RELATIVELY EXTENDING MODULES

SEPTIMIU CRIVEI

To Professor Fred Van Oystaeyen on the occasion of his 60th birthday

Abstract. We investigate a generalization of extending modules relative to a class of modules and a proper class of short exact sequences of modules.

1. Introduction

Let $\mathcal{A}$ be a class of modules closed under isomorphisms and containing the zero module. Smith, Huynh and Dung studied in [19] and [20] the class $d\mathcal{A}$ consisting of modules A with the property that every submodule B of A is contained in a direct summand C of A such that $C/B \in \mathcal{A}$. This class has been useful, among other things, for providing some general characterizations of modules which are direct sums of a module from the class $\mathcal{A}$ and a module from some other class. In order to point out previous interest in dealing with such direct sum decompositions, we mention the studies of Chatters [5] on rings such that every cyclic module is the direct sum of a projective module and a module of Krull dimension at most an ordinal α, Huynh and Dan [13] on rings such that every cyclic module is a direct sum of a projective module and an Artinian module, and Al-Khazzi and Smith [1] on modules which are direct sums of a semisimple and a Noetherian module. A certain class of type $d\mathcal{A}$ includes the extending modules, which are defined as modules with the property that every submodule is essential in a direct summand or, equivalently, every closed submodule is a direct summand [10]. Indeed, every extending module is in $d\mathcal{S}$, where $\mathcal{S}$ is the class of singular modules. Extending modules have been generalized in [8] to $\mathbb{E}$-extending modules by using instead of direct summands (i.e. splitting short exact sequences) elements of a proper class $\mathbb{E}$ of short exact sequences of modules. As particular cases, our framework also included purely extending introduced by Fuchs [11] and then studied by Clark [6].

In the present paper we generalize the class $d\mathcal{A}$ using proper classes of short exact sequences. We study the members of this new class of modules, which will be called $\mathbb{E}$-$\mathcal{A}$-extending modules. They generalize extending modules, but also $\mathbb{E}$-extending modules, since every $\mathbb{E}$-extending module is $\mathbb{E}$-$\mathcal{S}$-extending, where $\mathcal{S}$ is the class of singular modules. We show that natural classes of modules and precovers fit well into the theory of extending modules at this level of generality. The paper is organized as follows. In Section 2 we introduce and make use of suitable closures of modules, which are related to natural classes of modules and module

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approximations. In Section 3 we introduce and give some basic properties of $\mathbb{E}$-$\mathcal{A}$-extending modules. In Section 4 we collect needed properties of some classes generalizing the class $\mathrm{Add}(M)$ of direct summands of direct sums of copies of a module M, and we show that every module has a cover relative to the class of pure homomorphic images of direct sums of copies of M. The main characterizations of Σ-$\mathbb{E}$-$\mathcal{A}$-extending modules are established in Section 5 as well as various corollaries of them for natural classes of modules, non-singular modules or relative complemented modules. Motivated by the fact that the closed submodules actually coincide with the direct summands in the case of extending modules, we consider strongly $\mathbb{E}$-$\mathcal{A}$-extending modules, and we establish characterizations of Σ-strongly $\mathbb{E}$-$\mathcal{A}$-extending modules. Finally, for a Σ-$\mathbb{E}$-extending module M, we show in Section 6 that certain M-generated extensions preserve modules in a class generalizing the class $\mathrm{Add}(M)$.

Throughout R is an associative ring with non-zero identity, all modules are unital right R-modules, and M is a module. By a class of modules we mean a class of modules closed under isomorphisms and containing the zero module. Denote by Mod-R the category of right R-modules, by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules, and by $\mathrm{Gen}(M)$ the class of M-generated modules. Also, $\mathcal{A}$ will be a class of modules in Mod-R. As usual, M is said to be Σ-$\mathcal{P}$ if every direct sum of copies of M has the property $\mathcal{P}$. A submodule L of M is called *closed* in M if L has no proper essential extension in M. Given submodules $K \subseteq L \subseteq M$, we say that L is a *closure* of K in M if K is an essential submodule of L and L is a closed submodule of M. A module $N \in \sigma[M]$ is called *M-singular* if $N \cong L/K$ for some essential submodule K of a module $L \in \sigma[M]$. The modules in the torsionfree class of the torsion theory in $\sigma[M]$ generated by the M-singular modules are called *non-M-singular* [7, p.72].

2. Relatively closed submodules

We introduce the notions of closed submodule and closure of a submodule relative to a class of modules. We employ some special classes of modules associated to $\mathcal{A}$, which play an important part in the theory of natural classes of modules, very much investigated in recent years. A class of modules is called *natural* if it is closed under submodules, direct sums and essential extensions (or injective hulls) (see [9]). Let us denote by $\mathcal{F}'(\mathcal{A})$ the class consisting of all modules which do not contain any non-zero submodule embeddable in some module of $\mathcal{A}$, and by $\mathcal{F}(\mathcal{A})$ the class consisting of all modules having no non-zero submodule in $\mathcal{A}$. It is clear that $\mathcal{F}'(\mathcal{A}) = \mathcal{F}(\mathcal{A})$ if $\mathcal{A}$ is closed under submodules. A key result states that a class of modules closed under submodules is natural if and only if it is of the form $\mathcal{F}'(\mathcal{A})$ [9, Theorem 2.3.15]. If $\mathcal{S}$ is the class of M-singular modules, then $\mathcal{F}'(\mathcal{S}) = \mathcal{F}(\mathcal{S})$ is the class of non-M-singular modules. If $\mathcal{A}$ is the torsion class of a hereditary torsion theory τ in $\sigma[M]$, then $\mathcal{F}'(\mathcal{A}) = \mathcal{F}(\mathcal{A})$ is the class of τ-torsionfree modules.

