $\overline{(T(A_0) \cup S(A_0) \cup A_1)} = A_0$.

Example 16.1.2. Let $(X, +, \mathbb{R}, \tau)$ be a linear topological space, $Y \in P_{cl,cv}(X)$ and $T, S : Y \to P(Y)$ such that $T \circ S = S \circ T$. Then there exists $A_0 \subset Y$ such that:

(a) $A_1 \subset A_0$;

(b) $A_0 = \overline{co}A_0$;

(c) $A_0 \in I(T) \cap I(S)$;

(d) $\overline{co}(T(A_0) \cup S(A_0) \cup A_1) = A_0$.

The first problems in the common fixed point theory for multivalued operators are the following.

Let X be a nonempty set and $T, S : X \to P(X)$. In which conditions we have:

(P1) $F_T \cap F_S \neq \emptyset$;

(P2) $F_{\widehat{T}} \cap F_{\widehat{S}} \neq \emptyset$;

(P3) $F_T = F_S \neq \emptyset$;

(P4) $F_{\widehat{T}} = F_{\widehat{S}} \neq \emptyset$;

(P5) $F_T = F_S = \{x^*\}$;

(P6) $(SF)_T \cap (SF)_S \neq \emptyset$;

(P7) $(SF)_T = (SF)_S \neq \emptyset$;

(P8) $(SF)_T = (SF)_S = \{x^*\}$;

(P9) $F_T = (SF)_T = F_S = (SF)_S = \{x^*\}$;

(P10) $F_T \neq \emptyset$, $F_S \neq \emptyset$, $T \circ S = S \circ T \;\Rightarrow\; F_T \cap F_S \neq \emptyset$;

(P11) $(SF)_T \neq \emptyset$, $(SF)_S \neq \emptyset$, $T \circ S = S \circ T \;\Rightarrow\; (SF)_T \cap (SF)_S \neq \emptyset$.

In this chapter we shall consider, in principal, the problems (P10) and (P11).

16.2 Common fixed point structures

Let X be a nonempty set.

Definition 16.2.1. A fixed point structure $(X, \mathbf{S}(X), M^0)$ is with the common fixed point property iff

$$Y \in \mathbf{S}(X),\ T, S \in M^0(Y),\ T \circ S = S \circ T \;\Rightarrow\; F_T \cap F_S \neq \emptyset.$$

Definition 16.2.2. A strict fixed point structure $(X, \mathbf{S}(X), M^0)$ is with the common strict fixed point property iff

$$Y \in \mathbf{S}(X),\ T, S \in M^0(Y),\ T \circ S = S \circ T \ \Rightarrow\ (SF)_T \cap (SF)_S \neq \emptyset.$$

Definition 16.2.3. Let X be a nonempty set, $\mathbf{S}(X) \subset P(X)$, $\mathbf{S}(X) \neq \emptyset$ and $(PM)^0 \subset (P\mathbb{M})^0 := \{(T, S) \mid T, S \in \mathbb{M}(Y),\ Y \subset X\}$. A triple $(X, \mathbf{S}(X), (PM)^0)$ is a common fixed point structure iff

$$Y \in \mathbf{S}(X),\ (T, S) \in (PM)^0(Y) \ \Rightarrow\ F_T \cap F_S \neq \emptyset.$$

Definition 16.2.4. A triple $(X, \mathbf{S}(X), (PM)^0)$ is a common strict fixed point structure iff

$$Y \in \mathbf{S}(X),\ (T, S) \in (PM)^0(Y) \ \Rightarrow\ (SF)_T \cap (SF)_S \neq \emptyset.$$

Example 16.2.1. Trivial f.p.s. (Example 11.2.1) is a f.p.s. with the common fixed point property.

Example 16.2.2. Let (X, d) be a complete metric space, $\mathbf{S}(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) \mid T$ is a multivalued contraction with $(SF)_T \neq \emptyset\}$. The triple $(X, P_{cl}(X), M^0(Y))$ is a s.p.p.s. with the common strict fixed point property.

Indeed, from Theorem 8.5.1 in I.A. Rus [37] (p.87) $(X, P_{cl}(X), M^0(Y))$ is a s.f.p.s. Let $Y \in P_{cl}(X)$ and $T, S \in M^0(Y)$ such that $T \circ S = S \circ T$. We have $F_T = (SF)_T = \{x^*\}$, $F_S = (SF)_S = \{y^*\}$. From $T \circ S = S \circ T$ it follows $x^* = y^*$.

