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Associated classes of modules

IULIU CRIVEI AND SEPTIMIU CRIVEI

Abstract. Let $\mathcal{C}$ be a non-empty class of modules closed under isomorphic copies. We consider some classes of modules associated to $\mathcal{C}$. Among them, we study two important classes in the theory of natural and conatural classes of modules, namely the class consisting of all modules having no non-zero submodule in $\mathcal{C}$, as well as its dual.

1. **Introduction**

Throughout R is an associative ring with non-zero identity and all modules are unitary right R-modules. Also, $\mathcal{C}$ is a class of modules, always non-empty and closed under isomorphic copies. For modules A, B, C, we denote by $A \leq B$ (respectively $A < B$, $A \trianglelefteq B$, $A << B$) the fact that A is a submodule (respectively proper, essential, superfluous submodule) of B. Also, we denote by $A \hookrightarrow B$ a monomorphism from A to B and by $B \twoheadrightarrow C$ an epimorphism from B to C.

Consider the following classes associated to $\mathcal{C}$:

$$\begin{aligned}
\mathcal{F}(\mathcal{C}) &= \{A \mid 0 \neq B \leq A \Longrightarrow B \notin \mathcal{C}\},\\
\mathcal{T}(\mathcal{C}) &= \{A \mid B < A \Longrightarrow A/B \notin \mathcal{C}\},\\
\mathcal{F}'(\mathcal{C}) &= \{B \mid A \text{ submodule of } B, M \in \mathcal{C}, A \hookrightarrow M \Longrightarrow A = 0\},\\
\mathcal{T}'(\mathcal{C}) &= \{B \mid C \text{ homomorphic image of } B, M \in \mathcal{C}, M \twoheadrightarrow C \Longrightarrow C = 0\},\\
\mathcal{H}(\mathcal{C}) &= \{A \mid A \notin \mathcal{C}, \text{ but } 0 \neq B \leq A \Longrightarrow A/B \in \mathcal{C}\}\\
\mathcal{S}(\mathcal{C}) &= \{A \mid A \notin \mathcal{C}, \text{ but } B < A \Longrightarrow B \in \mathcal{C}\}\\
\mathcal{H}'(\mathcal{C}) &= \{A \mid 0 \neq B \leq A \Longrightarrow A/B \in \mathcal{C}\}\\
\mathcal{S}'(\mathcal{C}) &= \{A \mid B < A \Longrightarrow B \in \mathcal{C}\}
\end{aligned}$$

The first four classes are important bricks in the theory of hereditary and cohereditary classes and, in particular, in the theory of natural and conatural classes [7]. The last four ones arise naturally and include various examples, such as simple modules or almost finitely generated modules [9]. We shall establish some properties of these classes.

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2. Classes related to natural and conatural classes

The terminology of natural classes appeared in the beginning of the 1990s, allowing unification and simplification of some previous results in ring and module theory. A *natural class* is defined as a class of modules closed under isomorphic copies, submodules, direct sums and injective hulls. They have been studied by J. Dauns [5], S.S. Page and Y. Zhou [8] in the 1990s. In recent years, in a series of articles further developed in a recent monograph [7], J. Dauns and Y. Zhou have created a powerful theory of what is thought to be the new generation of ring and module theory.

Note that since projective covers of modules do not exist in general, the notion of natural class previously defined cannot be always dualized (this is possible for instance in the case of modules over perfect rings). The class $\mathcal{C}$ is called a *conatural class* if the condition

$$(*)\ \forall M \twoheadrightarrow N \neq 0, \text{ there exist } C \in \mathcal{C},\ K \neq 0 \text{ and } N \twoheadrightarrow K \twoheadleftarrow C$$

implies $M \in \mathcal{C}$ [1]. In general one only has that, if $\mathcal{C}$ is a conatural class, then $\mathcal{C}$ is closed under homomorphic images and superfluous epimorphisms [1, Theorem 24].

Alternatively, natural classes in Mod-R may be seen as the skeleton of the class of all hereditary classes (closed under submodules) in Mod-R. This point of view allows one to introduce conatural classes, as the skeleton of the class of all cohereditary classes (closed under homomorphic images) in Mod-R. This was the approach of A. Alvarado García, H. Rincón and J. Ríos Montes [1].