Definition 2.1. Let A be a module and B a submodule of A. Then a submodule C of A is called *$\mathcal{A}$-dense in A* if $A/C \in \mathcal{A}$, and *$\mathcal{A}$-closed in A* if $A/C \in \mathcal{F}(\mathcal{A})$. A submodule C of A containing B is called an *$\mathcal{A}$-closure of B in A* if B is $\mathcal{A}$-dense in C and C is $\mathcal{A}$-closed in A.

If $\mathcal{S}$ is the class of M-singular modules, then any closure of a submodule of a non-M-singular module is an $\mathcal{S}$-closure. Our terminology of $\mathcal{A}$-closures completely agrees with that of τ-closures when $\mathcal{A}$ is the torsion class of a torsion theory τ in $\sigma[M]$. Let B be a submodule of a module $A \in \sigma[M]$. Then B is called *τ-dense* (*τ-closed*) in A if A/B is τ-torsion (τ-torsionfree) [12]. A submodule C of A containing B is called a *τ-closure* of B in A if B is τ-dense in C and C is τ-closed in A.

Lemma 2.2. *Let $\mathcal{A}$ be the torsion class of a torsion theory τ in $\sigma[M]$, A a module and B, C submodules of A with $B \subseteq C$. Then:*

(i) C is $\mathcal{A}$-closed in A if and only if it is τ-closed in A.

(ii) C is an $\mathcal{A}$-closure of B in A if and only if it is a τ-closure of B in A.

Proof. Straightforward. □

For the sake of brevity, let us say that a module M has the *$\mathcal{A}$-closure property* if every submodule of M has an $\mathcal{A}$-closure in M. A module may have or may not have the $\mathcal{A}$-closure property, as we see in the following example.

Example 2.3. (i) If $\mathcal{A}$ is the torsion class of a torsion theory τ in $\sigma[M]$, then it is well known that every submodule of a module $A \in \sigma[M]$ has a τ-closure in A, and so an $\mathcal{A}$-closure in A by Lemma 2.2.

(ii) A module $A \in \sigma[M]$ is called *M-corational* if $\mathrm{Hom}(M, A/C) = 0$ for every submodule C of A [7, p. 84]. These modules have a natural interpretation in the language of relative closed submodules. It is easy to see that if $\mathcal{A}$ is closed under homomorphic images, A is a module and C is a submodule of A, then C is $\mathcal{A}$-closed in A if and only if $\mathrm{Hom}(X, A/C) = 0$ for every $X \in \mathcal{A}$. Then it follows that a module $A \in \sigma[M]$ is M-corational if and only if every submodule of A is $\mathrm{Gen}(M)$-closed in A. Hence clearly every M-corational module has the $\mathrm{Gen}(M)$-closure property.

(iii) Let $\mathcal{Z}$ be the class consisting of the zero modules and the simple modules. Also, let A be a module and B a submodule of A such that the socle of A/B consists of a non-zero module, say D/B, and A/D is simple. Then it is easy to see that B does not have a $\mathcal{Z}$-closure in A.

Let $\mathcal{X}$ be any class of modules and A a module. Then $f \in \mathrm{Hom}(X, A)$, with $X \in \mathcal{X}$, is called an *$\mathcal{X}$-precover* of A if the induced abelian group homomorphism $\mathrm{Hom}(X', X) \to \mathrm{Hom}(X', A)$ is surjective for every $X' \in \mathcal{X}$. An $\mathcal{X}$-precover $f \in \mathrm{Hom}(X, A)$ of A is called an *$\mathcal{X}$-cover* if every endomorphism $g : X \to X$ with $fg = f$ is an automorphism (e.g. [22]). The existence of relative closures can be related to the existence of monic relative (pre)covers (i.e., relative (pre)covers which are monomorphisms) in the following way.

Theorem 2.4. *Let A be a module and B a submodule of A. Consider the following statements:*

(i) B has an $\mathcal{A}$-closure in A;

(ii) A/B has a monic $\mathcal{A}$-(pre)cover.

Then (i)⇒(ii), provided $\mathcal{A}$ is closed under homomorphic images, and (ii)⇒(i), provided $\mathcal{A}$ is closed under extensions.

Proof. (i)⇒(ii) Assume that $\mathcal{A}$ is closed under homomorphic images, and let C be an $\mathcal{A}$-closure of B in A. Then there is a short exact sequence $0 \to C/B \xrightarrow{f} A/B \xrightarrow{g} A/C \to 0$, where $C/B \in \mathcal{A}$ and $A/C \in \mathcal{F}(\mathcal{A})$. Let $h : X \to A/B$ be a homomorphism with $X \in \mathcal{A}$. By hypothesis we have $\mathrm{Im}(gh) \in \mathcal{A}$, whence $gh = 0$, and so $\mathrm{Im}\, h \subseteq \mathrm{Im}\, f$. Then h factors through f, showing that $f : C/B \to A/B$ is an $\mathcal{A}$-(pre)cover of A/B.

