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FINITELY ACCESSIBLE CATEGORIES, GENERALIZED MODULE CATEGORIES AND APPROXIMATIONS

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Abstract. We review some properties of finitely accessible categories related to approximations, and we analyze the relationship between approximations in a finitely accessible additive category and in its associated generalized module category.

1. Introduction

The starting point of approximation theory for modules may be considered the classical result stating that any module embeds (minimally) in an injective module (the injective hull) [11]. Except for some very special cases, for instance when the ring R is of finite representation type (then the modules are direct sums of indecomposable modules), it is virtually impossible to describe all the modules over R. That is why one approximates arbitrary modules with modules in some classes and uses the properties of that class in order to study the entire category of modules. Such a technique, suggested by the previous study of injective hulls begun by Eckmann and Schopf [11] and of projective covers by the study of Bass [4] on perfect rings, appears in the early 1980s. In this sense, the research by Auslander and Smalø [3] in the case of finitely generated modules over finite dimensional algebras, and that by Enochs [13] for arbitrary modules set the base for a modern general theory of (pre)envelopes and (pre)covers. Apart from the classical module-theoretic setting, there have been considered various categorical frameworks for problems concerning approximations, especially for the existence of flat covers, see [2], [12], [14] or [15].

In the present note we place ourselves in the context of a finitely accessible additive category. Any finitely accessible additive category $\mathcal{C}$ is equivalent to the full subcategory $\mathrm{Fl}(A)$ of the category $\mathrm{Mod}(A)$ of unitary right A-modules consisting of flat right A-modules, where A is a ring with enough idempotents called the functor ring of $\mathcal{C}$ (e.g., see [10]). Note also that the category $\mathrm{Mod}(A)$ is equivalent to the functor category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ of all contravariant functors from the full subcategory $\mathrm{fp}(\mathcal{C})$ of finitely presented objects of $\mathcal{C}$ to the category Ab of abelian groups, and this equivalence

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restricts to one between the full subcategories of flat objects of $\mathrm{Mod}(A)$ and $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$.

Functor ring (functor category) techniques may be employed in order to relate properties of modules over rings with enough idempotents (objects in a functor category) to properties of objects in finitely accessible additive categories (e.g., see [10], [18]). We shall show how approximations of objects in a finitely accessible additive category and of objects in its associated functor category (or of unitary modules over its functor ring) may be related. Previously, Herzog [20] has used such an approach to prove the existence of a pure-injective envelope for any object in a finitely accessible additive category.

2. The setting

Throughout the paper $\mathcal{C}$ will be an additive category. We prepare the setting, explaining the needed terminology: functor rings and functor categories, finitely accessible categories, purity in finitely accessible categories, special functors.

Functor ring and functor category. Assume that the class of finitely presented objects of $\mathcal{C}$ is skeletally small and let $\mathcal{U} = (U_i)_{i\in I}$ be the family of representative classes of all finitely presented objects of $\mathcal{C}$. We associate a ring $A = A_{\mathcal{U}}$ to the family $\mathcal{U}$ in the following way (e.g., see [10], [17]):

$$A = \bigoplus_{i\in I} \bigoplus_{j\in J} \mathrm{Hom}(U_i, U_j)$$

as abelian group, and the multiplication is given by the rule: if $f \in \mathrm{Hom}(U_i, U_j)$ and $g \in \mathrm{Hom}(U_k, U_l)$, then $fg = f \circ g$ if $i = l$ and zero otherwise. Then A is a ring with enough idempotents [16], say $A = \bigoplus_{i\in I} e_i A = \bigoplus_{i\in I} Ae_i$. The idempotents e_i are the elements of A which are the identity on U_i and zero elsewhere, and they form a complete family of pairwise orthogonal idempotents. The ring A constructed above is called the *functor ring* of $\mathcal{C}$. Denote by $\mathrm{Mod}(A)$ the category of unitary right A-modules and note that the family $(e_iA)_{i\in I}$ is a family of finitely generated projective generators of $\mathrm{Mod}(A)$. $\mathrm{Mod}(A)$ is also refered to as a generalized module category.