Example 16.2.3. Let X be a Banach, $\mathbf{S}(X) := P_{cp,cv}(X)$ and $(PM)^0(Y) := \{(T, S) \mid T, S : Y \to P_{cp,cv}(Y)$ are u.s.c. and $T(x) \cap S(x) \neq \emptyset,\ \forall\, x \in Y\}$. Then the triple $(X, P_{cp,cv}(X), (PM)^0)$ is a common fixed point structure.

Indeed, let $Y \in P_{cp,cv}(X)$ and $T, S : Y \to P_{cp,cv}$ u.s.c. such that $T(x) \cap S(x) \neq \emptyset$. The operator $T \cap S$ satisfies the conditions of the fixed point theorem of Bohnenblust-Karlin. It is clear that $F_T \cap F_S = F_{T\cap S} \neq \emptyset$.

Example 16.2.4. (M. Avram (1975)) Let (X, d) be a complete metric space, $\mathbf{S}(X) := P_{b,cl}(X)$ and $(PM)^0(Y) := \{(T, S) \mid T, S : Y \to P_{b,cl}(Y)$ and there exist $a, b, c \in \mathbb{R}_+$, $a + 2b + 4c < 1$ such that $\delta(T(x), S(y)) \leq ad(x, y) + b[\delta(x, T(x)) + \delta(y, S(y))] + c[\delta(x, S(y)) + \delta(y, T(x))],\ \forall\ x, y \in Y\}$. Then $(X, P_{b,cl}(X), (PM)^0)$ is a common strict fixed point structure.

Remark 16.2.1. For other examples of common fixed point structures and of common strict fixed point structures see A. Sîntămărian (2006, Sc. Math. Japonicae).

Remark 16.2.2. To give examples of f.p.s. with the common fixed point property is another open problem in the f.p.s. theory.

16.3 (θ, φ)-contraction pair

Let X be a nonempty set, $Y \subset X$ and $\theta : Z \to \mathbb{R}_+$, where $Z \subset P(X)$, $Z \neq \emptyset$.

Definition 16.3.1. A pair of operators $T, S : Y \to P(Y)$ is a (θ, φ)-contraction pair iff

(i) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function;

(ii) $A \in P(Y) \cap Z \ \Rightarrow\ T(A) \cup S(A) \in Z$;

(iii) $\theta(T(A) \cup S(A)) \leq \varphi(\theta(A)),\ \forall\ A \in I(T) \cap I(S) \cap Z$.

If $l \in [0, 1[$ and $\varphi(t) = lt$, then by (θ, l)-contraction pair we understand a $(\theta, l(\cdot))$-contraction pair.

We have

Theorem 16.3.1. *Let $(X, \mathbf{S}(X), M^0)$ be a f.p.s. with the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, \mathbf{S}(X), M)$.*

Let $Y \in \eta(Z)$ and $T, S \in M^0(Y)$. We suppose that:

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $T \circ S = S \circ T$;

(iii) the pair (T, S) is a (θ, φ)-contraction pair.

Then, $F_T \cap F_S \neq \emptyset$.

Proof. Let $Y_1 := \eta(T(Y) \cup S(Y)), \ldots, Y_{n+1} := \eta(T(Y_n) \cup S(Y_n))$, $n \in \mathbb{N}$. From Lemma 16.1.2 we have that $Y_n \in I(T) \cap I(S)$, $n \in \mathbb{N}$. From the conditions (ii) and (iii) we have that

$$\theta(Y_{n+1}) = \theta(\eta(T(Y_n) \cup S(Y_n))) = \theta(T(Y_n) \cup S(Y_n))$$

$$\leq \varphi(\theta(Y_n)) \leq \cdots \leq \varphi^{n+1}(\theta(Y)) \to 0 \text{ as } n \to \infty.$$

From the condition (i) it follows that $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$ and $\theta(Y_\infty) = 0$. It is clear that $\eta(Y_\infty) = Y_\infty$ and $Y_\infty \in I(T) \cap I(S)$. These imply that $Y_\infty \in S(X)$ and so, $F_T \cap F_S \neq \emptyset$.

Theorem 16.3.2. *Let $(X, \mathbf{S}(X), M^0)$ be a f.p.s. with the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, \mathbf{S}(X), M^0)$. Let $Y \in F_\eta$ and $T, S \in M^0(Y)$ with $T(Y) \cup S(Y) \in Z$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $T \circ S = S \circ T$;

(iii) the pair (T, S) is a (θ, φ)-contraction pair.

