THE MANEV-TYPE PROBLEMS: A TOPOLOGICAL VIEW

FERENC SZENKOVITS
Babes-Bolyai University,
Faculty of Mathematics and Computer Science,
Str. M. Kogalniceanu 1, RO-3400 Cluj-Napoca, Romania.
e-mail: fszenko@math.ubbcluj.ro

CRISTINA STOICA
University of Victoria,
Department of Mathematics and Statistics,
Victoria, B.C., V8W 3P4, Canada,
e-mail: cstoica@math.uvic.ca

VASILE MIOC
Astronomical Institute of the Romanian Academy,
Astronomical Observatory Bucharest,
Str. Cuţitul de Argint 5, RO-75212 Bucharest, Romania.
e-mail: vmioc@roastro.astro.ro; obsastr@mail.soroscj.ro

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Abstract

The Manev-type two-body problem (associated to a force function of the form $A/r + B/r^2$), which models several problems of nonlinear particle dynamics, is being tackled from the standpoint of topology. The corresponding mechanical system is fully characterized, and the first integrals of energy and angular momentum are pointed out. These integrals are used to settle the invariant manifolds and the bifurcation set for the whole allowed interplay among the field parameters, the total energy level, and the angular momentum. The orbits on each manifold are interpreted in terms of physical motion. Besides recovering motions characteristic to classical models, entirely new types of motion are found.

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1. INTRODUCTION

The two-body problem associated to a potential of the form $A/r+B/r^2$ (with r = distance between particles, and A, B = real parameters) has a long history. Newton was the first to consider it while attempting to explain the Moon's apsidal motion. In his *Principia* (Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2), he proved that a force deriving from such a potential (with $A > 0$, $B > 0$) entails a *precessionally elliptic* relative orbit. In other words, fixing one particle at the origin of an inertial frame, the other particle will move (with respect to this frame) along an ellipse whose focal axis rotates in the plane of motion. Later, this potential was tackled by Clairaut, who eventually abandoned it in favour of the Newtonian one.

One knows that the perihelion advance of the planets (especially that of Mercury) cannot be fully explained within the framework of Newton's classical law, even resorting to the perturbation theory. Both before and after relativity there were proposed lots of laws (many of them empirical) which usually answered this question, but failed to explain other issues (as the secular motion of the Moon's perigee). As to the general relativity, it succeeded in explaining well such phenomena from quantitative and qualitative point of view. Unfortunately, this powerful theory, which brought momentous contributions to physics and astronomy, is not of much help for celestial mechanics. All attempts to formulate a meaningful relativistic n-body problem have failed to provide valuable results.

A hard problem arose then: to find a model able to respond to the theoretical needs of celestial mechanics (by keeping the simplicity of the Newtonian model), and also to bring the necessary corrections such that near-collision orbits match theory with observations. A model was needed which could offer equally good justifications of observed celestial dynamical phenomena as the relativity, however without leaving the realm of classical mechanics.

Such a model is that based on the above $A/r + B/r^2$ potential; it was proposed in the twenties by the Bulgarian physicist G. Manev (Maneff in his papers written in French or German). In a few words, this is its story: in *Électricité et optique*, Poincaré (1901) had remarked that the Lorentzian theory concerning the dynamics of moving bodies (on which special relativity would be founded) did not fulfil the action-reaction principle. In 1903, M. Abraham introduced the notion of *quantity of electromagnetic movement*; on this basis, F. Hasenöhrl proved in 1907 the correctness of Lorentz's contraction principle. A year later, using the concept defined by Abraham, M. Planck stated a more general action-reaction principle (verified by the special relativity) from which Newton's third law followed as a theorem. Resorting to all these results, Manev pointed out the fact that, while used to classical mechanics, Planck's action-reaction principle naturally led to a potential of the type $A/r + B/r^2$. He considered this model to be a classical alternative to relativity, and tackled on this basis the well-known relativistic tests (Maneff 1924, 1925, 1930a,b). In the corresponding central force problem with unit mass for the "satellite" particle, Manev's law gives $A = \mu$, $B = 3\mu^2/(2c^2)$, where μ is the Newtonian gravitational constant G_N multiplied by the sum of masses, and c is the speed of light.

Fallen into oblivion for half a century, then pointed out by Hagihara (1975) as providing the same good theoretical approximations as the relativity (at the solar system level, at least), Manev's model was recently reconsidered by Diacu (1993). Mioc and Stoica (1995a,b) regularized the motion equations of the two-body problem by Sundman-type transformations, giving the general solution, while Diacu et al. (1995) obtained the analytic solution and the local flow near collision. Stoica and Mioc (1996a) showed that the physical motion with nonzero angular momentum can be represented by precessional conic sections.

The Manev potential for arbitrary *positive* values of A and B was also approached. Lacomba et al. (1991) applied the KAM theory to a perturbed potential of this kind, proving a crucial result: under a slight perturbation, not necessarily Hamiltonian, *most* invariant cylinders and tori are topologically preserved. Casasayas et al. (1993) computed the Melnikov integral associated to nonhyperbolic equilibria, while Diacu (1996) proved that Manev's case represents the only bifurcation of the flow among all quasihomogeneous potentials. Stoica and Mioc (1996b,c) and Mioc and Stoica (1997) depicted geometrically the problem in different phase planes, while Delgado et al. (1996) provided the complete analytic, geometric, and physical description of the global flow. The anisotropic Manev model (important for the understanding of the connections between classical and quantum mechanics) was investigated by Craig et al. (1996).

