Relatively lifting modules

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Abstract. We consider a generalization of lifting modules relative to a class $\mathcal{A}$ of modules and a proper class $\mathbb{E}$ of short exact sequences of modules. These modules will be called $\mathbb{E}$-$\mathcal{A}$-lifting. We establish characterizations of modules with the property that every direct sum of copies of them is $\mathbb{E}$-$\mathcal{A}$-lifting.

2000 Mathematics Subject Classification: 16S90, 16D80.

Keywords: Proper class, torsion theory, lifting module.

Received 25 August 2007
Revised 29 June 2008

Communicated by Nanqing Ding

1 Introduction

Let $\mathcal{A}$ be a class of modules closed under isomorphisms and containing the zero module. Al-Khazzi and Smith studied in [1] the class $d^*\mathcal{A}$ consisting of modules A with the property that every submodule B of A contains a direct summand C of A such that $B/C \in \mathcal{A}$. The main motivation for their study was to offer a general setting for decomposing certain modules into a direct sum of a module in $\mathcal{A}$ and some other module. The modules in the class $d^*\mathcal{A}$ may also be seen as relative versions of the extensively investigated lifting modules (e.g., see [3]), that is, modules A such that every submodule B of A contains a direct summand C of A such that B/C is superfluous in A/C. Lifting modules have been generalized in [4] to $\mathbb{E}$-lifting modules by using instead of direct summands (i.e. splitting short exact sequences) elements of a proper class $\mathbb{E}$ of short exact sequences in the sense of Buchsbaum [2] or Mishina and Skornjakov [7].

In the present paper we put together the ideas from [1] and [4] in order to generalize the class $d^*\mathcal{A}$ by using such proper classes. The members of this new class of modules will be called $\mathbb{E}$-$\mathcal{A}$-lifting modules. They generalize lifting modules, but also $\mathbb{E}$-lifting modules, since every $\mathbb{E}$-lifting module is $\mathbb{E}$-$\mathcal{S}$-lifting, where $\mathcal{S}$ is the class of small modules. We also consider a specialization of this notion, called strongly $\mathbb{E}$-$\mathcal{A}$-lifting module. We see the class $\mathcal{A}$ as a cogenerating class for a torsion theory τ in the category $\sigma[M]$ and we establish characterizations of Σ-(strongly) $\mathbb{E}$-$\mathcal{A}$-lifting modules, that is, modules for which every direct sum of copies is (strongly) $\mathbb{E}$-$\mathcal{A}$-lifting. As a consequence, we deduce that

*The author acknowledges the support of the grant CEEX-ET 19/2005.

if a module M is Σ-$\mathcal{A}$-lifting, then every submodule of a direct sum of copies of M is a direct sum of a module in $\mathcal{A}$ and a module in the class $\mathrm{Add}(M)$ of direct summands of direct sums of copies of M. Finally, for a Σ-$\mathbb{E}$-lifting module M, we show that the property that a module belongs to a class generalizing the class $\mathrm{Add}(M)$ is lifted by certain epimorphisms.

Throughout R is an associative ring with non-zero identity and all modules are unital right R-modules. By a class of modules we mean a class of modules closed under isomorphisms and containing the zero module. Throughout M will be a module and $\mathcal{A}$ a class of modules in the category Mod-R of right R-modules. As usual, M is said to be Σ-$\mathcal{P}$ (respectively $\prod$-$\mathcal{P}$) if every direct sum (respectively direct product) of copies of M has the property $\mathcal{P}$. Denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules. By τ we denote a (not necessarily hereditary) torsion theory in $\sigma[M]$ and by $t(A)$ we denote the torsion submodule of a module A. Let $\mathcal{X}$ be any class of modules and A be a module. Then $f \in \mathrm{Hom}(A, X)$, with $X \in \mathcal{X}$, is called an *$\mathcal{X}$-preenvelope* of A if the induced abelian group homomorphism $\mathrm{Hom}(X, X') \to \mathrm{Hom}(A, X')$ is surjective for every $X' \in \mathcal{X}$.

For further terminology concerning lifting modules and torsion theories the reader is referred to [3] and [10].

2 Relatively coclosed modules

Let us give the following definitions and some basic related properties.

Definition 2.1. (i) A submodule C of a module A is called *$\mathcal{A}$-dense* in A if $A/C \in \mathcal{A}$.

(ii) A module C is called *$\mathcal{A}$-coclosed* if $C/C' \notin \mathcal{A}$ for every $C' < C$.