Denote by $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ the category of all contravariant functors from the full subcategory $\mathrm{fp}(\mathcal{C})$ of finitely presented objects of $\mathcal{C}$ to the category Ab of abelian groups. A family of finitely generated projective generators of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is given by the representable functors. It is well known that there is an equivalence of categories between $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ and $\mathrm{Mod}(A)$ (see [17], [25]).

Finitely accessible categories. Let us recall some terminology on finitely accessible categories. The category $\mathcal{C}$ is called *finitely accessible* (or *locally finitely presented* in the terminology of [8]) if the class of finitely presented objects is skeletally small, $\mathcal{C}$ has direct limits, and every object of $\mathcal{C}$ is a direct limit of finitely presented objects [23]. Following [23], we reserve the name *locally finitely presented* for a category which is finitely accessible and cocomplete (i.e., it has all colimits), equivalently finitely accessible and complete (i.e., it has all limits).

A finitely accessible additive category $\mathcal{C}$ may be identified with a full subcategory of the functor category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$. Denote by $\mathrm{Fl}(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ the full subcategory of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ consisting of the flat objects.

Theorem 2.1. [8, 1.4 Theorem] *Let $\mathcal{C}$ be a finitely accessible category and consider the covariant Yoneda functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$, which sends an object Z of $\mathcal{C}$ to the functor*

$$H_Z = \mathrm{Hom}(-, Z)|_{\mathrm{fp}(\mathcal{C})},$$

and a morphism $f : X \to Y$ in $\mathcal{C}$ to the natural transformation

$$H_f = \mathrm{Hom}(-, f) : H_X \to H_Y.$$

Then H induces an equivalence between the categories $\mathcal{C}$ and $\mathrm{Fl}(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$.

The covariant Yoneda functor may have a left adjoint in some finitely accessible categories.

Theorem 2.2. [8, 2.2] *Let $\mathcal{C}$ be a locally finitely presented category. Then the covariant Yoneda functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ has a left adjoint.*

In particular, if $\mathcal{C}$ is locally coherent finitely accessible, then H has a left adjoint.

Purity in finitely accessible categories. The framework of an accessible category in the sense of [1] and, in particular, of a finitely accessible additive category [8], is a natural one in which to consider purity.

By a *sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$ we mean a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$ with $gf = 0$. Such a sequence in $\mathcal{C}$ is called *pure exact* if it induces an exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z) \to 0$$

for every finitely presented object P of $\mathcal{C}$ [8]. Then f and g are called *pure monomorphism* and *pure epimorphism* respectively. Note that in a pure exact sequence the morphisms form a kernel-cokernel pair, and so pure monomorphisms and pure epimorphisms are indeed monomorphisms and epimorphisms respectively.

Purity has a nice behaviour through the Yoneda functor, as we may see in the following result (see [8]).

Theorem 2.3. *Let $0 \to X \to Y \to Z \to 0$ be a sequence in $\mathcal{C}$. Then it is pure exact in $\mathcal{C}$ if and only if*

$$0 \to H_X \to H_Y \to H_Z \to 0$$

is exact in $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$. *In this case the latter sequence is pure exact in* $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$.

Separable and Maschke functors. There are special functors that have good properties with respect to approximations, among them separable and Maschke functors. We recall their definitions from [22] and [7].

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between categories $\mathcal{C}$ and $\mathcal{D}$. Consider the associated natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))$$

defined by

$$\mathcal{F}_{C,C'}(f) = F(f)$$

for every morphism $f : C \to C'$ in $\mathcal{C}$.

The functor F is called *separable* if $\mathcal{F}$ splits as a natural transformation, that is, there exists a natural transformation

$$\mathcal{L} : \mathrm{Hom}_{\mathcal{D}}(F(-), F(-)) \to \mathrm{Hom}_{\mathcal{E}}(-,-)$$

such that $\mathcal{L} \circ \mathcal{F} = 1_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$.

The functor F is called *Maschke* if for every object M of $\mathcal{C}$, for every morphism $i : C \to C'$ in $\mathcal{C}$ with $F(i) : F(C) \to F(C')$ a split monomorphism in $\mathcal{D}$, and for every morphism $f : C \to M$ in $\mathcal{C}$, there exists a morphism $g : C' \to M$ in $\mathcal{C}$ such that $f = gi$.