Then, $F_T \cap F_S \neq \emptyset$.

Proof. We apply Theorem 16.3.1 to the pair $T, S : \eta(T(Y) \cup S(Y)) \to P(\eta(T(Y) \cup S(Y)))$.

In a similar way we have

Theorem 16.3.3. *Let $(X, \mathbf{S}(X), M^0)$ be a s.f.p.s. with the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, \mathbf{S}(X), M^0)$. Let $Y \in \eta(Z)$ and $T, S \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $T \circ S = S \circ T$*;*

(iii) the pair (T, S) *is a* (θ, φ)*-contraction pair.*

Then, $(SF)_T \cap (SF)_S \neq \emptyset$*.*

Theorem 16.3.4. *Let* $(X, \mathbf{S}(X), M^0)$ *be a s.f.p.s. with the common fixed point property and* (θ, η) $(\theta : Z \to \mathbb{R}_+)$ *a compatible pair with* $(X, \mathbf{S}(X), M^0)$*. Let* $Y \in F_\eta$ *and* $T, S \in M^0(Y)$ *with* $T(Y) \cup S(Y) \in Z$*. We suppose that:*

(i) $\theta|_{\eta(Z)}$ *has the intersection property;*

(ii) $T \circ S = S \circ T$*;*

(iii) the pair (T, S) *is a* (θ, φ)*-contraction pair.*

Then, $(SF)_T \cap (SF)_S \neq \emptyset$*.*

16.4 θ-condensing pair

Let X be a nonempty set, $Y \subset X$ and $\theta : Z \to \mathbb{R}_+$ where $Z \subset P(X)$, $Z \neq \emptyset$.

Definition 16.4.1. A pair $T, S : Y \to P(Y)$ is a θ-condensing pair iff:

(i) $A_i \in Z,\ i \in I,\ \bigcap\limits_{i \in I} A_i \neq \emptyset \ \Rightarrow\ \bigcap\limits_{i \in I} A_i \in Z$;

(ii) $A \in P(Y) \cap Z \ \Rightarrow\ T(A) \cup S(A) \in Z$;

(iii) $\theta(T(A) \cup S(A)) < \theta(A)$, for all $A \in I(T) \cap I(S) \cap Z$ such that $\theta(A) \neq \emptyset$.

We have

Theorem 16.4.1. *Let* $(X, \mathbf{S}(X), M^0)$ *be a f.p.s. with common fixed point property and* (θ, η) *a compatible pair with* $(X, \mathbf{S}(X), M^0)$*. Let* $Y \in \eta(Z)$ *and* $T, S \in M^0(Y)$*.*

We suppose that:

(i) $x \in Y,\ A \in Z$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$*;*

(ii) $T \circ S = S \circ T$*;*

(iii) the pair (T, S) *is* θ*-condensing pair.*

Then, $F_T \cap F_S \neq \emptyset$*.*

Proof. Let $x_0 \in Y$. By Lemma 16.1.3 there exists $A_0 \subset Y$ such that $x_0 \in A_0$, $A_0 \in F_\eta \cap I(T) \cap I(S) \cap Z$ and $\eta(T(A_0) \cup S(A_0) \cup \{x_0\}) = A_0$. From the condition (iii) we have that $\theta(A_0) = 0$. This implies that $A_0 \in \mathbf{S}(X)$. So, $F_T \cap F_S \neq \emptyset$.

From the proof of Theorem 16.4.1 we have

Theorem 16.4.2. *Let $(X, \mathbf{S}(X), M^0)$ be a f.p.s. with common fixed point property and (θ, η) a compatible pair with $(X, \mathbf{S}(X), M^0)$. Let $Y \in F_\eta$ and $T, S \in M^0(Y)$ with $T(Y) \cup S(Y) \in Z$.*

We suppose that:

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $T \circ S = S \circ T$;

(iii) the pair (T, S) is θ-condensing pair.

Then, $F_T \cap F_S \neq \emptyset$.

Theorem 16.4.3. *Let $(X, \mathbf{S}(X), M^0)$ be a s.f.p.s. with common fixed point property and (θ, η) a compatible pair with $(X, \mathbf{S}(X), M^0)$. Let $Y \in \eta(Z)$ and $T, S \in M^0(Y)$. We suppose that:*

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $T \circ S = S \circ T$;

(iii) the pair (T, S) is θ-condensing pair.

Then, $(SF)_T \cap (SF)_S \neq \emptyset$.