Definition 2.2. Let A be a module. A submodule C of A is called an *$\mathcal{A}$-coclosure of A* if C is an $\mathcal{A}$-dense submodule of A and C is $\mathcal{A}$-coclosed.

Lemma 2.3. *Let $\mathcal{A}$ be closed under submodules and C be a module. Then C is $\mathcal{A}$-coclosed if and only if $\mathrm{Hom}(C, Y) = 0$ for every $Y \in \mathcal{A}$.*

Proof. Suppose that C is $\mathcal{A}$-coclosed. Let $Y \in \mathcal{A}$ and $f \in \mathrm{Hom}(C, Y)$. Since $\mathrm{Im}\, f \subseteq Y \in \mathcal{A}$ and $C/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$, we must have $\mathrm{Ker}\, f = C$, because otherwise C is not $\mathcal{A}$-coclosed. Hence $f = 0$, and so $\mathrm{Hom}(C, Y) = 0$. Conversely, suppose that $\mathrm{Hom}(C, Y) = 0$ for every $Y \in \mathcal{A}$ and let $C' < C$. Then $\mathrm{Hom}(C, C/C') \neq 0$, hence $C/C' \notin \mathcal{A}$. Thus C is $\mathcal{A}$-coclosed. $\square$

The following well known technical result on torsion theories will be useful.

Lemma 2.4. *Let $\mathcal{A} \subseteq \sigma[M]$ be closed under submodules and $\tau = (\mathcal{T}, \mathcal{F})$ be cogenerated by $\mathcal{A}$. Then $\mathcal{F} = \{N \mid \forall 0 \neq L \leq N, \exists L' < L : L/L' \in \mathcal{A}\}$.*

Now we see $\mathcal{A}$ as a cogenerating class of τ.

Lemma 2.5. *Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$ and let C be a module.*

(i) If C is τ-torsion, then C is $\mathcal{A}$-coclosed.

(ii) If $\mathcal{A}$ is closed under submodules and C is $\mathcal{A}$-coclosed, then C is τ-torsion.

Proof. (i) Clear.

(ii) Let $D = t(C)$. If $D \neq C$, then by Lemma 2.4 the τ-torsionfree module C/D has a proper submodule C'/D such that $C/C' \cong (C/D)/(C'/D) \in \mathcal{A}$, contradicting the fact that C is $\mathcal{A}$-coclosed. Hence $D = C$, so that C is τ-torsion. □

Corollary 2.6. *Let $\mathcal{A} \subseteq \sigma[M]$ be the torsionfree class of τ, C be a module and B be a submodules of C. Then:*

(i) C is $\mathcal{A}$-coclosed if and only if it is τ-torsion.

(ii) B is an $\mathcal{A}$-coclosure of C if and only if C is τ-torsion and C/B is τ-torsionfree.

For the sake of brevity, let us say that a module M has the $\mathcal{A}$-*coclosure property* if every submodule of M has an $\mathcal{A}$-coclosure. In the following example we see that there are modules with the $\mathcal{A}$-coclosure property, but also without it.

Example 2.7. (i) Recall that a module $A \in \sigma[M]$ is called M-*rational* if $\mathrm{Hom}(C, M) = 0$ for every submodule C of A [3, p. 84]. Let $\mathrm{Cogen}(M)$ be the class of M-cogenerated modules, that is, the class of modules K for which there exists a monomorphism from K to some direct product M^I. Then by Lemma 2.3 it follows easily that a module $A \in \sigma[M]$ is M-rational if and only if every submodule of A is $\mathrm{Cogen}(M)$-coclosed. Moreover, clearly every M-rational module has the $\mathrm{Cogen}(M)$-coclosure property.

(ii) Let $\mathcal{Z}$ be the class consisting of the zero modules and the simple modules. Also, let B be a module with the radical consisting of a simple module, say D, which is also a maximal submodule of B (and so B is a module of composition length 2). We claim that B does not have a $\mathcal{Z}$-coclosure. Suppose the contrary and denote by C a $\mathcal{Z}$-coclosure of B. Then $B/C \in \mathcal{Z}$, hence C could be either B or D. But neither B nor D is $\mathcal{Z}$-coclosed, because we have $B/D \in \mathcal{Z}$ and $D \in \mathcal{Z}$. This is a contradiction, so the claim follows.