(ii)⇒(i) Assume that $\mathcal{A}$ is closed under extensions, let $f : C/B \to A/B$ be a monic $\mathcal{A}$-(pre)cover of A/B, and consider the induced short exact sequence $0 \to C/B \to A/B \to A/C \to 0$. Let K be a submodule of A/C with $K \in \mathcal{A}$. By a pullback we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C/B \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\beta"] \arrow[dl, dashed, "\alpha"'] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C/B \arrow[r, "f"] & A/B \arrow[r] & A/C \arrow[r] & 0
\end{tikzcd}$$

Since $C/B, K \in \mathcal{A}$, we have $Z \in \mathcal{A}$. Then there is a homomorphism $\alpha : Z \to C/B$ such that $f\alpha = \beta$. Note that α is a monomorphism, because so is β. We have $f\alpha j = \beta j = f$, and so $\alpha j = 1_{C/B}$. Hence α is an isomorphism, whence $K = 0$. Therefore, $A/C \in \mathcal{F}(\mathcal{A})$ and so C is an $\mathcal{A}$-closure of B in A. □

Corollary 2.5. *Consider the following statements for a module M:*

(i) M has the Σ-$\mathcal{A}$-closure ($\mathcal{A}$-closure) property;

(ii) Every M-generated (M-cyclic) module has a monic $\mathcal{A}$-(pre)cover.

Then (i)⇒(ii), provided $\mathcal{A}$ is closed under homomorphic images, and (ii)⇒(i), provided $\mathcal{A}$ is closed under extensions.

Thus the theory of module approximations may be used to obtain modules or rings with the (Σ-)$\mathcal{A}$-closure property. For instance, denote by Inj and $FPInj$ the classes of injective and FP-injective modules respectively.

Corollary 2.6. *(i) A right hereditary ring R is right Noetherian if and only if it has the Σ-Inj-closure property.*

(ii) Every right semihereditary ring R has the Σ-$FPInj$-closure property.

Proof. (i) If R is right hereditary, then Inj is closed under homomorphic images. Then use Corollary 2.5 and the fact that R is right hereditary right Noetherian if and only if every module has a monic injective cover [18, Corollary 4.12].

(ii) If R is right semihereditary, then $FPInj$ is closed under homomorphic images. Then use Corollary 2.5 and the fact that R is right semihereditary if and only if every module has a monic FP-injective cover [18, Corollary 4.13]. □

3. Relatively extending modules and proper classes

Throughout $\mathbb{E}$ will be a proper class of short exact sequences in Mod-R in the sense of Buchsbaum [4] or Mishina and Skornjakov [14]. We now recall its definition.

Definition 3.1. Let $\mathbb{E}$ be a class of short exact sequences in Mod-R. If an exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ belongs to $\mathbb{E}$, then f is called an $\mathbb{E}$-*monomorphism* and g is called an $\mathbb{E}$-*epimorphism.* Also, $\operatorname{Im} f$ is called an $\mathbb{E}$-*submodule* of L and N is called an $\mathbb{E}$-*homomorphic image* of L.

The class $\mathbb{E}$ is called a *proper class* if it has the following properties:

P1. $\mathbb{E}$ is closed under isomorphisms;

P2. $\mathbb{E}$ contains all splitting short exact sequences;

P3. the class of $\mathbb{E}$-monomorphisms is closed under composition;

if f, f' are monomorphisms and $f'f$ is an $\mathbb{E}$-monomorphism, then f is an $\mathbb{E}$-monomorphism;

P4. the class of $\mathbb{E}$-epimorphisms is closed under composition;

if g, g' are epimorphisms and gg' is an $\mathbb{E}$-epimorphism, then g is an $\mathbb{E}$-epimorphism.

Example 3.2. Some examples of proper classes, which will be referred by us in the sequel, are the following (e.g., see [7]):

(i) The class $\mathbb{E}_s$ of all splitting short exact sequences in Mod-R.

(ii) The class $\mathbb{E}^{\mathcal{X}}$ of all short exact sequences in Mod-R on which the functor $\mathrm{Hom}(X, -)$ is exact for every $X \in \mathcal{X}$, where $\mathcal{X}$ is any class of modules in Mod-R. Its elements are called $\mathcal{X}$-pure exact sequences. For the class $\mathcal{X} = \mathcal{P}$ of finitely presented modules, one has the classical pure exact sequences. Taking $\mathcal{X} = \mathcal{G}$ to be the class of finitely generated modules, one has the finitely split exact sequences in the sense of Azumaya [2]. Clearly, we have $\mathbb{E}_s \subseteq \mathbb{E}^{\mathcal{G}} \subseteq \mathbb{E}^{\mathcal{P}}$.

We introduce the following definition.

Definition 3.3. A module A is called $\mathbb{E}$-$\mathcal{A}$-*extending* if every submodule B of A is contained in an $\mathbb{E}$-submodule C of A such that B is $\mathcal{A}$-dense in C.

We call $\mathbb{E}_s$-$\mathcal{A}$-extending modules simply $\mathcal{A}$-*extending*. Note that the class of $\mathcal{A}$-extending modules is exactly the class $d\mathcal{A}$ mentioned in the introduction. By analogy with purely extending modules in the sense of Clark [6], let us call $\mathbb{E}^{\mathcal{P}}$-$\mathcal{A}$-extending modules *purely* $\mathcal{A}$-*extending*. Also, we call $\mathbb{E}^{\mathcal{G}}$-$\mathcal{A}$-extending modules *finitely* $\mathcal{A}$-*extending*. Clearly, every $\mathcal{A}$-extending module is finitely $\mathcal{A}$-extending, which furthermore is purely $\mathcal{A}$-extending.