It is known that natural classes and conatural classes form complete Boolean lattices. If $\mathcal{C}$ is a natural class, then its complement is $\mathcal{F}(\mathcal{C})$, whereas if $\mathcal{C}$ is a conatural class, then its complement is $\mathcal{T}(\mathcal{C})$ [1]. Moreover, natural classes and conatural classes are characterized as follows, properties that show the strong relationship between them and the considered associated classes and motivates the interest in their study.

Theorem 2.1. [8, Proposition 4] *A hereditary class $\mathcal{C}$ is a natural class if and only if $\mathcal{C} = \mathcal{F}(\mathcal{F}(\mathcal{C}))$ if and only if $\mathcal{F}(\mathcal{F}(\mathcal{C})) \subseteq \mathcal{C}$.*

Theorem 2.2. [1, Theorem 23] *A cohereditary class $\mathcal{C}$ is a conatural class if and only if $\mathcal{C} = \mathcal{T}(\mathcal{T}(\mathcal{C}))$ if and only if $\mathcal{T}(\mathcal{T}(\mathcal{C})) \subseteq \mathcal{C}$.*

Now we establish several properties of our classes, giving proofs only for the classes $\mathcal{T}'(\mathcal{C})$ and $\mathcal{T}(\mathcal{C})$. Note first that if $\mathcal{C}$ is hereditary, then $\mathcal{F}'(\mathcal{C}) = \mathcal{F}(\mathcal{C})$ and, if $\mathcal{C}$ is cohereditary, then $\mathcal{T}'(\mathcal{C}) = \mathcal{T}(\mathcal{C})$.

Denote $\mathcal{C}^{\perp} = \{Y \mid \mathrm{Hom}_R(\mathcal{C}, Y) = 0\}$ and ${}^{\perp}\mathcal{C} = \{X \mid \mathrm{Hom}_R(X, \mathcal{C}) = 0\}$. The final part of the following result completes [6, Theorem 3.1].

Theorem 2.3. *(i) $\mathcal{T}'(\mathcal{C})$ is closed under homomorphic images, extensions and superfluous epimorphisms.*

(ii) Let $B \in \mathcal{T}'(\mathcal{C})$ and $A \leq B$ with $A \in \mathcal{C}$. Then $A << B$.

(iii) If $\mathcal{C}$ is hereditary, then ${}^{\perp}\mathcal{C} = \mathcal{T}(\mathcal{C})$ and $\mathcal{C} \subseteq \mathcal{T}(\mathcal{C})^{\perp}$.

(iv) If $\mathcal{C}$ is hereditary, then $\mathcal{T}(\mathcal{C})$ is a torsion class. If $\mathcal{C}$ is also closed under essential extensions, then $\mathcal{T}(\mathcal{C})$ is a hereditary torsion class.

(v) Let $\mathcal{C}$ be a natural class. Then $\mathcal{C}$ cogenerates a hereditary torsion theory, namely $(\mathcal{T}(\mathcal{C}), \mathcal{F}(\mathcal{T}(\mathcal{C})))$, and $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is a natural class.

Proof. (i) Clearly, $\mathcal{T}'(\mathcal{C})$ is cohereditary.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, C \in \mathcal{T}'(\mathcal{C})$. We may assume that $A \leq B$. Let D' be a homomorphic image of B, say B/D, and let $M \in \mathcal{C}$ and $M \twoheadrightarrow D'$. Then there exists $M \twoheadrightarrow B/D \twoheadrightarrow B/(A+D) \cong (B/A)/((A+D)/A)$, the last one being a homomorphic image of C. Since $C \in \mathcal{T}'(\mathcal{C})$, it follows that $B/(A+D) = 0$, hence $A + D = B$. Now $D' \cong B/D = (A+D)/D \cong A/(A \cap D)$ is a homomorphic image of $A \in \mathcal{T}'(\mathcal{C})$. Then $D' = 0$. Thus $\mathcal{T}'(\mathcal{C})$ is closed under extensions.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $f(A) << B$ and $C \in \mathcal{T}'(\mathcal{C})$. We may assume that $A \leq B$. Let D' be a homomorphic image of B, say B/D, $M \in \mathcal{C}$ and $M \twoheadrightarrow D'$. As above, it follows that $A + D = B$. But since $A << B$, we get $D = B$, hence $D' = 0$. Thus $B \in \mathcal{T}'(\mathcal{C})$. Hence $\mathcal{T}'(\mathcal{C})$ is closed under superfluous epimorphisms.