One could say: to determine the motion generated by the $A/r + B/r^2$ potential is an obsolete problem; once determined the exact analytic solution, to continue the study is useless. Indeed, the problem can be found as an exercise in classical textbooks as those of Moulton (1923, p.96, Problem 4) or Goldstein (1980, p.123, Problem 14). On the one hand, leaving aside the fact that the respective statement is incorrect in Goldstein's case or covers a very restricted area in Moulton's case, the above quoted results show how complex the problem is in reality. On the other hand, the analytic solution (e.g. Delgado et al. 1996; Diacu et al. 1996) has a form intricate enough, that hides the nice geometric properties of the model.

The goal of this paper is to tackle the Manev-type class of problems (for *any* values - positive, negative, or zero - of the field parameters A and B) from a topological standpoint. The importance of such an analysis for physics (and not only) is emphasized by the multitude of concrete situations modellable in this way. The motion in certain relativistic fields, truncating the negligible terms, is such a situation. To Fock's field correspond the expressions (Mioc 1994): $A = (2E^2 - 1)\mu$, $B = 3(\mu E/c)^2$ (where $E = 1 + h/c^2$, h being the total energy per unit mass of the orbiting particle), therefore $A > 0$, $B > 0$. For the field endowed with the Reissner-Nordström metric, we have $A > 0$, $B < 0$ ($A = \mu$, $B = -G_N Q^2/(8\pi\epsilon_0 c^2)$, where Q = electric charge of the field source, ϵ_0 = electric permitivity of vacuum). A photogravitational field is featured by $A = \mu - \sigma L/(4\pi mc)$ (where σ and m are the cross-section and the mass of the orbiting particle, while L stands for the luminosity of the central body), and B can be zero or nonzero as the gravitational component of the field is Newtonian or not. In its turn, A can be positive/zero/negative as the Newtonian part of the gravitational force is stronger than/equal to/weaker than the repulsive radiative force. If L is not constant (a variable star, for instance; see Saslaw 1978; Mioc and Radu 1992; Selaru et al. 1993), we face a perturbed Manev-type potential. The Coulombian field in the second approximation (Sommerfeld 1951; Belenkii 1981) also joins this model. Implications in astrophysics (Ureche 1995), mechanics (especially the case $A = 0$, $B > 0$; see Moser 1975; McGehee 1981; Diacu 1990), celestial mechanics and dynamical astronomy (e.g. Diacu et al. 1995; Stoica and Mioc 1996b,d; Mioc and Stoica 1997) are to be added to the above arguments.

The present paper attempts to provide a unifying point of view for several such problems of nonlinear particle dynamics (including the classical models of Kepler, Coulomb, Manev, and the force-free motion) by resorting to the tools of topology and to the program outlined by Smale (1970a,b) (see also Iacob 1973; Abraham and Marsden 1981). Fully characterizing the corresponding mechanical system, we use the first integrals of energy and angular momentum to settle the invariant manifolds and the bifurcation set for the whole interplay among field parameters, total energy, and angular momentum. The orbits on each manifold are interpreted in terms of physical motion. This provides a global geometric and physical picture of the Manev-type problems.

2. BASIC EQUATIONS

It is clear that the Manev-type two-body problem can be reduced to a central force problem (e.g. Arnold 1976). Within this framework, the motion of the particle is confined to a plane. We shall use polar coordinates (r, θ), and follow the treatment presented by Abraham and Marsden (1981, p.656).

The mechanical system with symmetry which describes the problem is (M, K, V, G), where:

$M = (0, \infty) \times S^1$ is the space of the coordinates (r, θ), regarded as a Riemannian manifold endowed with the metric

$$\left\langle (r_1, \theta_1, \dot{r}_1, \dot{\theta}_1), (r_2, \theta_2, \dot{r}_2, \dot{\theta}_2) \right\rangle = \dot{r}_1 \dot{r}_2 + r_1 r_2 \dot{\theta}_1 \dot{\theta}_2,$$

dots marking time-differentiation;

K is the kinetic energy of the metric above, whose expression on the cotangent bundle T^*M is

$$K(r, \theta, p_r, p_\theta) = (p_r^2 + p_\theta^2/r^2)/2, \tag{1}$$

p_r, p_θ denoting the momenta;

V is the potential energy, given by

$$V(r,\theta) = -A/r - B/r^2; \tag{2}$$

$G = SO(2) \cong S^1$ is the Lie group that acts on M by rotations ($\cong$ denoting isomorphism). Observe that G acts by isometries and leaves V invariant (cf. Abraham and Marsden 1981).

The Hamiltonian of the system is

$$H(r,\theta,p_r,p_\theta) = (p_r^2 + p_\theta^2/r^2)/2 - A/r - B/r^2. \tag{3}$$

The momentum mapping $J : T^*M \longrightarrow \mathbf{R}$ is given by $J(r,\theta,p_r,p_\theta) = p_\theta$, and is invariant under the action of G.

Consider $\mathbf{x} = (r,\theta) \in M$ and the mapping $J_\mathbf{x} : T^*_\mathbf{x}M \longrightarrow \mathbf{R}$. The expression $J_\mathbf{x} : (p_r,p_\theta) \longmapsto p_\theta$ of this mapping shows that $J_\mathbf{x}$ is surjective for all $\mathbf{x} \in M$. In other words,

$$\Lambda = \{\mathbf{x} \in M \mid \; J_\mathbf{x} : T^*_\mathbf{x}M \longrightarrow \mathbf{R} \text{ is not surjective}\} = \emptyset.$$

Note that $dJ = dp_\theta$, therefore J has no critical points on T^*M.