Example 3.4. (i) Every semisimple module is $\mathbb{E}$-$\mathcal{A}$-extending, and every $\mathcal{C}$-extending module is extending, where $\mathcal{C}$ is the class of semisimple modules in $\sigma[M]$ [19, Proposition 1.5].

(ii) Let τ be a torsion theory in $\sigma[M]$. A module A is called τ-*complemented* if every submodule of A is τ-dense in a direct summand of A [21]. If $\mathcal{A}$ is the torsion class of τ, then $\mathcal{A}$-extending means τ-complemented. If τ is generated by $\mathcal{A}$ and hereditary, then every τ-torsionfree $\mathcal{A}$-extending module is extending.

(iii) A module A is called $\mathbb{E}$-*extending* if every submodule of A is essential in an $\mathbb{E}$-submodule of A [8]. For the proper classes $\mathbb{E}_s$, $\mathbb{E}^{\mathcal{G}}$ and $\mathbb{E}^{\mathcal{P}}$, the corresponding notions are extending, finitely extending and purely extending modules respectively. If $A \in \sigma[M]$ is $\mathbb{E}$-extending, then it is clearly $\mathbb{E}$-$\mathcal{S}$-extending, where $\mathcal{S}$ is the class of M-singular modules. In particular, every extending module is $\mathcal{S}$-extending, every finitely extending module is finitely $\mathcal{S}$-extending, and every purely extending module is purely $\mathcal{S}$-extending.

We shall need the following result, whose proof is similar to the corresponding one for extending modules.

Lemma 3.5. *(i) Let A be an $\mathbb{E}$-$\mathcal{A}$-extending module. Then every $\mathcal{A}$-closed submodule of A is an $\mathbb{E}$-submodule.*

(ii) Let A be a module with the $\mathcal{A}$-closure property such that every $\mathcal{A}$-closed submodule of A is an $\mathbb{E}$-submodule. Then A is $\mathbb{E}$-$\mathcal{A}$-extending.

(iii) The class of $\mathbb{E}$-$\mathcal{A}$-extending modules is closed under homomorphic images.

4. The classes $\mathbb{E}\mathrm{Add}(M)$

Following [8], we denote by $\mathbb{E}\mathrm{Add}(M)$ the class of modules N for which there is an $\mathbb{E}$-epimorphism from some direct sum $M^{(I)}$ of copies of M to N. For instance,

$\mathbb{E}_s\mathrm{Add}(M)$ is the class $\mathrm{Add}(M)$ of direct summands of direct sums of copies of M, hence $\mathbb{E}_s\mathrm{Add}(R)$ is the class of projective modules. Also, $\mathbb{E}^{\mathcal{G}}\mathrm{Add}(R)$ is the class of finitely projective modules in the sense of Azumaya [2], and $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(R)$ is the class of flat modules. Clearly, every projective module is finitely projective, and every finitely projective module is flat.

We need the following lemma, which follows because the composition of two $\mathbb{E}$-epimorphisms is an $\mathbb{E}$-epimorphism, and $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ is closed under direct sums.

Lemma 4.1. [8, Lemma 3.1] *(i) The class* $\mathbb{E}\mathrm{Add}(M)$ *is closed under* $\mathbb{E}$-*homomorphic images.*

(ii) The class $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ *is closed under direct limits.*

Let us recall from [8] a generalization of direct projectivity and its basic characterization, which are useful for studying $\mathbb{E}$-$\mathcal{A}$-extending modules. We include a proof for the reader's convenience.

Definition 4.2. A module M is called $\mathbb{E}$-*direct projective* if, for every $\mathbb{E}$-homomorphic image X of M, every epimorphism $M \to X$ is an $\mathbb{E}$-epimorphism.

Lemma 4.3. [8, Lemma 4.3] *A module M is Σ-$\mathbb{E}$-direct projective if and only if for every $U \in \mathrm{Gen}(M)$ and every $V \in \mathbb{E}\mathrm{Add}(M)$, every epimorphism $U \to V$ is an $\mathbb{E}$-epimorphism.*

Proof. Suppose that M is Σ-$\mathbb{E}$-direct projective. Let $U \in \mathrm{Gen}(M)$, $V \in \mathbb{E}\mathrm{Add}(M)$ and let $f : U \to V$ be an epimorphism. Then there is an epimorphism $g : M^{(I)} \to U$ and an $\mathbb{E}$-epimorphism $h : M^{(J)} \to V$. Consider the epimorphism $fgp : M^{(I)} \oplus M^{(J)} \to V$, where $p : M^{(I)} \oplus M^{(J)} \to M^{(I)}$ is the projection epimorphism. Since V is an $\mathbb{E}$-homomorphic image of $M^{(I)} \oplus M^{(J)}$ and M is Σ-$\mathbb{E}$-direct projective, fgp is an $\mathbb{E}$-epimorphism, hence f is an $\mathbb{E}$-epimorphism. The converse is clear. $\square$

Now let us discuss the existence of certain relative covers, that will be useful in the last section, but may also be of independent interest. In [8] the question was raised as to whether the class $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ of pure homomorphic images of direct sums of copies of M is (pre)covering, in the sense that every module has an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-(pre)cover. Since the class $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ is closed under direct limits by Lemma 4.1, if it is precovering, then it is covering by [22, Theorem 2.2.8]. For $M = R$, the class $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(R)$ of flat modules is known to be covering, this being the positively solved Flat Cover Conjecture [3]. More generally, it was showed that if M is finitely presented, then every module has an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-cover [8, Proposition 3.3]. Now we give a complete positive answer to our question, following an idea from [17]. First we need some preliminary results.