(ii) Let D be such that $A + D = B$. Then $A/(A \cap D) \cong (A+D)/D = B/D$, hence there exists $A \twoheadrightarrow B/D$. Since $A \in \mathcal{C}$ and $B \in \mathcal{T}'(\mathcal{C})$, it follows that $D = B$. Hence $A << B$.

(iii) Let $A \in \mathcal{T}(\mathcal{C})$, $B \in \mathcal{C}$ and $0 \neq f \in \mathrm{Hom}_R(A, B)$. Since $\mathcal{C}$ is closed under submodules, $\mathrm{Im}\, f \in \mathcal{C}$. But since $\mathrm{Ker}\, f \neq A \in \mathcal{T}(\mathcal{C})$, we have $\mathrm{Im}\, f \cong A/\mathrm{Ker}\, f \notin \mathcal{C}$, a contradiction. Thus $\mathrm{Hom}_R(\mathcal{T}(\mathcal{C}), \mathcal{C}) = 0$. Hence we have $\mathcal{T}(\mathcal{C}) \subseteq^{\perp} \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{T}(\mathcal{C})^{\perp}$.

Now let $A \in^{\perp} \mathcal{C}$. If $A = 0$, we are done, so that assume $A \neq 0$. Let $B < A$ and suppose that $A/B \in \mathcal{C}$. Then $\mathrm{Hom}_R(A, A/B) = 0$. But the natural homomorphism $p : A \to A/B$ is non-zero, a contradiction. Hence $A/B \notin \mathcal{C}$, so that $A \in \mathcal{T}(\mathcal{C})$.

(iv) The first part follows by (iii). In order to show that $\mathcal{T}(\mathcal{C})$ is hereditary, let $A \in \mathcal{T}(\mathcal{C})$ and $B \leq A$. Suppose that $B \notin \mathcal{T}(\mathcal{C})$, hence there exists $C \in \mathcal{C}$ and a non-zero homomorphism $f : B \to C$. Taking the injective hull $j : C \to E$ of C, it follows that there is a non-zero homomorphism $h : A \to E$ extending jf. But this contradicts the fact that $A \in \mathcal{T}(\mathcal{C})$ and $E \in \mathcal{C}$.

(v) By (iii) and (iv), $\mathcal{T}(\mathcal{C})$ is the torsion class of the torsion theory cogenerated by $\mathcal{C}$, while $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is its torsionfree class. Now $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is a natural class. □

In a dual manner one obtains the following result. Note that in case R is right perfect we have a characterization of conatural classes as follows: $\mathcal{C}$ is a conatural class if and only if $\mathcal{C}$ is closed under homomorphic images, projective covers and direct sums of simple modules [2, Theorem 17].

Theorem 2.4. *(i) $\mathcal{F}'(\mathcal{C})$ is closed under submodules, extensions and essential extensions.*

(ii) Let $B \in \mathcal{F}'(\mathcal{C})$ and $A \leq B$ be such that $B/A \in \mathcal{C}$. Then $A \trianglelefteq B$.

(iii) If $\mathcal{C}$ is cohereditary, then $\mathcal{C}^{\perp} = \mathcal{F}(\mathcal{C})$ and $\mathcal{C} \subseteq^{\perp} \mathcal{F}(\mathcal{C})$.

(iv) If $\mathcal{C}$ is cohereditary, then $\mathcal{F}(\mathcal{C})$ is a torsionfree class. If R is right perfect and $\mathcal{C}$ is also closed under superfluous epimorphisms, then $\mathcal{F}(\mathcal{C})$ is a cohereditary torsionfree class.

(v) Let R be right perfect and $\mathcal{C}$ a conatural class. Then $\mathcal{C}$ generates a cohereditary torsion theory, namely $(\mathcal{T}(\mathcal{F}(\mathcal{C})), \mathcal{F}(\mathcal{C}))$, and $\mathcal{T}(\mathcal{F}(\mathcal{C}))$ is a conatural class.