The problem admits the first integrals of energy and angular momentum (e.g. Delgado et al. 1996; Diacu et al. 1996), respectively:

$$H(r,\theta,p_r,p_\theta) = K(r,\theta,p_r,p_\theta) + V(r) = h, \tag{4}$$

$$J(r,\theta,p_r,p_\theta) = p_\theta = C, \tag{5}$$

where h and C stand for the integration constants of energy and angular momentum.

3. EFFECTIVE POTENTIAL ENERGY

Eliminating p_θ between (3)+(4) and (5), one gets

$$p_r^2 = 2(h - V_C), \tag{6}$$

in which

$$V_C\,(r) = V(r) + C^2/(2r^2) = -A/r + (C^2 - 2B)/(2r^2) \tag{7}$$

denotes the so-called *effective potential energy.*

Settled the constant angular momentum C, one sees by (6) that the real motion is possible only in the domains $h \geq V_C\,(r)$, where h is a fixed total energy level. Concretely, using (6) and (7), and denoting by $r_{cr} = (C^2 - 2B)/A$ the point where $dV_C\,(r)\,/dr = 0$, and

$$h_{cr} = V_C(r_{cr}) = -A^2/[2(C^2 - 2B)],$$

the real motion is *not* possible in the cases

$$\begin{gathered}\{A > 0,\ C^2 - 2B > 0,\ h < h_{cr}\}\,,\ \{A = 0,\ C^2 - 2B > 0,\ h \leq 0\}\,,\\ \{A = 0,\ C^2 - 2B = 0,\ h < 0\}\,,\ \{A < 0,\ C^2 - 2B \geq 0,\ h \leq 0\}\,.\end{gathered}$$

The graph of the function $V_C = V_C\,(r)$ for the whole allowed interplay among A, B, and C is plotted in Figure 1 (see also Stoica 1995).

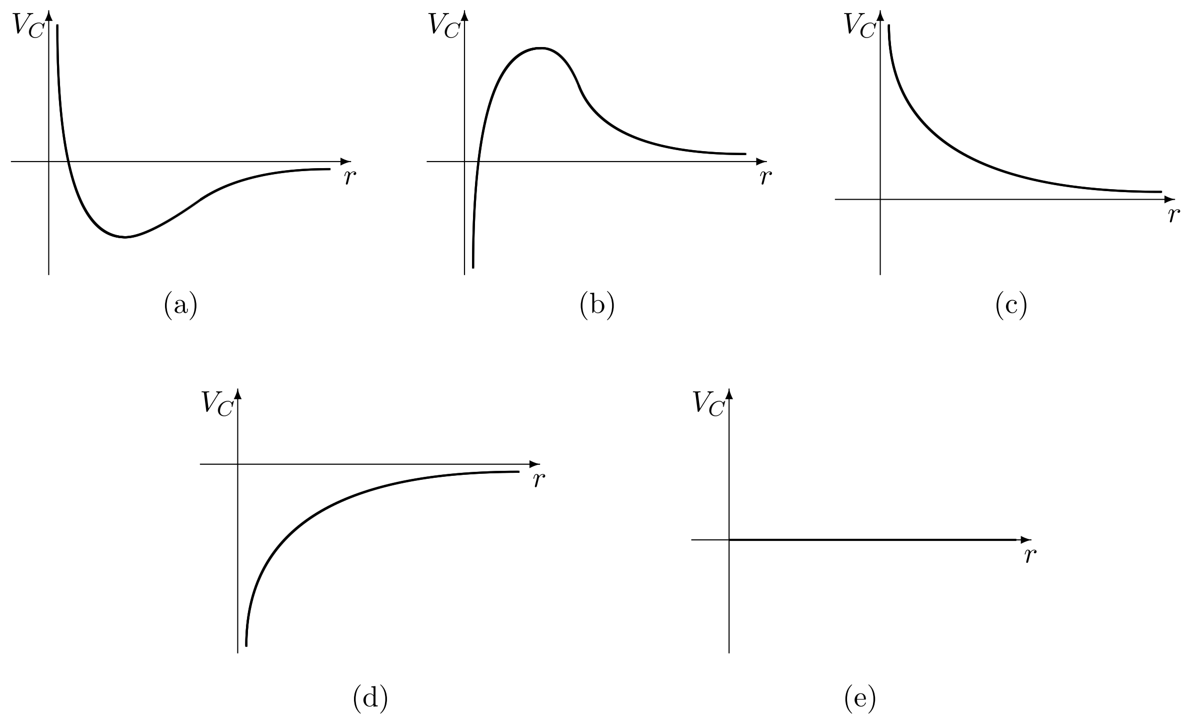

Figure 1: The graph of the effective potential energy: (a) $A > 0$, $C^2 - 2B > 0$; (b) $A < 0$, $C^2 - 2B < 0$; (c) $A = 0$, $C^2 - 2B > 0$, or $A < 0$, $C^2 - 2B \geq 0$; (d) $A > 0$, $C^2 - 2B \leq 0$, or $A = 0$, $C^2 - 2B < 0$; (e) $A = 0$, $C^2 - 2B = 0$.

4. GEOMETRY OF THE MOTION

To study the motion, we use the invariant manifolds $I_{h,C} = (H \times J)^{-1}(h, C)$, whose defining equations are (4) and (5). Obviously, the topological type of $I_{h,C}$ depends on the condition $h \geq V_C(r)$. Using the graphs of Figure 1, and observing their significance as regards the allowed domains for r to have real motion, we are able to identify the invariant manifolds diffeomorphic ($\approx$) with $I_{h,C}$ on which the phase curves lie. All possible cases are synthesized in Table 1.