3 Relatively lifting modules and proper classes

Recall the definition of a proper class of short exact sequences (e.g., see [3, 10.1]).

Definition 3.1. Let $\mathbb{E}$ be a class of short exact sequences in Mod-R. If an exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ belongs to $\mathbb{E}$, then f is called an $\mathbb{E}$-*monomorphism* and g is called an $\mathbb{E}$-*epimorphism*. Also, $\mathrm{Im}\, f$ is called an $\mathbb{E}$-*submodule* of L and N is called an $\mathbb{E}$-*homomorphic image* of L.

The class $\mathbb{E}$ is called a *proper class* if it has the following properties:

P1. $\mathbb{E}$ is closed under isomorphisms;

P2. $\mathbb{E}$ contains all splitting short exact sequences;

P3. the class of $\mathbb{E}$-monomorphisms is closed under composition;

if f, f' are monomorphisms and $f'f$ is an $\mathbb{E}$-monomorphism, then f is an $\mathbb{E}$-monomorphism;

P4. the class of $\mathbb{E}$-epimorphisms is closed under composition;

if g, g' are epimorphisms and gg' is an $\mathbb{E}$-epimorphism, then g is an $\mathbb{E}$-epimorphism.

Example 3.2. Some examples of proper classes are the following (e.g., see [3]):

(i) The class $\mathbb{E}_s$ of all splitting short exact sequences in Mod-R.

(ii) The class $\mathbb{E}^{\mathcal{X}}$ of all short exact sequences in Mod-R on which the functor $\mathrm{Hom}(X, -)$ is exact for every $X \in \mathcal{X}$, where $\mathcal{X}$ is any class of modules in Mod-R. Its elements are called $\mathcal{X}$-pure exact sequences. For the class $\mathcal{X} = \mathcal{P}$ of finitely presented modules, one has the classical pure exact sequences.

Throughout, $\mathbb{E}$ will be a proper class of short exact sequences in Mod-R. We introduce the following definition.

Definition 3.3. A module A is called $\mathbb{E}$-$\mathcal{A}$-*lifting* if every submodule B of A contains an $\mathbb{E}$-submodule C of A such that C is $\mathcal{A}$-dense in B.

For $\mathbb{E} = \mathbb{E}_s$, we call $\mathbb{E}_s$-$\mathcal{A}$-lifting modules simply $\mathcal{A}$-*lifting*. Note that the class of $\mathcal{A}$-lifting modules is exactly the class $d^*\mathcal{A}$ from the introduction.

Example 3.4. (i) Every semisimple module is $\mathbb{E}$-$\mathcal{A}$-lifting.

(ii) Let $\mathcal{O}$ be the class of zero modules. Then a module is $\mathcal{O}$-lifting if and only if it is semisimple. Also, a module is $\mathbb{E}^{\mathcal{P}}$-$\mathcal{O}$-lifting if and only if it is regular in the sense of [11, Chapter 37].

(iii) Recall that a module A is called τ-*supplemented* if every submodule B of A contains a direct summand C of A such that B/C is τ-torsion [6]. If $\mathcal{A}$ is the torsion class of τ, then $\mathcal{A}$-lifting means τ-supplemented.

(iv) Recall that a module $A \in \sigma[M]$ is called M-*small* if A is superfluous in some module $A' \in \sigma[M]$. Also, recall that a module A is called $\mathbb{E}$-*lifting* if every submodule B of A contains an $\mathbb{E}$-submodule C of A such that B/C is superfluous in A/C [4]. If $A \in \sigma[M]$ is $\mathbb{E}$-lifting, then it is clearly $\mathbb{E}$-$\mathcal{S}$-lifting, where $\mathcal{S}$ is the class of M-small modules. In particular, every lifting module is $\mathcal{S}$-lifting.

(v) Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$ and suppose that τ is cohereditary. If A is τ-torsion $\mathcal{A}$-lifting, then it is lifting. Indeed, if B is a submodule of A, then it contains some direct summand C of A such that $B/C \in \mathcal{A}$, hence B/C is τ-torsionfree. We claim that $X = B/C$ is superfluous in $Y = A/C$. If $Z < Y$, then we have $Z + X \neq Y$, because otherwise the non-zero module $X/(Z \cap X)$ would be both τ-torsion, being isomorphic to Y/Z, and τ-torsionfree, because τ is cohereditary. This shows that A is lifting.