In what follows, let us see other connections between our classes. The second part of the next result was established for a natural class $\mathcal{C}$ in [6, Lemma 3.3].

Theorem 2.5. *If either $\mathcal{T}(\mathcal{C})$ is hereditary or $\mathcal{C}$ is closed under essential extensions, then*

$$\mathcal{T}(\mathcal{C}) = \{A \mid B < A \Longrightarrow A/B \in \mathcal{F}(\mathcal{C})\} \subseteq \mathcal{F}(\mathcal{C}).$$

Proof. Denote $\mathcal{A} = \{A \mid B < A \Longrightarrow A/B \in \mathcal{F}(\mathcal{C})\}$. Note that the inclusion $\mathcal{A} \subseteq \mathcal{T}(\mathcal{C})$ holds for any class $\mathcal{C}$. Indeed, let $A \in \mathcal{A}$ and $B < A$. Then $A/B \in \mathcal{F}(\mathcal{C})$, hence $A/B \notin \mathcal{C}$. Thus $A \in \mathcal{T}(\mathcal{C})$. Now we show the converse inclusion.

(i) Suppose that $\mathcal{T}(\mathcal{C})$ is hereditary. Let $A \in \mathcal{T}(\mathcal{C})$ and $B < A$. Let us prove that $A/B \in \mathcal{F}(\mathcal{C})$. Let $0 \neq D/B \leq A/B$. By hypothesis, we have $D \in \mathcal{T}(\mathcal{C})$. Since $B < D$, it follows that $D/B \notin \mathcal{C}$. Thus $A/B \in \mathcal{F}(\mathcal{C})$, whence $A \in \mathcal{A}$.

(ii) Suppose that $\mathcal{C}$ is closed under essential extensions. Let $A \in \mathcal{T}(\mathcal{C}) \setminus \mathcal{A}$. Then there exists $B < A$ such that $A/B \notin \mathcal{F}(\mathcal{C})$, whence there exists $0 \neq D/B \leq A/B$ such that $D/B \in \mathcal{C}$. Let D'/B be a complement of D/B in A/B. Then $D/B \cap D'/B = 0$ and $D/B + D'/B \trianglelefteq A/B$. It follows that $(D/B + D'/B)/(D'/B) \trianglelefteq (A/B)/(D'/B)$, that is, $D/B \trianglelefteq A/D'$. Since $D/B \in \mathcal{C}$, we get $A/D' \in \mathcal{C}$ by the hypothesis on $\mathcal{C}$. Having noted that $D' < A$, we have $A \notin \mathcal{T}(\mathcal{C})$, a contradiction. Hence $\mathcal{T}(\mathcal{C}) \subseteq \mathcal{A}$. $\square$

Dually, one has the following result. Note that if the ring is right perfect, then every submodule of a module has a supplement.

Theorem 2.6. *If either $\mathcal{F}(\mathcal{C})$ is cohereditary or R is right perfect and $\mathcal{C}$ is closed under superfluous epimorphisms, then*

$$\mathcal{F}(\mathcal{C}) = \{A \mid 0 \neq B \leq A \Longrightarrow B \in \mathcal{T}(\mathcal{C})\} \subseteq \mathcal{T}(\mathcal{C}).$$

3. **Other associated classes**

As before, let $\mathcal{C}$ be a class of modules. We have considered in the introduction the classes $\mathcal{H}(\mathcal{C})$, $\mathcal{S}(\mathcal{C})$, $\mathcal{H}'(\mathcal{C})$ and $\mathcal{S}'(\mathcal{C})$. Now we mention some examples covered by such general associated classes.

Example 3.1. (1) If $\mathcal{C} = \{0\}$, then $\mathcal{H}(\mathcal{C})$ consists of the simple modules.

(2) If $\mathcal{C}$ is the torsion class for a hereditary torsion theory τ in Mod-R, then $\mathcal{H}(\mathcal{C})$ consists of the τ-cocritical modules.

(3) If $\mathcal{C}$ is the class of commutative perfect rings, then $\mathcal{H}'(\mathcal{C})$ consists of the almost perfect rings [3].