Theorem 4.4. [3, Theorem 5] *Let σ be a purity projectively generated by a set of modules. Then for each cardinal λ, there is a cardinal κ such that for any module M and any submodule L of M with $|M| \geq \kappa$ and $|M/L| \leq \lambda$, L contains a non-zero σ-pure submodule of M.*

Lemma 4.5. [17, Lemma 4.7] *Let M be a module and let $\mathcal{A}$ be a class of modules closed under direct sums and $\mathcal{B} \subset \mathcal{A}$ a set such that any homomorphism $A \to M$ with $A \in \mathcal{A}$ factors through a module in $\mathcal{B}$. Then M has an $\mathcal{A}$-precover.*

Proof. One shows that $\bigoplus_{B \in \mathcal{B}} B^{\mathrm{Hom}(B,M)} \to M$ is an $\mathcal{A}$-precover of M. $\square$

Lemma 4.6. *Let M be a module with $|M| = \lambda$ and let κ be the cardinal from Theorem 4.4. Then any homomorphism $A \to M$ with $A \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ factors through a module $B \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ with $|B| < \kappa$.*

Proof. Let $f_0 : A \to M$ be a homomorphism with $A \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ large enough and let $K_0 = \mathrm{Ker}\, f_0$. Since $|A/K_0| \leq |M|$, A/K_0 is small enough, hence K_0 contains a non-zero pure submodule L_0 of A by Theorem 4.4. Then $A/L_0 \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ by Lemma 4.1. If A/L_0 is not small enough, then repeat the process with the induced homomorphism $f_1 : A/L_0 \to M$. Let $K_1/L_0 = \mathrm{Ker}\, f_1$. Since A/K_1 is small enough, K_1/L_0 contains a non-zero pure submodule L_1/L_0 of A/L_0 by Theorem 4.4. Then $A/L_1 \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ by Lemma 4.1. If A/L_1 is not small enough, then continue the process with the induced homomorphism $f_2 : A/L_1 \to M$. Thus we arrive at $B = \varinjlim A/L_i$. Now we have $B \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ by Lemma 4.1, $|B| < \kappa$, and the required factorization. □

Theorem 4.7. *Every module has an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-cover.*

Proof. We have seen that in order to derive the conclusion it is enough to show that every module has an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-precover. Let M be a module and take a set X with $|X| = \kappa$, where κ is the cardinal from Theorem 4.4. Form the set of all subsets of X, and then for each such subset consider all binary operations on it, which form a set. Then find all scalar multiplications on the union of the above sets, and in this way get a set $\mathcal{B}'$. Some of the elements of $\mathcal{B}'$ are modules and choose from them those which are in $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$. Then we obtain a set $\mathcal{B}$ and $\bigoplus_{B\in\mathcal{B}} B^{\mathrm{Hom}(B,M)}$ is again a set.

We claim that $\bigoplus_{B\in\mathcal{B}} B^{\mathrm{Hom}(B,M)} \to M$ is an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-precover of M. First note that the class $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ is closed under direct sums. Let $f : A \to M$ be a homomorphism with $A \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$. By Lemma 4.6, f factors through a module $B \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ with $|B| < \kappa$, and consequently through a module in $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$ isomorphic to one in $\mathcal{B}$. Now by Lemma 4.5, it follows that $\bigoplus_{B\in\mathcal{B}} B^{\mathrm{Hom}(B,M)} \to M$ is an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-precover of M. □

5. Σ-$\mathbb{E}$-$\mathcal{A}$-EXTENDING MODULES

Now we can characterize Σ-$\mathbb{E}$-$\mathcal{A}$-extending modules.

Theorem 5.1. *Consider the following statements:*

(a) *M is Σ-$\mathbb{E}$-$\mathcal{A}$-extending;*
(b) *Every module in $\mathrm{Add}(M)$ is $\mathbb{E}$-$\mathcal{A}$-extending;*
(c) *Every M-generated module N has an $\mathbb{E}$-submodule $Y \in \mathcal{A}$ such that $N/Y \in \mathbb{E}\mathrm{Add}(M)$;*
(d) *Every M-generated module in $\mathcal{F}'(\mathcal{A})$ is in $\mathbb{E}\mathrm{Add}(M)$;*
(e) *Every M-generated module in $\mathcal{F}'(\mathcal{A})$ is $\mathbb{E}$-$\mathcal{A}$-extending.*

Then the following implications hold:

(1) *For every module M, (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).*
(2) *If M is Σ-$\mathbb{E}$-direct projective, then (c)$\Rightarrow$(a).*
(3) *If M is Σ-$\mathbb{E}$-direct projective, has the Σ-$\mathcal{A}$-closure property, and $\mathcal{A}$ is closed under submodules, then (d)$\Rightarrow$(e).*
(4) *If $M \in \mathcal{F}'(\mathcal{A})$, then (e)$\Rightarrow$(a).*

Proof. (1) (a)⇔(b) The first implication follows by Lemma 3.5 and the converse is obvious.

(b)⇒(c) Let N be an M-generated module and take an epimorphism $f : M^{(I)} \to N$ with $K = \operatorname{Ker} f$. Since $M^{(I)}$ is $\mathbb{E}$-$\mathcal{A}$-extending, K is contained in an $\mathbb{E}$-submodule L of $M^{(I)}$ such that $L/K \in \mathcal{A}$. Then N has a submodule $Y \cong L/K \in \mathcal{A}$ which is an $\mathbb{E}$-submodule of N, because L/K is an $\mathbb{E}$-submodule of $M^{(I)}/K$. Also, we have $N/Y \cong M^{(I)}/L \in \mathbb{E}\mathrm{Add}(M)$.