Theorem 16.4.4. *Let $(X, \mathbf{S}(X), M^0)$ be a s.f.p.s. with common fixed point property and (θ, η) a compatible pair with $(X, \mathbf{S}(X), M^0)$. Let $Y \in F_\eta$ and $T, S \in M^0(Y)$ with $T(Y) \cup S(Y) \in Z$. We suppose that:*

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $T \circ S = S \circ T$;

(iii) the pair (T, S) is θ-condensing pair.

Then, $(SF)_T \cap (SF)_S \neq \emptyset$.

16.5 References

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Chapter 17

Fixed point property and coincidence property

17.1 Fixed point structure with the coincidence property

Definition 17.1.1. A f.p.s. $(X, S(X), M^0)$ on a nonempty set X is with the coincidence property iff

$$Y \in S(X),\ P, Q \in M^0(Y),\ P \circ Q = Q \circ P \ \Rightarrow\ C(P, Q) \neq \emptyset.$$

Example 17.1.1. Each f.p.s. with the common fixed point property is a f.p.s. with the coincidence property.

Example 17.1.2. Let $(X, S(X), M)$ be a f.p.s. with the coincidence property, in the case of singlevalued operators. For $Y \in S(X)$, let

$$M^0(Y) := \{P : Y \multimap Y \mid P(x) := \{p(x)\},\ p \in M(Y)\}.$$

Then $(X, S(X), M^0(Y))$ is a f.p.s. with the coincidence property.

It is an open problem to establish if a given f.p.s. is or not with the coincidence property. So, we have

Problem 17.1.1. Which are the f.p.s. with the coincidence property?

This problem has the following particular cases.

Problem 17.1.2. For which Banach spaces X the f.p.s. of Bohnenblust-Karlin, $(X, P_{cp,cv}(X), M^0)$ is with the coincidence property?

Problem 17.1.3. For which complete metric spaces, (X, d), the f.p.s. of multivalued contractions, $(X, P_{cl}(X), M^0)$, is with the coincidence property?

Problem 17.1.4. Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $P : Y \to P_{cl,cv}(X)$ a multivalued operator. We suppose that

(i) P is u.s.c.;

(ii) there exists $n_0 \in \mathbb{N}^*$ such that $\overline{P^{n_0}(Y)} \in P_{cp}(Y)$.

Does P have a fixed point?

Remark 17.1.1. For the case of singlevalued operators see Problem 8.4.1, Horn's conjecture and Schauder's conjecture.

We have

Theorem 17.1.1. *Let X be a nonempty set and $Z \subset P(X)$ such that $A, B \in Z$ implies $A \cup B \in Z$. Let $(X, S(X), M^0)$ be a f.p.s. with the coincidence property, (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $P, Q \in M^0(Y)$ such that $P \circ Q = Q \circ P$. We suppose that:*

(i) $\theta(A \cup B) = \max(\theta(A), \theta(B)),\ \forall\ A, B \in Z$;

(ii) P and Q are θ-condensing operators;

(iii) $F_P \in Z$ and $P(F_P) = F_P$.

Then $C(P, Q) \neq \emptyset$.

Proof. First of all we remark that condition (ii) implies that $F_P \neq \emptyset$ and (ii) and (iii) imply that $\theta(F_P) = 0$.

Let $A_1 := F_P$. From Lemma 16.1.3 there exists $A_0 \subset Y$ such that $A_1 \subset A_0$, $A_0 \in I(P) \cap I(Q) \cap F_\eta$ and $\eta(P(A_0) \cup Q(A_0) \cup F_P) = A_0$. If $\theta(A_0) > 0$, we have

$$\theta(A_0) = \theta[\eta(P(A_0) \cup Q(A_0) \cup F_P)] = \theta(P(A_0) \cup Q(A_0) \cup F_P)$$
$$= \theta(P(A_0) \cup Q(A_0)) = \max(\theta(P(A_0)), \theta(Q(A_0))) < \theta(A_0).$$

So, $\theta(A_0) = 0$. Hence, $A_0 \in F_\eta \cap Z_\theta$. But the pair (θ, η) is a compatible pair with $(X, S(X), M^0)$. From this we have that $A_0 \in S(X)$ and $P|_{A_0}, Q|_{A_0} \in M^0(A_0)$. These imply that $C(P, Q) \neq \emptyset$.

17.2 Coincidence producing singlevalued operators

Definition 17.2.1. Let $(X, S(X), M)$ be a l.f.p.s. on a nonempty set X and $Y \in S(X)$. An operator $p : Y \to Y$ is coincidence producing w.r.t. $(X, S(X), M)$ iff $q \in M(Y)$ implies that $C(p, q) \neq \emptyset$.