Lemma 3.5. *(i) Let A be an $\mathbb{E}$-$\mathcal{A}$-lifting module. Then every $\mathcal{A}$-coclosed submodule of A is an $\mathbb{E}$-submodule.*

(ii) Let A be a module with the $\mathcal{A}$-coclosure property such that every $\mathcal{A}$-coclosed submodule of A is an $\mathbb{E}$-submodule. Then A is $\mathbb{E}$-$\mathcal{A}$-lifting.

(iii) The class of $\mathbb{E}$-$\mathcal{A}$-lifting modules is closed under submodules.

Proof. (i) Let B be an $\mathcal{A}$-coclosed submodule of A. Then B contains an $\mathbb{E}$-submodule C of A such that $B/C \in \mathcal{A}$. Then $B = C$, because otherwise we would have $B/C \notin \mathcal{A}$ since B is $\mathcal{A}$-coclosed. Hence B is an $\mathbb{E}$-submodule of A.

(ii) Let B be a submodule of A and C be an $\mathcal{A}$-coclosure of B. Then $B/C \in \mathcal{A}$ and C is $\mathcal{A}$-coclosed, so that C is an $\mathbb{E}$-submodule. Hence A is $\mathbb{E}$-$\mathcal{A}$-lifting.

(iii) Let A be an $\mathbb{E}$-$\mathcal{A}$-lifting module and D be a submodule of A. Let B be a submodule of D. Then B contains an $\mathbb{E}$-submodule C of A such that $B/C \in \mathcal{A}$. Then C is an $\mathbb{E}$-submodule of D, showing that D is $\mathbb{E}$-$\mathcal{A}$-lifting. □

4 Σ-$\mathbb{E}$-$\mathcal{A}$-lifting modules

Following [4], we denote by $\mathbb{E}\mathrm{Prod}(M)$ (respectively $\mathbb{E}\mathrm{Prod}'(M)$) the class of modules K for which there is an $\mathbb{E}$-monomorphism from K to some direct product M^I(respectively direct sum $M^{(I)}$) of copies of M and by $\mathrm{Cogen}(M)$ (respectively $\mathrm{Cogen}'(M)$) the class of modules K for which there exists a monomorphism from K to some M^I (respectively $M^{(I)}$). For instance, $\mathbb{E}_s\mathrm{Prod}(M)$ (respectively $\mathbb{E}_s\mathrm{Prod}'(M)$) is the class $\mathrm{Prod}(M)$ (respectively $\mathrm{Add}(M)$) of direct summands of direct products (respectively direct sums) of copies of M.

We need the following lemma, whose proof is straightforward taking into account that the composition of two $\mathbb{E}$-monomorphisms is again an $\mathbb{E}$-monomorphism.

Lemma 4.1. *[4, Lemma 3.1] The classes* $\mathbb{E}\mathrm{Prod}(M)$ *and* $\mathbb{E}\mathrm{Prod}'(M)$ *are both closed under* $\mathbb{E}$*-submodules.*

Recall that a module is called *direct injective* if for every direct summand X of M, every monomorphism $X \to M$ splits (for instance, see [5, 2.11]). In our context, we need a generalization of direct injectivity with respect to proper classes, which was considered in [4].

Definition 4.2. A module M is called $\mathbb{E}$-*direct injective* if, for every $\mathbb{E}$-submodule X of M, every monomorphism $X \to M$ is an $\mathbb{E}$-monomorphism.

Σ-$\mathbb{E}$-direct injective modules may be characterized as follows. We sketch a proof for the reader's convenience.

Lemma 4.3. *[4, Lemma 4.6] A module M is Σ-$\mathbb{E}$-direct injective if and only if for every $U \in \mathrm{Cogen}'(M)$ and every $V \in \mathbb{E}\mathrm{Prod}'(M)$, every monomorphism $V \to U$ is an $\mathbb{E}$-monomorphism.*

Proof. Suppose first that M is Σ-$\mathbb{E}$-direct injective. Let $U \in \mathrm{Cogen}'(M)$ and $V \in \mathbb{E}\mathrm{Prod}'(M)$ and let $f : V \to U$ be a monomorphism. Then there exist a monomorphism $g : U \to M^{(I)}$ and an $\mathbb{E}$-monomorphism $h : V \to M^{(J)}$. Let us consider the monomorphism $igf : V \to M^{(I)} \oplus M^{(J)}$, where $i : M^{(I)} \to M^{(I)} \oplus M^{(J)}$ is the inclusion monomorphism. Since we may see V as an $\mathbb{E}$-submodule of $M^{(I)} \oplus M^{(J)}$ and M is Σ-$\mathbb{E}$-direct injective, igf is an $\mathbb{E}$-monomorphism, hence f has to be an $\mathbb{E}$-monomorphism. The converse is clear. □

Now we can establish our main result on Σ-$\mathbb{E}$-$\mathcal{A}$-lifting modules.