The symbols used in Table 1 are:

$0 \to 0$: orbits ejecting from collision and then tending back to collision;

$0 \to \infty$: orbits ejecting from collision and tending to infinity;

$\infty \to 0$: orbits coming from infinity and tending to collision;

$\infty \to \infty$: orbits coming from infinity and then tending back to infinity;

SE: stable equilibrium;

UE: unstable equilibrium;

P/QP: periodic/quasiperiodic orbits.

To have a deeper insight in the geometry of the problem, les us establish and plot the bifurcation sets $\Sigma_{H \times J}$, defined as the sets of couples $(h, C) \in \mathbf{R}^2$ for which the energy-momentum mapping $H \times J$ fails to be locally trivial, in other words, those points in whose neighbourhood the topological type of the invariant manifolds is changing (see e.g. Abraham and Marsden 1981). For this purpose we shall use Table 1.

Table 1: The invariant manifolds $I_{h,C}$ and the motions on them.

Conditions			$I_{h,C} \approx$	**Manifold**	**Type of motion**
$A > 0$	$C^2 > 2B$	$h < h_{cr}$	$\emptyset$	–	–
		$h = h_{cr}$	S^1	circle	SE
		$h_{cr} < h < 0$	$S^1 \times S^1$	torus	P, QE
		$h \geq 0$	$S^1 \times \mathbf{R}$	cylinder	$\infty \to \infty$
	$C^2 \leq 2B$	$h < 0$	$S^1 \times \mathbf{R}$	cylinder	$0 \to 0$
		$h \geq 0$	$S^0 \times S^1 \times \mathbf{R}$	two disjont cylinders	$0 \to \infty$ $\infty \to 0$
$A = 0$	$C^2 > 2B$	$h \leq 0$	$\emptyset$	–	–
		$h > 0$	$S^1 \times \mathbf{R}$	cylinder	$\infty \to \infty$
	$C^2 = 2B$	$h < 0$	$\emptyset$	–	–
		$h = 0$	$S^1 \times \mathbf{R}$	cylinder	SE
		$h > 0$	$S^0 \times S^1 \times \mathbf{R}$	two disjont cylinders	$0 \to \infty$ $\infty \to 0$
	$C^2 < 2B$	$h < 0$	$S^1 \times \mathbf{R}$	cylinder	$0 \to 0$
		$h \geq 0$	$S^0 \times S^1 \times \mathbf{R}$	two disjont cylinders	$0 \to \infty$ $\infty \to 0$
$A < 0$	$C^2 \geq 2B$	$h \leq 0$	$\emptyset$	–	–
		$h > 0$	$S^1 \times \mathbf{R}$	cylinder	$\infty \to \infty$
	$C^{<}2B$	$h \leq 0$	$S^1 \times \mathbf{R}$	cylinder	$0 \to 0$
		$0 < h < h_{cr}$	$S^0 \times S^1 \times \mathbf{R}$	two disjoint cylinders	$0 \to 0$ $\infty \to \infty$
		$h = h_{cr}$	$(S^1 \times \mathbf{R}) \cup (S^1 \times \mathbf{R})$ $(S^1 \times \mathbf{R}) \cap (S^1 \times \mathbf{R})$ $\approx S^1$	two cylinders whose intersection is a circle	$0 \to$ UE, UE $\to 0$ $\infty \to$UE, UE$\to \infty$ UE
		$h > h_{cr}$	$S^0 \times S^1 \times \mathbf{R}$	two disjont cylinders	$0 \to \infty$ $\infty \to 0$

Consider $A > 0$. If $B > 0$, the bifurcation set is (Figure 2a):

$$\Sigma_{H\times J} = \Sigma'_{H\times J} \cup \{(h, C) \in \mathbf{R}^2 \mid |C| = \sqrt{2B}\} \cup \{(h, C) \in \mathbf{R}^2 \mid h = 0\}, \tag{8}$$

where $\Sigma'_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h = h_{cr}, |C| > \sqrt{2B}\}$ is the set of critical points for $H \times J$. If $B = 0$, we have (Figure 2b):

$$\Sigma_{H\times J} = \Sigma'_{H\times J} \cup \{(h, C) \in \mathbf{R}^2 \mid C = 0\} \cup \{(h, C) \in \mathbf{R}^2 \mid h = 0\},$$

with $\Sigma'_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h = h_{cr}, |C| \neq 0\}$. Finally, if $B > 0$, the bifurcation set, represented in Figure 2c, is

$$\Sigma_{H\times J} = \Sigma'_{H\times J} \cup \{(h, C) \in \mathbf{R}^2 \mid h = 0\},$$

in which $\Sigma'_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h = h_{cr}\}$.