Remark 17.2.1. In 1964, W. Holsztynski gave the following definition: Let X and Y be two topological spaces. A continuous operator $p : X \to Y$ is universal iff $q \in C(Y, Y)$ implies that $C(p, q) \neq \emptyset$. In 1967, H. Schirmer suggested to use "coincidence producing" instead "universal".

Example 17.2.1. Let $(\mathbb{R}, P_{cp,cv}(\mathbb{R}), M)$ be the f.p.s. of Brouwer on $\mathbb{R}$. Let $Y \in P_{cp,cv}(\mathbb{R})$, i.e., Y is compact interval of $\mathbb{R}$. Then each surjective and continuous function $f : Y \to Y$ is a coincidence producing function w.r.t. $(\mathbb{R}, P_{cp,cv}(\mathbb{R}), M)$.

Example 17.2.2. Let X be a Banach space, $S(X) := \{X\}$ and $M(X) := \{f \in C(X, X) \mid \overline{f(x)} \in P_{cp}(X)\}$. Then $(X, S(X), M)$ is a l.f.p.s. By M. Furi, M. Martelli and A. Vignoli (1978) a continuous operator $g : X \to X$ is a strong surjection iff it is coincidence producing w.r.t. $(X, S(X), M)$. So, each example of strong surjection is an example of producing operator.

Example 17.2.3. (H. Schirmer (1966)). Consider on $\mathbb{R}^n$ the following l.f.p.s.: $S(\mathbb{R}^n) := \{I^n\}$, where $I = [-1, 1]$ and $M(I^n) = C(I^n, I^n)$. If

$g \in C(I^n, I^n)$ maps the boundary of I^n essentially onto itself, then g is a coincidence producing function.

Example 17.2.4. (L.J. Chu and C.Y. Lin (2002)).
Let $(X, P_{cp,cv}(X), M)$ be the f.p.s. of Tychonoff on a locally convex Hausdorff topological vector space. Let $Y \in P_{cp,cv}(X)$. Then a Vietoris operator $g : Y \to Y$ is producing w.r.t. $(X, P_{cp,cv}(X), M)$.

By definition $g \in C(Y, Y)$ is Vietoris iff $g(y) = y$ and $g^{-1}(y)$ is acyclic w.r.t. Čech homology with the coefficients in Q (i.e., $H_q(g^{-1}(y)) = 0$ for $g \geq 1$ and $H_0(g^{-1}(y)) \approx Q$), for each $y \in Y$.

17.3 Coincidence producing multivalued operators

Definition 17.3.1. Let $(X, S(X), M^0)$ be a l.f.p.s. of a nonempty set X and $Y \in S(X)$. An operator $P : Y \multimap Y$ is coincidence producing w.r.t. $(X, S(X), M^0)$ iff $Q \in M^0(Y)$ implies that $C(P, Q) \neq \emptyset$.

Example 17.3.1. Let X be a locally convex Hausdorff topological vector space and $(X, P_{cp,cv}(X), M^0)$ the f.p.s. of Glicksberg-Fan. Let $Y \in P_{cp,cv}(X)$ and $P : Y \to P_{cp}(Y)$ such that $P^{-1}(y) \in P_{cv}(Y)$, for each $y \in Y$. Then the operator P is coincidence producing w.r.t. $(X, P_{cp,cv}(X), M^0)$.

Indeed, the following result of F.E. Browder (1968) is well known:

Theorem of Browder. *Let X and E be two locally convex Hausdorff topological vector spaces, $Y \in P_{cp,cv}(X)$ and $K \in P_{cp,cv}(E)$. Let $P, Q : Y \multimap K$ be two multivalued operators. We suppose that:*

(i) $P : Y \to P_{cl,cv}(K)$ is u.s.c.;

(ii) $Q : Y \to P_{cp}(K)$ is such that $Q^{-1}(y)$ is a nonempty convex subset of Y.

Then, $C(P, Q) \neq \emptyset$.

We take in the above theorem $E = X$ and $K = Y$.

The following problem is one of the basic problems in the coincidence theory.

Problem 17.3.1. Let $(X, S(X), M^0)$ be a fixed point structure, $Y \in S(X)$ and $P : Y \multimap Y$. In which conditions the multivalued operator P is coincidence producing w.r.t. $(X, S(X), M^0)$?