Theorem 4.4. *Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$. Consider the following statements:*

(a) M is Σ-$\mathbb{E}$-$\mathcal{A}$-lifting;

(b) Every module in $\mathrm{Add}(M)$ is $\mathbb{E}$-$\mathcal{A}$-lifting;

(c) Every $K \in \mathrm{Cogen}'(M)$ has an $\mathbb{E}$-homomorphic image $K/Y \in \mathcal{A}$ such that $Y \in \mathbb{E}\mathrm{Prod}'(M)$;

(d) Every τ-torsion module in $\mathrm{Cogen}'(M)$ is in $\mathbb{E}\mathrm{Prod}'(M)$;

(e) Every τ-torsion module in $\mathrm{Cogen}'(M)$ is $\mathbb{E}$-$\mathcal{A}$-lifting.

Then the following implications hold:

1. *For every module M, (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).*

2. *If M is Σ-$\mathbb{E}$-direct injective, then (c)$\Rightarrow$(a).*

3. *If M is Σ-$\mathbb{E}$-direct injective, has the Σ-$\mathcal{A}$-coclosure property, and $\mathcal{A}$ is closed under submodules, then (d)$\Rightarrow$(e).*

4. *If M is τ-torsion, then (e)$\Rightarrow$(a).*

Proof. (1) (a)$\Leftrightarrow$(b) Suppose that M is Σ-$\mathbb{E}$-$\mathcal{A}$-lifting and let $N \in \mathrm{Add}(M)$. Then there is a monomorphism $N \to M^{(I)}$. Now by Lemma 3.5 it follows that N is $\mathbb{E}$-$\mathcal{A}$-lifting. The converse is obvious.

(b)$\Rightarrow$(c) Let $K \in \mathrm{Cogen}'(M)$ and take a monomorphism $f : K \to M^{(I)}$. Since $M^{(I)}$ is $\mathbb{E}$-$\mathcal{A}$-lifting, $f(K)$ contains an $\mathbb{E}$-submodule L such that $f(K)/L \in \mathcal{A}$. If $Y = f^{-1}(L)$, then it follows that $K/Y \in \mathcal{A}$, $Y \in \mathbb{E}\mathrm{Prod}'(M)$ and Y is an $\mathbb{E}$-submodule of K.

(c)$\Rightarrow$(d) Clear.

(2) Assume that M is Σ-$\mathbb{E}$-direct injective.

(c)$\Rightarrow$(a) Let I be a set and K be a submodule of $M^{(I)}$. Then by hypothesis K has an $\mathbb{E}$-homomorphic image $K/Y \in \mathcal{A}$ such that $Y \in \mathbb{E}\mathrm{Prod}'(M)$. Then by Lemma 4.3, the inclusion monomorphism $Y \to M^{(I)}$ is an $\mathbb{E}$-monomorphism, hence Y is an $\mathbb{E}$-submodule of $M^{(I)}$. Thus $M^{(I)}$ is $\mathbb{E}$-$\mathcal{A}$-lifting.

(3) Assume that M is Σ-$\mathbb{E}$-direct injective, has the Σ-$\mathcal{A}$-coclosure property, and $\mathcal{A}$ is closed under submodules.

(d)$\Rightarrow$(e) Let K be a τ-torsion module in $\mathrm{Cogen}'(M)$ and consider a monomorphism $f : K \to M^{(I)}$. Let L be a proper submodule of K. Then $f(L)$ has an $\mathcal{A}$-coclosure, say C. It follows that C is τ-torsion by Lemma 2.5. Since $C \in \mathrm{Cogen}'(M)$, we have $C \in \mathbb{E}\mathrm{Prod}'(M)$ by hypothesis. Now by Lemma 4.3 the inclusion $C \to f(K)$ is an $\mathbb{E}$-monomorphism. Then the inclusion $f^{-1}(C) \to K$ is an $\mathbb{E}$-monomorphism. Since $f^{-1}(C)$ is $\mathcal{A}$-coclosed, it follows that K is $\mathbb{E}$-$\mathcal{A}$-lifting.