(4) If $\mathcal{C} = \{0\}$, then $\mathcal{S}(\mathcal{C})$ consists of the simple modules.

(5) If $\mathcal{C}$ is the class of finitely generated modules, then $\mathcal{S}(\mathcal{C})$ consists of the almost finitely generated (a.f.g.) modules [9].

(6) If $\mathcal{C}$ is the class of modules having maximal submodules, then $\mathcal{S}(\mathcal{C})$ consists of the a.m.s. modules [4].

Let us give some properties of these classes. We will give proofs only for the classes $\mathcal{H}(\mathcal{C})$ and $\mathcal{H}'(\mathcal{C})$, the other ones being dual.

Theorem 3.2. *(i) If $0 \in \mathcal{C}$, then every simple module belongs either to $\mathcal{C}$ or to $\mathcal{H}(\mathcal{C})$.*

(ii) $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{H}'(\mathcal{C})$. If $\mathcal{C}$ is closed under extensions, then $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{F}(\mathcal{C})$.

(iii) If $\mathcal{C}$ is closed under submodules and extensions, then $\mathcal{H}(\mathcal{C})$ is closed under non-zero submodules and every module in $\mathcal{H}(\mathcal{C})$ is uniform.

(iv) If $\mathcal{C}$ is hereditary, then $\mathcal{H}'(\mathcal{C})$ is closed under non-zero submodules.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules and assume that $\mathcal{C}$ is closed under extensions.

(v) If $A \in \mathcal{H}'(\mathcal{C})$, $f(A) \trianglelefteq B$ and every homomorphic image of C belongs to $\mathcal{C}$, then $B \in \mathcal{H}'(\mathcal{C})$.

Proof. (i) and (ii) Straightforward.

(iii) Let $A \in \mathcal{H}(\mathcal{C})$ and $0 \neq B \leq A$. Suppose that $B \in \mathcal{C}$. Since $A/B \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions, we have $A \in \mathcal{C}$, a contradiction. Hence $B \notin \mathcal{C}$. Now let $0 \neq D \leq B$. Then $B/D \leq A/D \in \mathcal{C}$, hence $B/D \in \mathcal{C}$. Therefore $B \in \mathcal{H}(\mathcal{C})$.

Let $A \in \mathcal{H}(\mathcal{C})$ and suppose that it is not uniform. Then there exist non-zero submodules B and D of A such that $B \cap D = 0$. Since $\mathcal{H}(\mathcal{C})$ is closed under non-zero submodules, we have $B, B + D \in \mathcal{H}(\mathcal{C})$. Hence $B \cong (B + D)/D \in \mathcal{C}$, a contradiction. Therefore A is uniform.

(iv) See (iii).

(v) We may assume that A is a submodule of B. Let D be a non-zero proper submodule of B. Then $A \cap D \neq 0$, hence $(A + D)/D \cong A/(A \cap D) \in \mathcal{C}$. Consider the exact sequence of modules $0 \to (A + D)/D \to B/D \to B/(A + D) \to 0$. Since $\mathcal{C}$ is closed under extensions, it follows that $B/D \in \mathcal{C}$. Therefore $B \in \mathcal{H}'(\mathcal{C})$. □

Theorem 3.3. *(i) If $0 \in \mathcal{C}$, then every simple module belongs either to $\mathcal{C}$ or to $\mathcal{S}(\mathcal{C})$.*

(ii) $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}'(\mathcal{C})$. If $\mathcal{C}$ is closed under extensions, then $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{T}(\mathcal{C})$.

(iii) If $\mathcal{C}$ is closed under homomorphic images and extensions, then $\mathcal{S}(\mathcal{C})$ is closed under proper homomorphic images and every module in $\mathcal{S}(\mathcal{C})$ is hollow.

(iv) If $\mathcal{C}$ is cohereditary, then $\mathcal{S}'(\mathcal{C})$ is closed under proper homomorphic images.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules and assume that $\mathcal{C}$ is closed under extensions.

(v) If $C \in \mathcal{S}'(\mathcal{C})$, $f(A) << B$ and every submodule of A belongs to $\mathcal{C}$, then $B \in \mathcal{S}'(\mathcal{C})$.