Consider now $A = 0$. If $B > 0$, the bifurcation set is (Figure 3a):

$$\Sigma_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid |C| = \sqrt{2B}\} \cup \{(h, C) \in \mathbf{R}^2 \mid h = 0\},$$

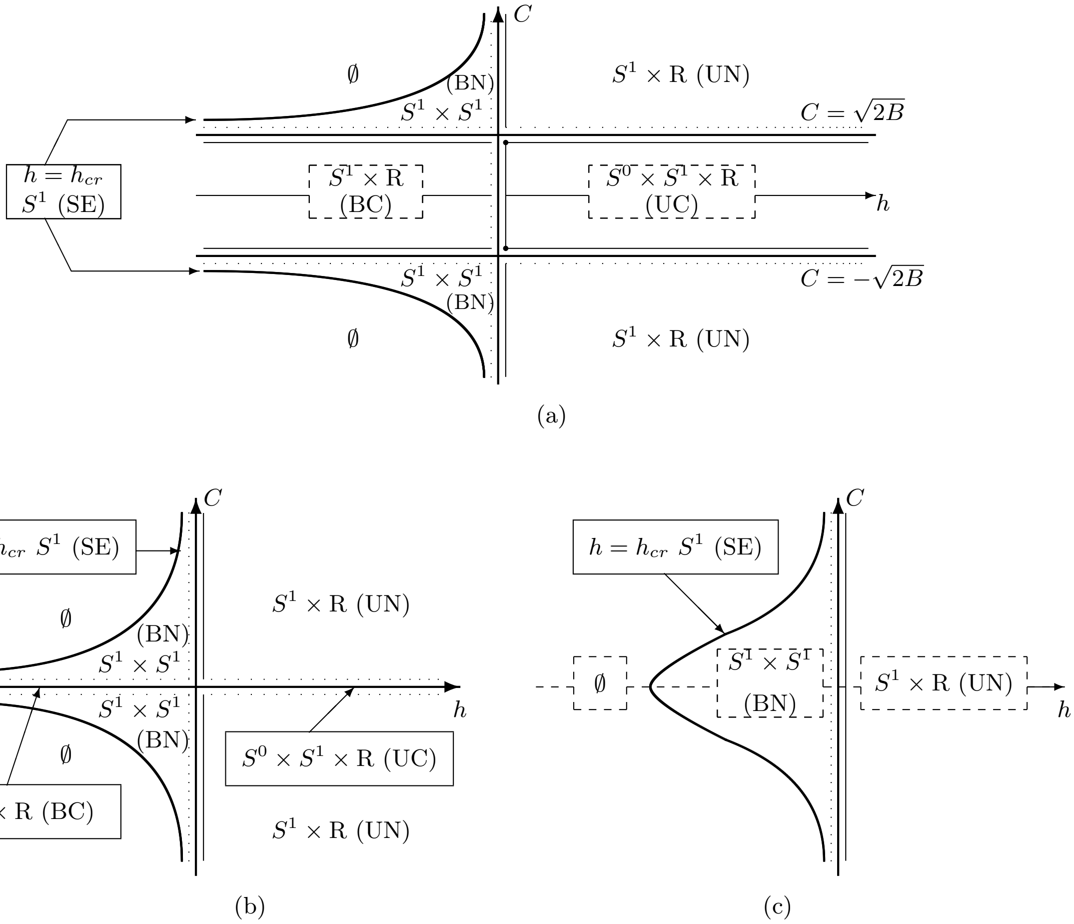

Figure 2: The bifurcation set for $A > 0$: (a) $B > 0$; (b) $B = 0$; (c) $B < 0$.

For $B = 0$, the bifurcation set is plotted in Figure 3b and reads

$$\Sigma_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h \geq 0, C = 0\} \cup \{(h, C) \in \mathbf{R}^2 \mid h = 0\},$$

while for $B < 0$ we have (Figure 3c):

$$\Sigma_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h = 0\}. \tag{9}$$

Lastly, consider $A < 0$. For the case $B > 0$, the bifurcation set is represented in Figure 4. Its expression is given by (8), but now the set of critical points reads $\Sigma'_{H\times J} = \{(h, C) \in \mathbf{R}^2 \mid h = h_{cr}, |C| < \sqrt{2B}\}$. If $B \leq 0$, the bifurcation set has the expression (9), and is plotted in Figure 3c, too.

To better understand the problem, the next step must necessarily consist of the interpretation of the above geometric portrait (Table 1 and Figures 2-4) in terms of physical motion. As a matter of fact, the general types of motion were also marked on Figures 2-4, under the abbreviations: ∅, BC, BN, UC, UN, BCas, UNas, SE, UE. Their significances will become clear in Section 5, where a detailed description of the features of each physical motion will be provided.