(4) Assume that M is τ-torsion.

(e)$\Rightarrow$(a) If M is τ-torsion, then every $M^{(I)}$ is τ-torsion, hence $\mathbb{E}$-$\mathcal{A}$-lifting. $\square$

For the proper class $\mathbb{E} = \mathbb{E}_s$ we obtain the following consequence.

Corollary 4.5. *Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$. Consider the following statements:*

(a) M is Σ-$\mathcal{A}$-lifting;

(b) Every module in $\mathrm{Add}(M)$ is $\mathcal{A}$-lifting;

(c) Every module in $\mathrm{Cogen}'(M)$ is a direct sum of a module in $\mathcal{A}$ and a module in $\mathrm{Add}(M)$;

(d) Every τ-torsion module in $\mathrm{Cogen}'(M)$ is in $\mathrm{Add}(M)$;

(e) Every τ-torsion module in $\mathrm{Cogen}'(M)$ is $\mathcal{A}$-lifting.

Then the following implications hold:

1. *For every module M, (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).*

2. *If M is Σ-direct injective, then (c)⇒(a).*

3. *If M is Σ-direct injective, has the Σ-$\mathcal{A}$-coclosure property, and the class $\mathcal{A}$ is closed under submodules, then (d)⇒(e).*

4. *If M is τ-torsion, then (e)⇒(a).*

Corollary 4.6. *Let τ be the torsion theory in* Mod-R *cogenerated by $\mathcal{A}$. Consider the following statements:*

(a) R is right Σ-$\mathcal{A}$-lifting;

(b) Every projective module is $\mathcal{A}$-lifting;

(c) Every submodule of a free module is a direct sum of a module in $\mathcal{A}$ and a projective module;

(d) Every τ-torsion submodule of a free module is projective;

(e) Every τ-torsion submodule of a free module is $\mathcal{A}$-lifting.

Then the following implications hold:

1. *For any R, (a)⇔(b)⇒(c)⇒(d).*

2. *If R is Σ-direct injective, then (c)⇒(a).*

3. *If R is Σ-direct injective, has the Σ-$\mathcal{A}$-coclosure property, and the class $\mathcal{A}$ is closed under submodules, then (d)⇒(e).*

4. *If R is τ-torsion, then (e)⇒(a).*

One may further particularize Theorem 4.4 to some classes $\mathcal{A}$, of which the classes of τ-supplemented or M-small modules (see Example 3.4) are of interest.

5 Σ-strongly $\mathbb{E}$-$\mathcal{A}$-lifting modules

Now let us consider a natural intermediate notion between those of $\mathcal{A}$-lifting module and $\mathbb{E}$-$\mathcal{A}$-lifting module.

Definition 5.1. A module M is called *strongly $\mathbb{E}$-$\mathcal{A}$-lifting* if M has the $\mathcal{A}$-coclosure property and the $\mathcal{A}$-coclosed submodules of M coincide with its $\mathbb{E}$-submodules.

Lemma 5.2. *Let A be a strongly $\mathbb{E}$-$\mathcal{A}$-lifting module and D be an $\mathcal{A}$-coclosed submodule ($\mathbb{E}$-submodule) of A. Then D is strongly $\mathbb{E}$-$\mathcal{A}$-lifting.*

Proof. By Lemma 3.5, D is $\mathbb{E}$-$\mathcal{A}$-lifting. Since A has the $\mathcal{A}$-coclosure property, then clearly so does any submodule of A. Finally, let B be an $\mathbb{E}$-submodule of D. Since D is an $\mathbb{E}$-submodule of A, B is an $\mathbb{E}$-submodule of A, and so an $\mathcal{A}$-closed submodule of A. Therefore, D is strongly $\mathbb{E}$-$\mathcal{A}$-lifting. □

In the following result we characterize Σ-strongly $\mathbb{E}$-$\mathcal{A}$-lifting modules.

Theorem 5.3. *Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$. Consider the following statements:*

(a) M is Σ-strongly $\mathbb{E}$-$\mathcal{A}$-lifting;

(b) Every module in $\mathrm{Add}(M)$ *is strongly $\mathbb{E}$-$\mathcal{A}$-lifting;*

(c) Every module in $\mathbb{E}\mathrm{Prod}'(M)$ *is strongly $\mathbb{E}$-$\mathcal{A}$-lifting;*

(d) Every τ-torsion module in $\mathrm{Cogen}'(M)$ *is strongly $\mathbb{E}$-$\mathcal{A}$-lifting;*

(e) $\mathbb{E}\mathrm{Prod}'(M)$ *consists of the τ-torsion modules in* $\mathrm{Cogen}'(M)$.