(c)⇒(d) Let N be an M-generated module in $\mathcal{F}'(\mathcal{A})$. By (c) there is an $\mathbb{E}$-submodule $Y \in \mathcal{A}$ such that $N/Y \in \mathbb{E}\mathrm{Add}(M)$. Since $N \in \mathcal{F}'(\mathcal{A})$, we have $Y = 0$ and so $N \in \mathbb{E}\mathrm{Add}(M)$, as required.

(2) Assume that M is Σ-$\mathbb{E}$-direct projective.

(c)⇒(a) Let I be a set and K a submodule of $M^{(I)}$. Then by hypothesis $M^{(I)}/K$ has an $\mathbb{E}$-submodule $Y/K \in \mathcal{A}$ such that $M^{(I)}/Y \cong (M^{(I)}/K)/(Y/K) \in \mathbb{E}\mathrm{Add}(M)$. Then by Lemma 4.3, the natural epimorphism $M^{(I)} \to M^{(I)}/Y$ is an $\mathbb{E}$-epimorphism, hence Y is an $\mathbb{E}$-submodule of $M^{(I)}$. Thus $M^{(I)}$ is $\mathbb{E}$-$\mathcal{A}$-extending.

(3) Assume that M is Σ-$\mathbb{E}$-direct projective, has the Σ-$\mathcal{A}$-closure property, and $\mathcal{A}$ is closed under submodules.

(d)⇒(e) Let N be an M-generated module in $\mathcal{F}'(\mathcal{A})$ and K a proper submodule of N. Let L be an $\mathcal{A}$-closure of K in N. Then N/L is M-generated and $N/L \in \mathcal{F}(\mathcal{A}) = \mathcal{F}'(\mathcal{A})$, hence by hypothesis we have $N/L \in \mathbb{E}\mathrm{Add}(M)$. Now the natural epimorphism $N \to N/L$ is an $\mathbb{E}$-epimorphism by Lemma 4.3. Thus L is an $\mathbb{E}$-submodule of N, so that N is $\mathbb{E}$-$\mathcal{A}$-extending.

(4) Assume that $M \in \mathcal{F}'(\mathcal{A})$.

(e)⇒(a) If $M \in \mathcal{F}'(\mathcal{A})$, then every $M^{(I)} \in \mathcal{F}'(\mathcal{A})$ because $\mathcal{F}'(\mathcal{A})$ is a natural class. Hence every $M^{(I)}$ is $\mathbb{E}$-$\mathcal{A}$-extending. □

Let us first give a widely applicable corollary for natural classes. Recall that for a natural class of modules $\mathcal{K}$, we have $\mathcal{F}'(\mathcal{K}) = \mathcal{F}(\mathcal{K})$ and $\mathcal{F}(\mathcal{F}(\mathcal{K})) = \mathcal{K}$ [9, Theorem 2.3.15].

Corollary 5.2. *Let $\mathcal{K}$ be a natural class of modules and consider the following statements:*

- (a) *M is Σ-$\mathbb{E}$-$\mathcal{F}(\mathcal{K})$-extending;*
- (b) *Every module in $\mathrm{Add}(M)$ is $\mathbb{E}$-$\mathcal{F}(\mathcal{K})$-extending;*
- (c) *Every M-generated module N has an $\mathbb{E}$-submodule $Y \in \mathcal{F}(\mathcal{K})$ such that $N/Y \in \mathbb{E}\mathrm{Add}(M)$;*
- (d) *Every M-generated module in $\mathcal{K}$ is in $\mathbb{E}\mathrm{Add}(M)$;*
- (e) *Every M-generated module in $\mathcal{K}$ is $\mathbb{E}$-$\mathcal{F}(\mathcal{K})$-extending.*

Then the following implications hold:

1. *For every module M, (a)⇔(b)⇒(c)⇒(d).*
2. *If M is Σ-$\mathbb{E}$-direct projective, then (c)⇒(a).*
3. *If M is Σ-$\mathbb{E}$-direct projective and has the Σ-$\mathcal{F}(\mathcal{K})$-closure property, then (d)⇒(e).*
4. *If $M \in \mathcal{K}$, then (e)⇒(a).*

For the class $\mathcal{S}$ of M-singular modules Theorem 5.1 yields a generalization of [8, Theorem 5.1] from $\mathbb{E}$-extending to $\mathbb{E}$-$\mathcal{S}$-extending modules. When $\mathcal{A}$ is the torsion class of a torsion theory τ in $\sigma[M]$, we have seen in Example 3.4 that $\mathcal{A}$-extending means τ-complemented. Then Theorem 5.1 extends [21, Proposition 10].

For $M = R$ and the proper classes $\mathbb{E}_s$, $\mathbb{E}^{\mathcal{G}}$ and $\mathbb{E}^{\mathcal{P}}$ we have the following consequence of Theorem 5.1.