Dually, one defines the notion of *dual Maschke* functor.

Let us note that every separable functor is a Maschke and dual Maschke functor [7, Proposition 3.3].

3. Approximations

Covers and envelopes. Let us recall the notions of (pre)cover and (pre)envelope (e.g., see [26]). By a class of objects in $\mathcal{C}$ we mean a class of objects closed under isomorphisms.

Let M be an object in $\mathcal{C}$ and $\mathcal{A}$ be a class of objects in $\mathcal{C}$. A morphism $f \in \mathrm{Hom}(A, M)$, with $A \in \mathcal{A}$, is called an $\mathcal{A}$*-precover* of M if the induced abelian group morphism

$$\mathrm{Hom}(A', A) \to \mathrm{Hom}(A', M)$$

is surjective for every $A' \in \mathcal{A}$.

An $\mathcal{A}$-precover $f \in \mathrm{Hom}(A, M)$ of M is called an $\mathcal{A}$*-cover* if every endomorphism $g : A \to A$ with $fg = f$ is an automorphism.

The class $\mathcal{A}$ is called *(pre)covering* if every object of $\mathcal{C}$ has an $\mathcal{A}$-cover.

An $\mathcal{A}$-(pre)cover $f \in \mathrm{Hom}(A, M)$ of M is said to *have the unique mapping property* if for every $f' \in \mathrm{Hom}(A', M)$ with $A' \in \mathcal{A}$, there exists a unique $g \in \mathrm{Hom}(A', A)$ such that $fg = f'$.

Dually, one defines the notions of *relative (pre)envelope*, *(pre)enveloping class* and *(pre)envelope with the unique mapping property.*

Properties in finitely accessible categories versus approximations. Properties in a finitely accessible category $\mathcal{C}$ may be deduced or expressed by means of approximations in its associated generalized module category $\mathrm{Mod}(A)$. Recall that a *pseudokernel* of a morphism $g : Y \to Z$ in $\mathcal{C}$ is a morphism $f : X \to Y$ with $gf = 0$ such that for every morphism $h : X' \to Y$ there is a morphism $\alpha : X' \to X$ such that $f\alpha = h$. Dually, one defines the notion of *pseudocokernel*.

We discuss the existence of (pseudo)kernels and (pseudo)cokernels in a finitely accessible category.

Proposition 3.1. [18, Lemma 2.2] *Every finitely accessible category has pseudokernels.*

This follows because a finitely accessible category $\mathcal{C}$ has pseudokernels if and only if every right A-module has a flat cover, which always holds (e.g., see [18, Lemma 2.2]). Note that since the class of flat right A-modules is closed under direct limits, the existence of flat precovers is equivalent to the existence of flat covers [26, Theorem 2.2.8].

Proposition 3.2. [18, Proposition 2.1] *Let $\mathcal{C}$ be a finitely accessible category. Then $\mathcal{C}$ has products if and only if $\mathcal{C}$ has pseudocokernels if and only if every right A-module has a flat preenvelope.*

Immediate categorical considerations show the following.

Proposition 3.3. *Let $\mathcal{C}$ be a finitely accessible category. Then $\mathcal{C}$ has kernels if and only if every right A-module has a flat cover with the unique mapping property.*

Proposition 3.4. *Let $\mathcal{C}$ be a finitely accessible category. Then $\mathcal{C}$ has cokernels if and only if every right A-module has a flat preenvelope with the unique mapping property.*

Preserving and reflecting approximations. Several functors have a good behaviour with respect to preserving or reflecting approximations.

Let $\mathcal{A}$ be a class of objects in $\mathcal{C}$. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is said to *preserve $\mathcal{A}$-(pre)covers* if whenever $f : A \to D$ is an $\mathcal{A}$-(pre)cover of an object D of $\mathcal{D}$, $F(f) : F(A) \to F(D)$ is an $F(\mathcal{A})$-(pre)cover of $F(D)$. The functor F *reflects $\mathcal{A}$-(pre)covers* if whenever $F(f) : F(A) \to F(D)$ is an $F(\mathcal{A})$-(pre)cover of $F(D)$ for some object D of $\mathcal{D}$, $f : A \to D$ is an $\mathcal{A}$-(pre)cover of D.