17.4 References

For general results in the coincidence theory of multivalued operators see: L. Górniewicz [20], O. Hadžić [21], S. Hu and N.S. Papageorgiou [22], G. Isac [23], A. Muntean [26], S. Park [28], A. Petruşel [30], I.A. Rus [35]. See also:

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Chapter 18

Fixed point theory for retractible multivalued operators

18.1 Fixed point theorems

Let X be a nonempty set and $Y \in P(X)$. An operator $\rho : X \to Y$ is a set-retraction if $\rho|_Y = 1_Y$. An operator $T : Y \to P(X)$ is retractible w.r.t. the retraction $\rho : X \to Y$ iff $F_{\rho\circ T} = F_T$.

We have

Lemma 18.1.1. *Let $(X, S(X), M^0)$ be a l.f.p.s. on X. Let $Y \in S(X)$, $\rho : X \to Y$ a retraction and $T : Y \to P(X)$ an operator. We suppose that:*

(i) $\rho \circ T \in M^0(Y)$;

(ii) T is retractible w.r.t. ρ.

Then, $F_T \neq \emptyset$.

Proof. Since $(X, S(X), M^0)$ is a f.p.s. and $Y \in S(X)$, from (i) we have that $F_{\rho\circ T} \neq \emptyset$. From (ii) it follows that $F_T \neq \emptyset$.

Theorem 18.1.1. *Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ is with the intersection property;

(ii) T is retractible w.r.t. ρ and $\rho \circ T \in M(Y)$;

(iii) ρ is (θ, l)-Lipschitz $(l \in \mathbb{R}_+)$;

(iv) T is a strong (θ, φ)-contraction;

(v) the function $l\varphi$ is a comparison function.

Then, $F_T \neq \emptyset$.

Proof. From the condition (iii), (iv) and (v) the operator $\rho \circ T : Y \multimap Y$ is a strong (θ, l_φ)-contraction. By Theorem 14.1.1 we have that $F_{\rho \circ T} \neq \emptyset$. From the condition (ii) it follows $F_T \neq \emptyset$.

Theorem 18.1.2. *Let $(X, S(X), M^0)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is retractible w.r.t. ρ and $\rho \circ T \in M(Y)$;

(iii) ρ is $(\theta, 1)$-Lipschitz;

(iv) T is strong θ-condensing.

Then, $F_T \neq \emptyset$.

Proof. From the conditions (iii) and (iv) the operator $\rho \circ T : Y \multimap Y$ is strong θ-condensing. By Theorem 15.1.1, we have that $F_{\rho \circ T} \neq \emptyset$. From the condition (ii) it follows that $F_T \neq \emptyset$.

From Theorem 18.1.1 we have

Theorem 18.1.3. (I.A. Rus (1991)) *Let X be a Hilbert space and $T : \overline{B}(0; R) \to P_{cp}(X)$ a multivalued operator. We suppose that:*

(i) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(T(x), T(y)) \leq \varphi(d(x, y)), \ \forall\ x, y \in \overline{B}(0; R);$$

(ii) T is retractible onto $\overline{B}(0;R)$ w.r.t. the radial retraction $\rho : X \to \overline{B}(0;R)$.
Then, $F_T \neq \emptyset$.

18.2 Strict fixed point theorems

As in section 18.1, we have:

Lemma 18.2.1. *Let $(X, S(X), M^0)$ be a l.s.f.p.s. on X. Let $Y \in S(X)$, $\rho : X \to Y$ a retraction and $T : Y \to P(X)$ a multivalued operator. We suppose that:*

(i) $\rho \circ T \in M^0(Y)$;

(ii) $(SF)_{\rho \circ T} = (SF)_T$.

Then, $(SF)_T \neq \emptyset$.

Theorem 18.2.1 *Let $(X, S(X), M^0)$ be a s.f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ is with the intersection property;

(ii) $(SF)_{\rho \circ T} = (SF)_T$ and $\rho \circ T \in M(Y)$;

(iii) ρ is (θ, l)-Lipschitz;

(iv) T is a strong (θ, φ)-contraction;

(v) the function l_φ is a comparison function.

Then, $(SF)_T \neq \emptyset$.

Theorem 18.2.2. *Let $(X, S(X), M^0)$ be a s.f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ an operator and $\rho : X \to Y$ a retraction. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $(SF)_{\rho \circ T} = (SF)_T$ and $\rho \circ T \in M(Y)$;

(iii) ρ is $(\theta, 1)$-Lipschitz;

(iv) T is strong θ-condensing.

Then, $(SF)_T \neq \emptyset$.

18.3 References

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