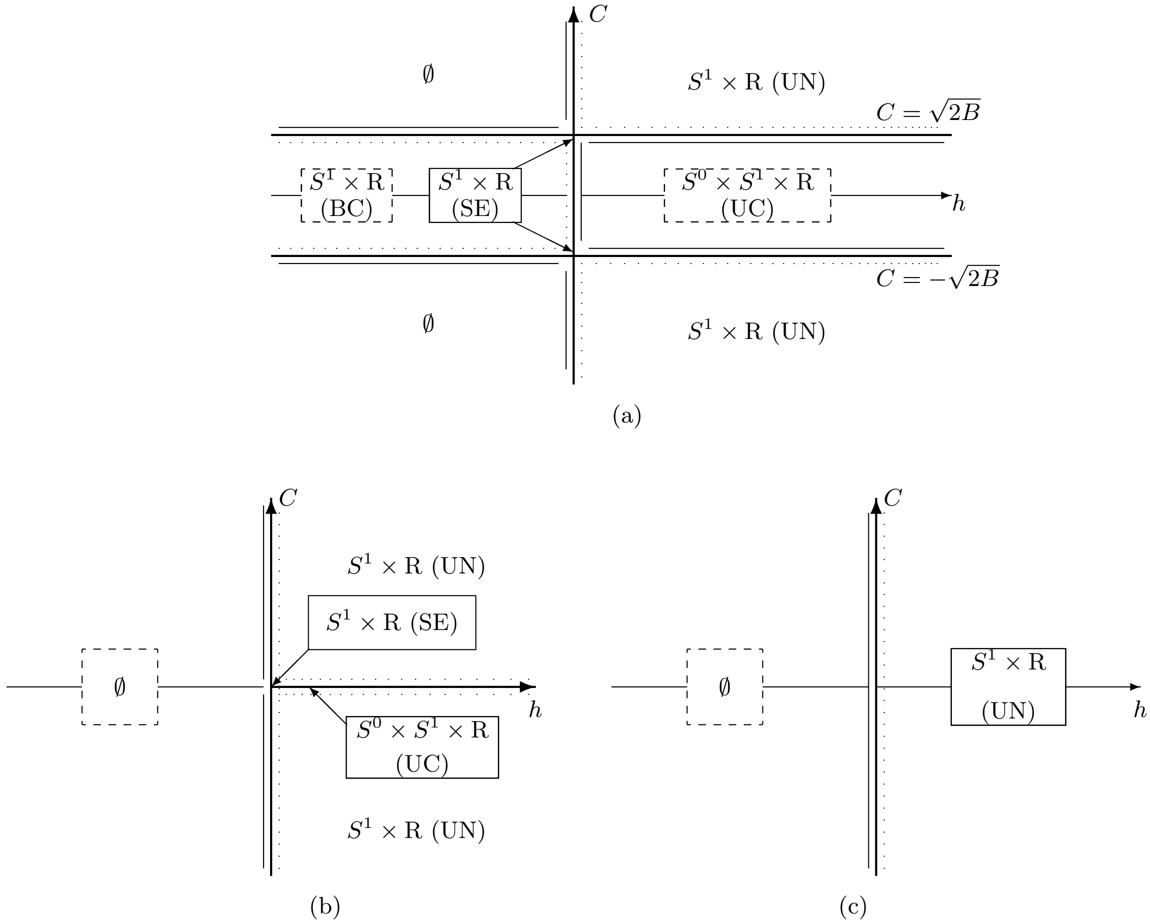

Figure 3: The bifurcation set for: (a) $A = 0$, $B > 0$; (b) $A = 0$, $B = 0$; (c) $A = 0$, $B < 0$, or $A < 0$, $B \leq 0$.

5. PHYSICAL INTERPRETATION

First of all, let us clear up the nature of the physical motion represented by the phase curves lying on the invariant manifolds $I_{h,C}$. It is obvious that all physical trajectories are confined to a plane, because we face a central force problem. As proved by Stoica and Mioc (1996b), Delgado et al. (1996), Diacu et al. (1996), or Mioc and Stoica (1997), if the $I_{h,C}$ manifolds are diffeomorphic with cylinders, then the helices winding around the cylinders physically mean spiral orbits ($C \neq 0$), except the cases with $B = 0$; the generatrices of the cylinders represent radial motion ($C = 0$). If the $I_{h,C}$ manifolds are diffeomorphic with tori, then the motion has a cyclic character: the closed curves winding around the tori represent periodic orbits, the unclosed curves which fill densely the tori mean quasiperiodic trajectories (in both situations $C \neq 0$), whereas the parallels on the tori correspond to radial librations ($C = 0$). Finally, the equilibria ($I_{h,C} \approx S^1$) physically mean circular motion (if $C \neq 0$) or rest (if $C = 0$).

Also, we must emphasize that all physical spiral trajectories can be regarded as precessional conic

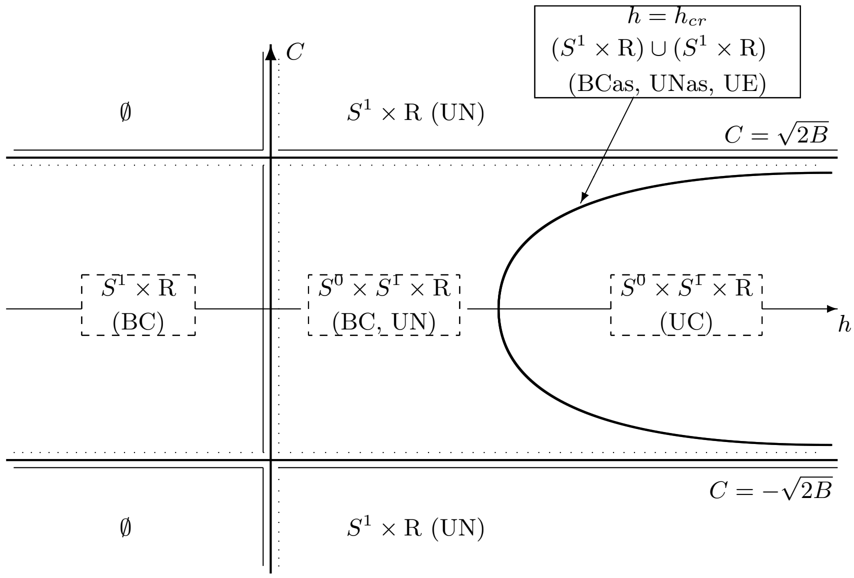

Figure 4: The bifurcation set for $A < 0$, $B > 0$.

sections, as Stoica and Mioc (1996a) showed. In other words, the curves for which $h < 0$ ($h = 0$, $h > 0$) physically represent precessional ellipses (parabolas, hyperbolas, respectively).

Another remark concerns the unbounded orbits (of the type $0 \to \infty$, $\infty \to 0$, $\infty \to \infty$,$\infty \to$UE, UE$\to \infty$). The asymptotic velocity at infinity is zero for precessional parabolas, and is positive (and, by (3) and (4), equal to $\sqrt{h}$) for precessional hyperbolas.