Corollary 5.3. *Consider the following statements:*

(a) *R is right (finitely, purely) Σ-$\mathcal{A}$-extending;*
(b) *Every projective module is (finitely, purely) $\mathcal{A}$-extending;*
(c) *Every module N has a direct summand (finitely split submodule, pure submodule) $Y \in \mathcal{A}$ such that N/Y is projective (finitely projective, flat);*
(d) *Every module in $\mathcal{F}'(\mathcal{A})$ is projective (finitely projective, flat);*
(e) *Every module in $\mathcal{F}'(\mathcal{A})$ is (finitely, purely) $\mathcal{A}$-extending.*

Then the following implications hold:

(1) *For any R, (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Rightarrow$(d).*
(2) *If R has the Σ-$\mathcal{A}$-closure property, and $\mathcal{A}$ is closed under submodules, then (d)$\Rightarrow$(e).*
(3) *If $R \in \mathcal{F}'(\mathcal{A})$, then (e)$\Rightarrow$(a).*

Let R be non-singular and let $\mathcal{S}$ be the class of singular modules. Then $\mathcal{S}$-extending means extending for R, and R has the Σ-$\mathcal{S}$-closure property, because the latter means here exactly the Σ-closure property. Then one obtains characterizations of a non-singular ring R to be right (finitely, purely) Σ-extending, which recover some known results for extending modules [10, Corollary 11.4] and purely extending modules [6, Proposition 2.1]. Similar characterizations hold for right (finitely, purely) Σ-τ-complemented rings, when τ is a faithful hereditary torsion theory in Mod-R.

Now let us consider a natural intermediate notion between those of $\mathcal{A}$-extending module and $\mathbb{E}$-$\mathcal{A}$-extending module, motivated by the fact that actually a module is extending if and only if its closed submodules coincide with its direct summands.

Definition 5.4. A module M is called *strongly $\mathbb{E}$-$\mathcal{A}$-extending* if M has the $\mathcal{A}$-closure property and the $\mathcal{A}$-closed submodules of M coincide with its $\mathbb{E}$-submodules.

Lemma 5.5. *Let A be a strongly $\mathbb{E}$-$\mathcal{A}$-extending module and D an $\mathcal{A}$-closed submodule ($\mathbb{E}$-submodule) of A. Then A/D is strongly $\mathbb{E}$-$\mathcal{A}$-extending.*

Proof. Straightforward. □

In the following result we characterize Σ-strongly $\mathbb{E}$-$\mathcal{A}$-extending modules.

Theorem 5.6. *Consider the following statements:*

(a) *M is Σ-strongly $\mathbb{E}$-$\mathcal{A}$-extending;*
(b) *Every module in $\mathrm{Add}(M)$ is strongly $\mathbb{E}$-$\mathcal{A}$-extending;*
(c) *Every module in $\mathbb{E}\mathrm{Add}(M)$ is strongly $\mathbb{E}$-$\mathcal{A}$-extending;*
(d) *Every M-generated module in $\mathcal{F}'(\mathcal{A})$ is strongly $\mathbb{E}$-$\mathcal{A}$-extending;*
(e) *$\mathbb{E}\mathrm{Add}(M)$ consists of the M-generated modules in $\mathcal{F}'(\mathcal{A})$.*

Then the following implications hold:

(1) *For every module M, (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c).*
(2) *If $\mathcal{A}$ is closed under submodules, then (a)$\Rightarrow$(e) and (a)$\Rightarrow$(d).*
(3) *If $M \in \mathcal{F}'(\mathcal{A})$, then (d)$\Rightarrow$(a).*

Proof. (1) (a)$\Rightarrow$(c) By Lemma 5.5.

(c)$\Rightarrow$(b)$\Rightarrow$(a) Clear.

(2) Assume that $\mathcal{A}$ is closed under submodules.

(a)⇒(e) By Theorem 5.1, every M-generated module in $\mathcal{F}'(\mathcal{A})$ is in $\mathbb{E}\mathrm{Add}(M)$. Conversely, let $N \in \mathbb{E}\mathrm{Add}(M)$ and take some $\mathbb{E}$-epimorphism $g : M^{(I)} \to N$. Then $K = \mathrm{Ker}\, g$ is an $\mathbb{E}$-submodule of $M^{(I)}$, hence $\mathcal{A}$-closed in $M^{(I)}$. Now any submodule $L \in \mathcal{A}$ of N is isomorphic to a submodule of $M^{(I)}/K \in \mathcal{F}'(\mathcal{A}) = \mathcal{F}(\mathcal{A})$, hence $L = 0$. Thus $N \in \mathcal{F}(\mathcal{A}) = \mathcal{F}'(\mathcal{A})$.

(a)⇒(d) By (c), every module in $\mathbb{E}\mathrm{Add}(M)$ is strongly $\mathbb{E}$-$\mathcal{A}$-extending. Then by (e) it follows that every M-generated τ-torsionfree module is strongly $\mathbb{E}$-$\mathcal{A}$-extending.

(3) Assume that $M \in \mathcal{F}'(\mathcal{A})$.

(d)⇒(a) If $M \in \mathcal{F}'(\mathcal{A})$, then every $M^{(I)} \in \mathcal{F}'(\mathcal{A})$ because $\mathcal{F}'(\mathcal{A})$ is a natural class. Hence every $M^{(I)}$ is strongly $\mathbb{E}$-$\mathcal{A}$-extending. □

As in the case of $\mathbb{E}$-$\mathcal{A}$-extending modules, one may obtain various corollaries of Theorem 5.6, for instance, for natural classes of modules, the class of M-singular modules, or the torsion class of a torsion theory in $\sigma[M]$.