Then the following implications hold:

1. *For every module M, (a)⇔(b)⇔(c).*
2. *If $\mathcal{A}$ is closed under submodules, then (a)⇒(e) and (a)⇒(d).*
3. *If M is τ-torsion, then (d)⇒(a).*

Proof. (1) (a)⇒(c) Let $K \in \mathbb{E}\mathrm{Prod}'(M)$. Then there is an $\mathbb{E}$-monomorphism $K \to M^{(I)}$. Now by Lemma 5.2, K is strongly $\mathbb{E}$-$\mathcal{A}$-lifting.

(c)⇒(b)⇒(a) Clear.

(2) Assume that $\mathcal{A}$ is closed under submodules.

(a)⇒(e) By Theorem 4.4, every τ-torsion module in $\mathrm{Cogen}'(M)$ is in $\mathbb{E}\mathrm{Prod}'(M)$. Conversely, let $K \in \mathbb{E}\mathrm{Prod}'(M)$ and take some $\mathbb{E}$-monomorphism $g : K \to M^{(I)}$. Then K is an $\mathbb{E}$-submodule of $M^{(I)}$, hence $\mathcal{A}$-coclosed in $M^{(I)}$. Now by Lemma 2.5 it follows that K is τ-torsion.

(a)⇒(d) By (c), every module in $\mathbb{E}\mathrm{Prod}'(M)$ is strongly $\mathbb{E}$-$\mathcal{A}$-lifting. Then by (e) it follows that every τ-torsion module in $\mathrm{Cogen}'(M)$ is strongly $\mathbb{E}$-$\mathcal{A}$-lifting.

(3) Assume that M is τ-torsion.

(d)⇒(a) If M is τ-torsion, then every $M^{(I)}$ is τ-torsion, hence strongly $\mathbb{E}$-$\mathcal{A}$-lifting. □

Corollary 5.4. *Let τ be cogenerated by $\mathcal{A} \subseteq \sigma[M]$. Consider the following statements:*

(a) M is Σ-strongly $\mathcal{A}$-lifting;

(b) Every module in $\mathrm{Add}(M)$ *is strongly $\mathcal{A}$-lifting;*

(c) Every τ-torsion module in $\mathrm{Cogen}'(M)$ *is strongly $\mathcal{A}$-lifting;*

(d) $\mathrm{Add}(M)$ *consists of the τ-torsion modules in* $\mathrm{Cogen}'(M)$.

Then the following implications hold:

1. *For every module M, (a)⇔(b).*
2. *If $\mathcal{A}$ is closed under submodules, then (a)⇒(d) and (a)⇒(c).*
3. *If M is τ-torsion, then (c)⇒(a).*

Corollary 5.5. *Let τ be the torsion theory in* Mod-R *cogenerated by a class $\mathcal{A}$ closed under submodules and suppose that R is τ-torsion. Then the following are equivalent:*

(a) R is right Σ-strongly $\mathcal{A}$-lifting;

(b) Every projective module is strongly $\mathcal{A}$-lifting;

(c) Every submodule of a free module is strongly $\mathcal{A}$-lifting;

In this case, we also have:

(d) A module is projective if and only if it is a submodule of a free module.

6 Σ-$\mathbb{E}$-$\mathcal{A}$-lifting modules and relative preenvelopes

An important result of Oshiro [8, Theorem II] says that if R is right Σ-extending, then the class of projective modules is closed under essential extensions. Motivated by this, we establish in our case a result with dual flavor. Thus, for a Σ-$\mathbb{E}$-$\mathcal{A}$-lifting module M and an epimorphism $Y \to Z$ we study when $Z \in \mathbb{E}\mathrm{Prod}'(M)$ implies $Y \in \mathbb{E}\mathrm{Prod}'(M)$.

The following condition on a module M will be useful:

($*$) For every proper submodules B, C, D of M with $D + B = M$ and C $\mathcal{A}$-dense in B we have $D + C = M$.

For instance, any hollow module clearly satisfies ($*$).

We need the following technical lemma.