Now, resorting to both Table 1 and Figures 1-4, we are able to depict the physical motion for the whole interplay among the parameters of the field, the total energy, and the angular momentum. The discussion will be performed case by case.

5.1. Case $A > 0$

Consider firstly that $.B > 0$; this situation models Manev's field or that of Fock, as showed in Section 1.

For $h < 0$ we differentiate two situations, according to the values of C. If $|C| \leq \sqrt{2B}$, the phase curves correspond physically to solutions which eject from collision and end in collision (the motion is bounded and collisional: hereafter BC). The particle moves radially if $C = 0$, and spirally else. In the latter case ($C \neq 0$) we meet the so-called *black hole effect* (nonrectilinear collision/ejection), impossible in the Newtonian field: the particle spirals around the centre, performing infinitely many rotations immediately before collision (after ejection). This effect was pointed out by Wintner (1941) or McGehee (1981) for the homogeneous potential with $A = 0$, $B > 0$, and by Diacu et al. (1995) for the Manev-type case with $A > 0$, $B > 0$. Delgado et al. (1996) have shown that for the Manev-type model the set of initial conditions leading to collision has *positive* measure.

The second situation with $h < 0$ is $|C| > \sqrt{2B}$. Here we also distinguish some cases. If $h < h_{cr}$ (< 0), the real motion is impossible ($\emptyset$ in Figures 2-4; see Table 1). If $h = h_{cr}$, the motion is circular and stable. If $h > h_{cr}$, the motion is periodic or quasiperiodic. The quasiperiodic solutions (impossible in the Newtonian two-body problem) correspond to precessional ellipses filling densely an annulus, whereas the periodic solutions are precessional orbits that close after a finite number of rotations. Delgado et al. (1996) and Diacu et al. (1996) have shown that, for a fixed C, most of the values taken by the frequency ratio (as continuous function of C) are irrational, hence - except for a set of measure zero of tori foliated by periodic orbits - most of the tori are generated by quasiperiodic orbits. In other words, the bounded noncollisional orbits (hereafter BN) are in general quasiperiodic.

For $h \geq 0$, if $|C| \leq \sqrt{2B}$, the phase curves represent unbounded and collisional solutions (hereafter UC) that eject from collision and tend to infinity, or conversely. The motion is radial for $C = 0$, and spiral (also presenting the black hole effect) if $C \neq 0$. In case $|C| > \sqrt{2B}$, the motion is unbounded and noncollisional (hereafter UN): the particle comes from infinity, reaches a minimum distance from the centre, then tends back to infinity without encountering collision. If $h = 0$, the trajectories are precessional parabolas; if $h > 0$, they are precessional hyperbolas.

Consider now that $B = 0$; we shall recover the classical Newtonian field and the Keplerian motion. A more general situation is that of a photogravitational field (with Newtonian gravitational component) in which the attraction exceeds the repulsive radiative force.

For $h < 0$, if $C = 0$, the motion is radial of the type BC; if $C \neq 0$, the motion is curvilinear of the type BN. In the latter case, if $h < h_{cr}$ (< 0), the real motion is not possible; if $h = h_{cr}$, the orbits are circular and stable; if $h > h_{cr}$, the particle moves on a Keplerian ellipse.

For $h \geq 0$, if $C = 0$, the trajectories are radial of the type UC (ejection-escape or infinity-collision); f $C \neq 0$, they are Keplerian parabolas ($h = 0$) or hyperbolas ($h > 0$), belonging to the type UN.

Lastly, put $B < 0$; this situation models the field endowed with the Reissner-Nordström metric. From an abstract standpoint of classical mechanics, we may interpret this case as a central field consisting of the sum of an inverse-square attracting force and an inverse-cubic repulsive force.

For $h < h_{cr}$ (< 0), no real motion is possible. For $h = h_{cr}$, the equilibrium solutions SE (see Table 1) physically represent stable circular orbits if $C \neq 0$, and stable rest if $C = 0$. For $h_{cr} < h < 0$, the motion is of the type BN: if $C \neq 0$, we have periodic or quasiperiodic orbits (but here the pericentre shift of the precessional ellipses is performed in the opposite sense as compared to that corresponding to $A > 0$, $B > 0$; see Stoica (1997)); if $C = 0$, we face a new type of motion, namely radial librations of the particle (which moves back and forth without possibility of escape or collision). For $h \geq 0$, the motion is UN: the particle comes from infinity and then tends back to infinity. This collisionless motion is radial if $C = 0$, and spiral if $C \neq 0$. Most of these types of motion are new: for radial collisionless motion, radial librations, or rest, the assertion is obvious; as to the precessional parabolas/hyperbolas with $C \neq 0$, they turn their convexity on the centre (unlike the case $A > 0$, $B > 0$, in which the concavity is turned on the centre).

5.2. Case $A = 0$

The first situation we tackle is $B > 0$. We recover the inverse-cubic force case considered by Newton or, much later, by Moser (1975) or McGehee (1981).

Let h be negative. If $|C| < \sqrt{2B}$, the physical motion is of type BC: radial if $C = 0$, spiral (with black hole effect) if $C \neq 0$. If $|C| \geq \sqrt{2B}$, then the real motion is impossible (see Table 1).

Let h be zero. If $|C| < \sqrt{2B}$, the orbits are of type UC: radial for $C = 0$, and spiral (with black hole effect) else; in other words, the particle ejects from collision and tends to infinity (radially or spirally), or conversely. If $|C| = \sqrt{2B}$, the motion is circular (of any radius) and stable, while for $|C| > \sqrt{2B}$ no real motion is possible.