6. Σ-$\mathbb{E}$-$\mathcal{A}$-extending modules and relatively dense extensions

A classical theorem of Oshiro states that for a right Σ-extending ring the class of projective modules is closed under essential extensions [15, Theorem II]. More generally, Gómez Pardo and Guil Asensio showed that for a Σ-extending module M the class $\mathrm{Add}(M)$ is closed under essential extensions [16, Theorem 2.3]. Let us establish such a property in our context.

By Theorem 5.6, for a torsion theory τ in Mod-R generated by $\mathcal{A}$, it follows that if R is τ-torsionfree right Σ-strongly $\mathbb{E}$-$\mathcal{A}$-extending, then the class $\mathbb{E}\mathrm{Add}(R)$ is the torsionfree class of τ. Moreover, if τ is hereditary, then $\mathbb{E}\mathrm{Add}(R)$ is closed under essential extensions (injective hulls). We give such a closure result in the case of $\mathcal{A}$-dense extensions. We consider the following condition on a module M, suggested by the behavior of essential extensions:

(∗) *For every non-zero submodules B, C, D of M with $D \cap B = 0$ and B $\mathcal{A}$-dense in C we have $D \cap C = 0$.*

Example 6.1. (i) Any uniform module trivially satisfies (∗).

(ii) For a torsion theory τ, recall that a module is called *τ-full* if it is τ-torsionfree and a submodule of M is τ-dense in M if and only if it is essential in M [12]. Let $\mathcal{A}$ be the torsion class of τ. Then it is easy to see that every τ-full module satisfies (∗). In particular, every τ-semicocritical module (i.e., non-zero module which is isomorphic to a finite direct sum of τ-cocritical modules) satisfies (∗).

We need the following technical lemma.

Lemma 6.2. *Let $p : M \to N$ be an epimorphism such that M is $\mathbb{E}$-$\mathcal{A}$-extending and N satisfies (∗). If there exists a non-zero submodule D of M such that $D \cap \mathrm{Ker}\, p = 0$ and $p(D)$ is $\mathcal{A}$-dense in N, then* $\mathrm{Ker}\, p$ *is an $\mathbb{E}$-submodule of M.*

Proof. We may assume that $K = \mathrm{Ker}\, p \neq 0$. Since M is $\mathbb{E}$-$\mathcal{A}$-extending, K is contained in an $\mathbb{E}$-submodule L of M such that $L/K \in \mathcal{A}$. Since $D \cap K = 0$, we have $D \cap L = 0$, whence it follows easily that $p(D) \cap p(L) = 0$. This together with the fact that $p(D)$ is $\mathcal{A}$-dense in N implies by hypothesis that $p(L) = p(L) \cap N = 0$, hence $L \subseteq K$. Thus $K = L$ is an $\mathbb{E}$-submodule of M. □

Theorem 6.3. *Let M be Σ-$\mathbb{E}$-$\mathcal{A}$-extending. If $j : X \to Y$ is a non-zero monomorphism such that $X \in \mathbb{E}\mathrm{Add}(M)$, Y is M-generated and satisfies $(*)$, $\mathrm{Im}\, j$ is $\mathcal{A}$-dense in Y and Y has an $\mathbb{E}\mathrm{Add}(M)$-precover, then $Y \in \mathbb{E}\mathrm{Add}(M)$.*

Proof. Let $p : C \to Y$ be an $\mathbb{E}\mathrm{Add}(M)$-precover of Y. Then j factors through p, hence there is a homomorphism $q : X \to C$ such that $pq = j$. Since there exists some $\mathbb{E}$-epimorphism $M^{(I)} \to C$, C is $\mathbb{E}$-$\mathcal{A}$-extending by Lemma 3.5. Since Y is M-generated, it follows that p is an epimorphism. Then we have $q(X) \neq 0$ and $Y/p(q(X)) = Y/j(X) \in \mathcal{A}$. It is easy to check that $q(X) \cap \mathrm{Ker}\, p = 0$, whence $\mathrm{Ker}\, p$ is an $\mathbb{E}$-submodule of C by Lemma 6.2. Since $C \in \mathbb{E}\mathrm{Add}(M)$, it follows by Lemma 4.1 that $Y \in \mathbb{E}\mathrm{Add}(M)$. $\square$

Corollary 6.4. *Let $j : X \to Y$ be a non-zero monomorphism such that Y is M-generated and satisfies $(*)$, and $\mathrm{Im}\, j$ is $\mathcal{A}$-dense in Y. Then:*

(i) If M is Σ-$\mathcal{A}$-extending and $X \in \mathrm{Add}(M)$, then $Y \in \mathrm{Add}(M)$.

(ii) If M is Σ-purely $\mathcal{A}$-extending and $X \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$, then $Y \in \mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$.

Proof. (i) Use the fact that every module has an $\mathrm{Add}(M)$-precover [18] and Theorem 6.3 for $\mathbb{E} = \mathbb{E}_s$.

(ii) Every module has an $\mathbb{E}^{\mathcal{P}}\mathrm{Add}(M)$-cover by Theorem 4.7. Now use Theorem 6.3 for $\mathbb{E} = \mathbb{E}^{\mathcal{P}}$. $\square$

Corollary 6.5. *(i) If R is right Σ-$\mathcal{A}$-extending, then every $\mathcal{A}$-dense extension satisfying $(*)$ of a non-zero projective module is projective.*

(ii) If R is right Σ-purely $\mathcal{A}$-extending, then every $\mathcal{A}$-dense extension satisfying $(*)$ of a non-zero flat module is flat.

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Faculty of Mathematics and Computer Science, "Babeş-Bolyai" University, Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania
E-mail address: crivei@math.ubbcluj.ro