Lemma 6.1. *Let $p : K \to M$ be a monomorphism such that M is $\mathbb{E}$-$\mathcal{A}$-lifting and K satisfies ($*$). If there exists a proper submodule D of M such that $D + \mathrm{Im}\, p = M$ and $p^{-1}(D) \in \mathcal{A}$, then $\mathrm{Im}\, p$ is an $\mathbb{E}$-submodule of M.*

Proof. We may assume that $N = \mathrm{Im}\, p$ is a proper submodule of M. Since M is $\mathbb{E}$-$\mathcal{A}$-lifting, N contains an $\mathbb{E}$-submodule L of M such that $N/L \in \mathcal{A}$. Since $D + N = M$, we have $D + L = M$, whence it follows easily that $p^{-1}(D) + p^{-1}(L) = K$. This and the fact that $p^{-1}(D) \in \mathcal{A}$ implies by hypothesis that $p^{-1}(L) = p^{-1}(L) + 0 = K$, whence $N \subseteq L$. Thus $N = L$ is an $\mathbb{E}$-submodule of M. $\square$

Theorem 6.2. *Let M be Σ-$\mathbb{E}$-$\mathcal{A}$-lifting. If $j : Y \to Z$ is a non-zero epimorphism such that $\mathrm{Ker}\, j \in \mathcal{A}$, $Z \in \mathbb{E}\mathrm{Prod}'(M)$, $Y \in \mathrm{Cogen}'(M)$ and Y satisfies ($*$) and has an $\mathbb{E}\mathrm{Prod}'(M)$-preenvelope, then $Y \in \mathbb{E}\mathrm{Prod}'(M)$.*

Proof. Let $p : Y \to E$ be an $\mathbb{E}\mathrm{Prod}'(M)$-preenvelope of Y. Then j factors through p, hence there is a homomorphism $q : E \to Z$ such that $qp = j$. Since there exists some $\mathbb{E}$-monomorphism $E \to M^{(I)}$, E is $\mathbb{E}$-$\mathcal{A}$-lifting by Lemma 3.5. Since $Y \in \mathrm{Cogen}'(M)$, it follows that p is a monomorphism. Then we have $\mathrm{Ker}\, q \neq E$ and $p^{-1}(\mathrm{Ker}\, q) = \mathrm{Ker}\, j \in \mathcal{A}$. It is easy to check that $\mathrm{Ker}\, q + \mathrm{Im}\, p = E$, whence $\mathrm{Im}\, p$ is an $\mathbb{E}$-submodule of E by Lemma 6.1. Since $E \in \mathbb{E}\mathrm{Prod}'(M)$, it follows by Lemma 4.1 that $Y \in \mathbb{E}\mathrm{Prod}'(M)$. $\square$

Corollary 6.3. *Let M be Σ-$\mathbb{E}$-$\mathcal{A}$-lifting. If $j : Y \to Z$ is a non-zero epimorphism such that $\mathrm{Ker}\, j \in \mathcal{A}$, $Z \in \mathrm{Add}(M)$, $Y \in \mathrm{Cogen}'(M)$ and Y satisfies ($*$) and has an $\mathrm{Add}(M)$-preenvelope, then $Y \in \mathrm{Add}(M)$.*

Note that one may replace direct sums with direct products in Theorem 6.2 and obtain a similar result. Every module has an $\mathrm{Add}(M)$-preenvelope if and only if $\mathrm{Add}(M)$ is closed under products, and in this case M is called *product-complete* [9]. Also, every module has a $\mathrm{Prod}(M)$-preenvelope [9]. Then we have the following corollary.

Corollary 6.4. *(i) Let M be product-complete Σ-$\mathbb{E}$-$\mathcal{A}$-lifting. If $j : Y \to Z$ is a non-zero epimorphism such that $\mathrm{Ker}\, j \in \mathcal{A}$, $Z \in \mathrm{Add}(M)$, $Y \in \mathrm{Cogen}'(M)$ and Y satisfies ($*$), then $Y \in \mathrm{Add}(M)$.*

(ii) Let M be $\prod$-$\mathbb{E}$-$\mathcal{A}$-lifting. If $j : Y \to Z$ is a non-zero epimorphism such that $\mathrm{Ker}\, j \in \mathcal{A}$, $Z \in \mathrm{Prod}(M)$, $Y \in \mathrm{Cogen}(M)$ and Y satisfies ($*$), then $Y \in \mathrm{Prod}(M)$.

Acknowledgements. The author would like to thank the referee for the useful comments and suggestions.

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