Proposition 3.5. [19, Proposition 3] *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between categories $\mathcal{C}$ and $\mathcal{D}$, and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$.*

(i) *If F is full, then F preserves $\mathcal{A}$-precovers.*
(ii) *If F is full and faithful, then F preserves $\mathcal{A}$-covers.*
(iii) *If F is separable, then F reflects $\mathcal{A}$-covers.*

If $\mathcal{C}$ is a finitely accessible category, then the covariant Yoneda functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is full and faithful. It is known and easy to see that a functor is full and faithful if and only if it is full and separable. Now we have the following corollary.

Corollary 3.6. *Let $\mathcal{C}$ be a finitely accessible category and $\mathcal{A}$ be a class of objects in $\mathcal{C}$. Then H preserves and reflects $\mathcal{A}$-covers.*

In case of a pair of adjoint functors, we have the following variations of the above results, obtained by using [19, Propositions 4 and 6] and the Rafael-type characterization of dual Maschke functors [7, Theorem 3.4].

Theorem 3.7. *Let (L, R) be an adjoint pair of covariant functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$ and $\mathcal{B}$ be a class of objects in $\mathcal{D}$ such that $R(\mathcal{A}) \subseteq \mathcal{B}$ and $L(\mathcal{B}) \subseteq \mathcal{A}$.*

(i) *If C is an object of $\mathcal{C}$ which has an $\mathcal{A}$-precover, then $R(C)$ has a $\mathcal{B}$-precover.*

(ii) *If R is dual Maschke and C is an object of $\mathcal{C}$ such that $R(C)$ has a $\mathcal{B}$-precover, then C has an $\mathcal{A}$-precover.*

As the covariant Yoneda functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is separable, it is also dual Maschke. We have seen that if $\mathcal{C}$ is a locally finitely presented category, then H has a left adjoint, say S. Now we obtain a corollary for locally finitely presented categories.

Corollary 3.8. *Let $\mathcal{C}$ be a locally finitely presented category, let $\mathcal{A}$ be a class of objects in $\mathcal{C}$ and $\mathcal{B}$ be a class of objects in $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ such that $S(\mathcal{A}) \subseteq \mathcal{B}$ and $H(\mathcal{B}) \subseteq \mathcal{A}$, and let X be an object of $\mathcal{C}$. Then X has an $\mathcal{A}$-precover if and only if $H(X)$ has a $\mathcal{B}$-precover.*

Similar results may be established for (pre)envelopes.

Applications. In practice, in order to obtain approximation results in a finitely accessible category $\mathcal{C}$, one may establish such properties in the associated functor category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ (or in the associated category $\mathrm{Mod}(A)$ of unitary modules over the functor ring A of $\mathcal{C}$), and afterwards pull them back in $\mathcal{C}$, using the covariant Yoneda functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ and the above results.

It is worth mentioning a couple of general results on the existence of covers and envelopes. Note that a class of modules is closed under direct sums and pure epimorphic images if and only if it is closed under direct limits and pure epimorphic images (see [25, 33.9]).

Theorem 3.9. [6], [5, Theorem 2.5], [21, Theorem 2.5] *Over a ring with identity, every class of modules closed under direct sums and pure epimorphic images is covering.*

Theorem 3.10. [24, Corollary 3.5] *Over a ring with identity, every class of modules closed under direct products and pure submodules is preenveloping.*

Both results are shown to hold in the context of a ring with enough idempotents as well, see [9]. Using some of the above properties and techniques, they have been extended to finitely accessible categories in [9].

Theorem 3.11. [9] *Let $\mathcal{C}$ be a finitely accessible category and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$ closed under direct limits and pure epimorphic images. Then $\mathcal{A}$ is a covering class.*

Theorem 3.12. [9] *Let $\mathcal{C}$ be a finitely accessible category with products and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$ closed under direct products and pure subobjects. Then $\mathcal{A}$ is a preenveloping class.*

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