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A NOTE ON GENERALIZED INTERIORS OF SETS

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Abstract. This article familiarizes the reader with generalized interiors of sets, as they are extremely useful when formulating optimality conditions for problems defined on topological vector spaces. Two original results connecting the quasi-relative interior and the quasi interior are included.

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1. INTRODUCTION

Let X be a topological vector space and let X^* be the topological dual space of X. Given a linear continuous functional $x^* \in X^*$ and a point $x \in X$, we denote by $\langle x^*, x\rangle$ the value of x^* at x.

The **normal cone** associated with a set $M \subseteq X$ is defined by

$$N_M(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \text{ for all } y \in M\} & \text{if } x \in M \\ \emptyset & \text{otherwise.} \end{cases}$$

Given a nonempty cone $C \subseteq X$, its **dual cone** is the set

$$C^+ := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in C\}.$$

We also use the following subset of C^+:

$$C^{+0} := \{x^* \in X^* : \langle x^*, x\rangle > 0 \text{ for all } x \in C\backslash\{0\}\}. \tag{1}$$

2. GENERALIZED INTERIORS OF SETS. DEFINITIONS AND BASIC PROPERTIES

Interior and generalized interior notions of sets are highly important when tackling optimization problems, as they are often used in formulating regularity conditions.

Let X be a nontrivial vector space, and let $M \subseteq X$ be a set. The **algebraic interior** of M is defined by

$$\operatorname{core} M := \left\{ \begin{array}{ll} x \in X : & \text{for each } y \in X \text{ there exists } \delta > 0 \text{ such that} \\ & x + \lambda y \in M \text{ for all } \lambda \in [0, \delta] \end{array} \right\}.$$

The **intrinsic core** of M is defined by

$$\operatorname{icr} M := \left\{ \begin{array}{ll} x \in X : & \text{for each } y \in \operatorname{aff}(M - M) \text{ there exists } \delta > 0 \\ & \text{such that } x + \lambda y \in M \text{ for all } \lambda \in [0, \delta] \end{array} \right\}$$

(see HOLMES G. B. [11], ZĂLINESCU C. [16]). From the definitions mentioned abobe it follows that

$$\operatorname{core} M \subseteq M \text{ and } \operatorname{core} M \subseteq \operatorname{icr} M.$$

When M is a convex set, then we have:

$$\operatorname{core} M = \{x \in M : \operatorname{cone}(M - x) = X\}, \text{ and}$$

$$\operatorname{icr} M = \{x \in M : \operatorname{cone}(M - x) \text{ is a linear subspace of } X\}.$$

When M is a convex cone, then the set $\{0\} \cup \operatorname{core} M$ is also a convex cone and

$$\operatorname{core} M = M + \operatorname{core} M.$$

Let X be a topological vector space, and let $M \subseteq X$ be a set. We denote by $\operatorname{int} M$ the **interior** of M and by $\operatorname{cl} M$ the **closure** of M. When M is a convex set, both sets $\operatorname{int} M$ and $\operatorname{cl} M$ are convex. Moreover, if M is convex and $\operatorname{int} M \neq \emptyset$, then we have

$$\operatorname{int} M = \operatorname{int}(\operatorname{cl} M) \text{ and } \operatorname{cl} M = \operatorname{cl}(\operatorname{int} M).$$

For each set $M \subseteq X$ it holds $\operatorname{int} M \subseteq \operatorname{core} M$. The equality $\operatorname{int} M = \operatorname{core} M$ is true if M is a convex set and one of the following conditions is fulfilled: $\operatorname{int} M \neq \emptyset$; X is a Banach space and M is closed; X is finite dimensional.

Consider now a separated topological vector space X, and let $M \subseteq X$ be a set. The **strong quasi-relative interior** of M is

$$\operatorname{sqri} M := \begin{cases} \operatorname{icr} M & \text{if } \operatorname{aff} M \text{ is closed} \\ \emptyset & \text{otherwise} \end{cases}$$

(see BORWEIN J. M., JEYAKUMAR V., LEWIS A. S. and WOLKOWICZ H. [2], JEYAKUMAR V. and WOLKOWICZ H.[12], ZĂLINESCU C. [16]). When the set M is convex, then

$$\operatorname{sqri} M = \{x \in M : \operatorname{cone}(M - x) \text{ is a closed linear subspace of } X\}.$$

The **quasi-relative interior** of an arbitrary set $M \subseteq X$ is

$$\operatorname{qri} M := \big\{x \in M : \operatorname{cl} \operatorname{cone}(M - x) \text{ is a linear subspace of } X\big\}$$

(see BORWEIN J. M. and LEWIS A. S. [3]).