Theorem 3.4. *(i) Let $A \in \mathcal{H}'(\mathcal{C})$, $B \in \mathcal{F}'(\mathcal{C})$ and let $f : A \to B$ be a non-zero homomorphism. Then f is a monomorphism.*

(ii) Let $M \in \mathcal{H}'(\mathcal{C})$, let $N \in \mathcal{F}(\mathcal{C})$ be quasi-injective and let $S = \mathrm{End}_R(N)$. Then $\mathrm{Hom}_R(M, N)$ is a simple left S-module.

Proof. (i) We have $\mathrm{Ker}\, f \neq A$. Suppose that $\mathrm{Ker}\, f \neq 0$. Then $\mathrm{Im}\, f \cong A/\mathrm{Ker}\, f \in \mathcal{C}$, because $A \in \mathcal{H}'(\mathcal{C})$. Since $\mathrm{Im}\, f \hookrightarrow A/\mathrm{Ker}\, f \in \mathcal{C}$ and $B \in \mathcal{F}'(\mathcal{C})$, it follows that $\mathrm{Im}\, f = 0$, a contradiction. Hence f is a monomorphism.

(ii) Let $0 \neq f \in \mathrm{Hom}_R(M, N)$. By (i), f is a monomorphism. Let $g \in \mathrm{Hom}_R(M, N)$. Since N is quasi-injective, there exists $h \in S$ such that $hf = g$. Hence $g \in Sf$, so that $\mathrm{Hom}_R(M, N) = Sf$. Thus $\mathrm{Hom}_R(M, N)$ is a simple left S-module. □

Theorem 3.5. *(i) Let $A \in \mathcal{T}'(\mathcal{C})$, $B \in \mathcal{S}'(\mathcal{C})$ and let $f : A \to B$ be a non-zero homomorphism. Then f is an epimorphism.*

(ii) Let $M \in \mathcal{S}'(\mathcal{C})$, let $N \in \mathcal{T}(\mathcal{C})$ be quasi-projective and let $S = \mathrm{End}_R(N)$. Then $\mathrm{Hom}_R(N, M)$ is a simple left S-module.

In the sequel, let us see some other properties of the class $\mathcal{H}(\mathcal{C})$, when $\mathcal{C}$ is closed under submodules and extensions.

Theorem 3.6. *Let $\mathcal{C}$ be closed under submodules and extensions. Let A be a non-zero uniform module that has a submodule $B \in \mathcal{H}(\mathcal{C})$. Then A has a unique maximal submodule that belongs to $\mathcal{H}(\mathcal{C})$.*

Proof. Denote by D_i the submodules of A that belong to $\mathcal{H}(\mathcal{C})$, where $1 \leq i \leq \omega$ and ω is some ordinal. We show that $D = \sum_{i \leq \omega} D_i \in \mathcal{H}(\mathcal{C})$ by transfinite induction on ω. For $\omega = 1$, the result is trivial. Suppose that $\omega > 1$ and that $E = \sum_{i<\omega} D_i \in \mathcal{H}(\mathcal{C})$. If $D_\omega \subseteq E$, then $D = E \in \mathcal{H}(\mathcal{C})$. Now suppose that $D_\omega \nsubseteq E$. By Theorem 3.2, we have $A \in \mathcal{F}(\mathcal{C})$, hence $D \notin \mathcal{C}$. Let $0 \neq F \leq D$. Then $(E + F)/F \cong E/(F \cap E) \in \mathcal{C}$, because $F \cap E \neq 0$. We also have

$$D/(E + F) = (E + F + D_\omega)/(E + F) \cong D_\omega/((E + F) \cap D_\omega) \in \mathcal{C}\,,$$

because $(E + F) \cap D_\omega \neq 0$. By the exactness of the sequence $0 \to (E + F)/F \to D/F \to D/(E + F) \to 0$ and the fact that the class $\mathcal{C}$ is closed under extensions, it follows that $D/F \in \mathcal{C}$. Hence $D \in \mathcal{H}(\mathcal{C})$. Clearly, D is the unique maximal submodule of A that belongs to $\mathcal{H}(\mathcal{C})$. □

A module satisfying the hypothesis of the above theorem does exist by the following result.