Let, finally, h be positive. For $|C| \leq \sqrt{2B}$, the type of motion is UC, too (see the previous case, the only difference consisting of the asymptotic velocity at infinity), radial if $C = 0$, spiral (with black hole effect) if $C \neq 0$. For $|C| > \sqrt{2B}$, the motion is spiral, of the type UN.

The situation $B = 0$ allows us to recover the trivial case of the force-free field. The centre continues to exist, but it does no longer act on the particle.

If $h < 0$, there is no real motion (see Table 1). If $h = 0$, we distinguish two situations: for $C = 0$ the particle is at a stable rest; for $C \neq 0$ the real motion is impossible. The case $h > 0$ also bifurcates. For $C = 0$ the motion is radial of the type UC: ejection-escape or infinity-collision; for $C \neq 0$ the motion is UN: the particle comes from infinity, in *rectilinear but nonradial motion*, and follows its collisionless path, tending finally to infinity.

The last situation in this subsection is $B < 0$. Table 1 shows that for $h \leq 0$ there is no real motion. For $h > 0$, the physical motion is of type UN: the particle comes from infinity and then tends back to infinity, radially (if $C = 0$) or spirally (if $C \neq 0$), without encountering collision.

5.3. Case $A < 0$

In this case, the richest situation is $B > 0$, which will present some features which had not appeared yet in the cases discussed till now. From the viewpoint of classical mechanics, we may interpret this case as a central field consisting of the sum of an inverse-square repulsive force and an inverse-cubic attracting force.

Suppose, for the beginning, that $h \leq 0$. If $|C| \leq \sqrt{2B}$, the motion is of the type BC: radial for $C = 0$, spiral (with black hole effect) for $C \neq 0$. If $|C| > \sqrt{2B}$, the real motion is not possible.

The case $0 < h < h_{cr}$ exhibits new features. For $|C| < \sqrt{2B}$, there are two coexisting types of motion: BC ($0 \to 0$) and UN ($\infty \to \infty$), radial or spiral (as $C = 0$ or $C \neq 0$, respectively). For $|C| \geq \sqrt{2B}$, the motion is of the type UN: the particle spirals on precessional hyperbolas, coming from infinity and then tending back to infinity.

The case $h = h_{cr}$ is the most rich as regards the kinds of orbits. If $|C| < \sqrt{2B}$, we encounter a lot of new situations. There are trajectories which start from collision and tend asymptotically to an unstable circle, or conversely (BCas on Figure 4); there also are trajectories which come from infinity and tend asymptotically to the same unstable circle (UNas on Figure 4); lastly, there is the unstable circle itself. All these motions cannot be encountered within the Newtonian framework; they are radial if $C = 0$ (with unstable rest corresponding to UE; see Table 1), and spiral (except the unstable circular orbit) if $C \neq 0$. Finally, if $|C| \geq \sqrt{2B}$, the physical orbits are precessional hyperbolas which come spiralling from infinity and then tend back to infinity.

The remaining case $h > h_{cr}$ is more simple. If $|C| < \sqrt{2B}$, the physical motion is of the type UC: radial for $C = 0$, spiral (with black hole effect) for $C \neq 0$. If $|C| \geq \sqrt{2B}$, the motion is UN and spiral.

The situation with $B \leq 0$ models the Coulombian field in different approximations; in the case $B = 0$, a photogravitational field (with Newtonian gravitational component) in which the repulsive radiative force exceeds the attraction is modelled, too.

For $h \leq 0$ there is no real motion. For $h > 0$, the motion is of the type UN: the particle comes from infinity and then tends back to infinity (radially if $C = 0$, or spirally, on a precessional hyperbola with the convexity turned on the centre, if $C \neq 0$).

6. CONCLUDING REMARKS

The Manev-type class of problems, topologically and physically described here, includes many physical (and astronomical) models, as particular cases, trying to build unifying bridges between them. Studying all possible motions within this framework, we recover known issues, characteristic to classical fields (Newtonian, Coulombian, radiative, force-free). But what is the most interesting is the fact that we found a lot of types of motion which cannot be met in the classical models.

Resuming, we survey the general features of the Manev-type model:

- the occurence of the black hole effect (collision/ejection with nonzero angular momentum);

- the set of initial data leading to collision has positive measure;

- the existence of quasiperiodic orbits, and the fact that, among the initial conditions corresponding to bounded noncollisional motion, the set leading to such orbits has positive measure;

- the motion with $B < 0$ is noncollisional (result also proved by Saari (1974), but within another framework and by a method different from that used by Delgado et al. (1996) or Diacu et al. (1996));

- the real motion with $h < 0$ is always bounded (the particle cannot escape).

Besides these characteristics, we have to mention some other types of motion which do not correspond to the classical Keplerian model. Such issues are: radial motion of the type $\infty \to \infty$; fixed or precessional conic sections with the convexity turned on the centre; rest. However they can be recovered in classical fields other than the Newtonian one. But there also are features as: radial librations, unstable circular orbits and trajectories asymptotic to them, unstable rest and radial motion asymptotic to it, coexistence of bounded collisional and unbounded noncollisional orbits for the same angular momentum and the same energy level; all these are characteristic only to the Manev-type model.

To end, we emphasize once again the usefulness of approaching the Manev-type problems (which bring a unifying view for many concrete physical problems), as well as the usefulness of the powerful tool of topology in this approach.

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