The **quasi interior** of a set $M \subseteq X$ is tightly connected to the quasi-relative interior and appeared in the literature prior to it. It is defined by

$$\operatorname{qi} M := \big\{x \in M : \operatorname{cl} \operatorname{cone}(M - x) = X\big\}.$$

When M is a convex set, then

$$\operatorname{qi} M \subseteq \operatorname{qri} M \text{ and } \operatorname{qri}\{x\} = \{x\} \text{ for all } x \in X.$$

Moreover, whenever $\operatorname{qi} M \neq \emptyset$, then $\operatorname{qi} M = \operatorname{qri} M$.

The following chain of inclusions holds for an arbitrary set $M \subseteq X$:

$$\operatorname{core} M \subseteq \operatorname{sqri} M \subseteq \operatorname{icr} M.$$

When X is a separated locally convex space, and $M \subseteq X$ is a convex set, then the following chains of inclusions hold:

$$\text{int}\, M \subseteq \text{core}\, M \subseteq \text{sqri}\, M \subseteq \text{icr}\, M \subseteq \text{qri}\, M \subseteq M; \tag{2}$$

$$\text{int}\, M \subseteq \text{core}\, M \subseteq \text{qi}\, M \subseteq \text{qri}\, M \subseteq M. \tag{3}$$

When $\text{int}\, M \neq \emptyset$, then all the generalized interior notions in (2) and (3) collapse in equality to $\text{int}\, M$.

Let us consider the case when $X = \mathbb{R}^n$, with $n \in \mathbb{N}$, and let $M \subseteq \mathbb{R}^n$ be a set. Then the **relative interior** of M is defined by

$$\text{ri}\, M := \{x \in \text{aff}\, M : \text{ there exists } \varepsilon > 0 \text{ such that } \mathcal{B}(x, \varepsilon) \cap \text{aff}\, M \subseteq M\},$$

where $\mathcal{B}(x, \varepsilon)$ is the closed ball centered at x with radius ε, in the Euclidian norm. In this finite dimensional setting, when M is a convex set, then the following equalities hold:

$$\text{qri}\, M = \text{sqri}\, M = \text{icr}\, M = \text{ri}\, M \tag{4}$$

(according to Borwein J. M. and Lewis A. S. [3], Gowda M. S. and Teboulle M. [8]). As well, the following chain of equalities proves to be valid:

$$\text{core}\, M = \text{qi}\, M = \text{int}\, M \tag{5}$$

(according to Limber M.A. and Goodrich R.K. [13], Rockafellar R. T. [14]).

Let now X be a separated locally convex space, and let $C \subseteq X$ be a nonempty closed convex cone. Then

$$\text{qi}(C^+) = C^{+0}$$

(see for example Boţ R. I., Grad S. M. and Wanka G. [6, Proposition 2.1.1]). The equality above is mainly the reason why the set C^{+0} defined by (1) is called the **quasi-relative interior of the dual cone of** C (even in the more general case when C is not closed).

We refer the reader to Boţ R. I. and Csetnek E. R. [4] for a recent synthesis with respect to regularity conditions by means of generalized interiors along with new achievements in the same area.

We continue by presenting some characterizations of the quasi interior and quasi-relative interior of convex sets in separated locally convex spaces.

Theorem 2.1 (Borwein J. M., Lewis A. S.[3]). *Let M be a convex subset of a separated locally convex space X, and let $x \in M$. Then*

$$x \in \text{qri}\, M \textit{ if and only if } N_M(x) \textit{ is a linear subspace of } X^*.$$

The following characterization of the quasi interior of a convex set was extended to separated locally convex spaces by Boţ R. I., Csetnek E. R. and Wanka G. [5].

Theorem 2.2. *Let M be a convex subset of a separated locally convex space X, and let $x \in M$. Then*

$$x \in \operatorname{qi} M \text{ if and only if } N_M(x) = \{0\}.$$

Some useful properties of the quasi-relative interior of a convex set are listed below. For their proofs we refer the reader to Borwein J. M. and Goebel R.[1], Borwein J. M. and Lewis A. S.[3], and Boţ R. I., Csetnek E. R. and Wanka G. [5].

Proposition 2.3. *Let M and N be convex subsets of a separated locally convex space X. Then the following statements are true:*

(i) $\operatorname{qri} M + \operatorname{qri} N \subseteq \operatorname{qri}(M + N)$;
(ii) $\operatorname{qri} M \times \operatorname{qri} N = \operatorname{qri}(M \times N)$;
(iii) $\operatorname{qri}(M - x) = (\operatorname{qri} M) - x$ *for all* $x \in X$;
(iv) $\operatorname{qri}(\alpha M) = \alpha \operatorname{qri} M$ *for all* $\alpha \in \mathbb{R}\backslash\{0\}$;
(v) $\lambda \operatorname{qri} M + (1-\lambda)M \subseteq \operatorname{qri} M$ *for all* $\lambda \in (0, 1]$, *whence* $\operatorname{qri} M$ *is a convex set;*
(vi) $\operatorname{qri}(\operatorname{qri} M) = \operatorname{qri} M$;
(vii) *if M is affine, then* $\operatorname{qri} M = M$;
(viii) *if* $\operatorname{qri} M \neq \emptyset$, *then* $\operatorname{cl} \operatorname{qri} M = \operatorname{cl} M$ *and* $\operatorname{cl} \operatorname{cone} \operatorname{qri} M = \operatorname{cl} \operatorname{cone} M$.