Proposition 3.7. *Let $\mathcal{C}$ be closed under submodules and extensions. Let A be a noetherian module such that $A \notin \mathcal{C}$. Then there exists a proper submodule D of A such that $A/D \in \mathcal{H}(\mathcal{C})$.*

Proof. Let $\mathcal{M}$ be the family of all submodules B of A such that $A/B \notin \mathcal{C}$. Clearly $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$. Since A is noetherian, $\mathcal{M}$ has a maximal element, say D, that is a proper submodule of A. Hence $A/D \notin \mathcal{C}$. Now let $D < F \leq A$. Then $(A/D)/(F/D) \cong A/F \in \mathcal{C}$ by the maximality of D. Therefore $A/D \in \mathcal{H}(\mathcal{C})$. □

Lemma 3.8. *Let $\mathcal{C}$ be closed under submodules and extensions. Let B be a uniform module that contains a submodule $A \in \mathcal{H}(\mathcal{C})$ such that $B/A \in \mathcal{F}(\mathcal{C})$. Then A is the maximal submodule of B that belongs to $\mathcal{H}(\mathcal{C})$.*

Proof. Suppose the contrary. Then there exists $A < D \leq B$ such that $D \in \mathcal{H}(\mathcal{C})$. Hence $D/A \in \mathcal{C}$, therefore $B/A \notin \mathcal{F}(\mathcal{C})$, a contradiction. □

With the same assumption on $\mathcal{C}$ to be closed under submodules and extensions, for $A \in \mathcal{H}(\mathcal{C})$, let us denote by $M_{\mathcal{C}}(A)$ the maximal submodule of the injective hull $E(A)$ of A that belongs to $\mathcal{H}(\mathcal{C})$. Also, denote by $\mathcal{M}_{\mathcal{C}}$ the class consisting of all modules $M_{\mathcal{C}}(A)$ for $A \in \mathcal{H}(\mathcal{C})$.

Theorem 3.9. *Let $\mathcal{C}$ be closed under submodules and extensions. Let $A, B \in \mathcal{M}_{\mathcal{C}}$ and let $f : A \to B$ be a non-zero homomorphism. Then f is an isomorphism.*

Proof. By Theorem 3.4, f is a monomorphism. There exists a homomorphism $g : E(A) \to E(B)$ that extends jf, where $j : B \to E(B)$ is the inclusion homomorphism. Since $A \trianglelefteq E(A)$, g is a monomorphism. But $E(B)$ is indecomposable, hence g is an isomorphism. Clearly $g(A) \subseteq j(B)$, whence $A \subseteq g^{-1}(B)$. We also have $g^{-1}(B) \in \mathcal{H}(\mathcal{C})$ and by the maximality of A it follows that $A = g^{-1}(B)$. Thus $g(A) = B$, whence $f(A) = B$. Therefore f is an isomorphism. □

Theorem 3.10. *Let $\mathcal{C}$ be closed under submodules and extensions. Let $D \in \mathcal{M}_{\mathcal{C}}$ and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of modules with $B \in \mathcal{H}(\mathcal{C})$. Then D is injective with respect to the above sequence.*

Proof. Let $u : A \to D$ be a homomorphism. We may assume that $u \neq 0$. By Theorem 3.4, u is a monomorphism, because $A \in \mathcal{H}(\mathcal{C})$. Let $v : D \to E(D)$ be the inclusion homomorphism. Then there exists a homomorphism $w : B \to E(D)$ such that $wf = vu$. Since $f(A) \trianglelefteq B$, w is a monomorphism. But $w(B) \in \mathcal{H}(\mathcal{C})$. By the maximality of D, it follows that $w(B) \subseteq D$. Now let $h : B \to D$ be the homomorphism defined by $h(b) = w(b)$ for every $b \in B$. Then $hf = u$, showing that D is injective with respect to the above sequence. □

Corollary 3.11. *Let $\mathcal{C}$ be closed under submodules and extensions. Then every module in $\mathcal{H}(\mathcal{C})$ is quasi-injective.*

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Department of Mathematics, Technical University, Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania

E-mail address: crivei@math.utcluj.ro

Faculty of Mathematics and Computer Science, "Babeş-Bolyai" University, Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania

E-mail address: crivei@math.ubbcluj.ro