Remark 2.4. For two convex subsets M and N of a separated locally convex X such that $M \subseteq N$, it holds

$$\operatorname{qi} M \subseteq \operatorname{qi} N, \tag{6}$$

a property which is no longer true for the quasi-relative interior. However, according to Cammaroto F. and Di Bella B. [7, Proposition 1.12], when $\operatorname{aff} M = \operatorname{aff} N$, then

$$\operatorname{qri} M \subseteq \operatorname{qri} N$$

holds. □

Remark 2.5. Let X be a separated locally convex space, and let $C \subseteq X$ be a convex cone. Then, using Proposition 2.3 (v) and Remark 2.4 we deduce that the equality

$$\operatorname{qi} C + C = \operatorname{qi} C \tag{7}$$

holds. □

3. A CONNECTION BETWEEN THE QUASI INTERIOR AND THE QUASI-RELATIVE INTERIOR OF A CONVEX SET

In this section we present two original results of the author, the first of them linking in a new way the quasi interior and the quasi-relative interior of a convex set.

Theorem 3.1 (Grad A.). *Let M be a convex subset of a separated locally convex space X, and let $x \in M$. Then*

$$x \in \operatorname{qi} M \Longleftrightarrow \begin{cases} 0 \in \operatorname{qi}(M - M) \\ x \in \operatorname{qri} M. \end{cases}$$

Proof. **Necessity**. From $x \in \operatorname{qi} M$ and Theorem 2.2 it follows that $N_M(x) = \{0\}$. On the other hand, since $\operatorname{qri} M = \operatorname{qi} M$, we obviously have $x \in \operatorname{qri} M$. Let now $x^* \in N_{M-M}(0)$. This means that

$$\langle x^*, y - z \rangle \leq 0 \text{ for all } y, z \in M.$$

By taking $z := x$, we get that

$$\langle x^*, y - x \rangle \leq 0 \text{ for all } y \in M,$$

which means that $x^* \in N_M(x)$. Hence we have $x^* = 0$, whence it holds $N_{M-M}(0) = \{0\}$. By Theorem 2.2 we conclude that $0 \in \operatorname{qi}(M - M)$.

Sufficiency. Let $x^* \in N_M(x)$, which means that

$$\langle x^*, y - x \rangle \leq 0 \text{ for all } y \in M.$$

From $x \in \operatorname{qri} M$ and Theorem 2.1 it follows that $N_M(x)$ is a linear subspace of X^*. Hence we have $-x^* \in N_M(x)$. Therefore

$$\langle -x^*, y - x \rangle \leq 0 \text{ for all } y \in M.$$

Hence we get $\langle x^*, y - x \rangle = 0$ for all $y \in M$, from which we obtain

$$\langle x^*, y - z \rangle = 0 \text{ for all } y, z \in M.$$

Thus we have $x^* \in N_{M-M}(0)$. Consequently, the given hypothesis that $0 \in \operatorname{qi}(M - M)$ and Theorem 2.2 imply that $x^* = 0$. Thus $N_M(x) = \{0\}$, and using again Theorem 2.2, we get $x \in \operatorname{qi} M$. □

The following result extends in a natural way to linear functional, the strict positivity of the product of two strictly positive numbers.

Proposition 3.2 (Grad A.). *Let C be a nonempty convex cone of a separated locally convex space X. Then, for all $x^* \in C^+ \backslash \{0\}$ and for all $x \in \operatorname{qi} C$, the following inequality holds:*

$$\langle x^*, x \rangle > 0. \tag{8}$$

Proof. The case when $\operatorname{qi} C = \emptyset$ is obvious.

We continue with the case when $\operatorname{qi} C \neq \emptyset$, and use an approach similar to the proof given by Boţ R. I., Grad S. M. and Wanka G [6, Proposition 2.1.1]. By contradiction we assume that there exist an $x_0^* \in C^+ \backslash \{0\}$ and an $x_0 \in \operatorname{qi} C$ such that

$$\langle x_0^*, x_0 \rangle = 0. \tag{9}$$

From the definition of the dual cone C^+ we know that

$$\langle x_0^*, x \rangle \geq 0 \text{ for all } x \in C. \tag{10}$$

Combining (9) and (10) we get

$$\langle -x_0^*, x - x_0 \rangle \leq 0 \text{ for all } x \in C,$$

which means that $-x_0^* \in N_C(x_0)$. As $x_0 \in \operatorname{qi} C$, according to Theorem 2.2, we have $N_C(x_0) = \{0\}$. This implies that $-x_0^* = 0$, which is a contradiction. Thus (8) holds. □

For more applications of the previous two results, we refer the reader to the recent articles Grad A. [9] and Grad A. [10].

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