ON THE OSOFSKY-SMITH THEOREM

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To Professor Patrick F. Smith on the occasion of his 65th birthday

Abstract. We recall a version of the Osofsky-Smith theorem in the context of a Grothendieck category and derive several consequences of this result. For example, it is deduced that every locally finitely generated Grothendieck category with a family of completely injective finitely generated generators is semisimple. We also discuss the torsion-theoretic version of the classical Osofsky theorem which characterizes semisimple rings as those ring whose every cyclic module is injective.

1. Introduction

In the late 1960s B. Osofsky showed her classical result which asserts that a ring is semisimple if and only if every cyclic module is injective [8, Theorem], [9, Corollary]. Among the categorical generalizations of the Osofsky Theorem, we mention the version established by J.L. Gómez Pardo, N.V. Dung and R. Wisbauer [6]. They showed that if $\mathcal{C}$ is a locally finitely generated Grothendieck category and M is a finitely presented object of $\mathcal{C}$ which is completely (pure-)injective and has a von Neumann regular endomorphism ring S, then S is a semisimple ring [6, Theorem 1]. In the early 1990s, B. Osofsky and P.F. Smith established a module counterpart of the original Osofsky Theorem. They proved that if M is a cyclic module with the property that every cyclic submodule of M is completely extending, then M is a finite direct sum of uniform modules [10]. As a consequence, if M is a module with every quotient of a cyclic submodule injective, then M is semisimple. In the same paper, B. Osofsky and P.F. Smith noted that their result still holds in a more general categorical setting.

The purpose of this paper is to discuss some categorical version of the Osofsky-Smith Theorem and to give several applications. We first consider the setting of a locally finitely generated Grothendieck category $\mathcal{C}$ and we deduce that if $\mathcal{C}$ has a family of completely injective

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finitely generated generators, then $\mathcal{C}$ is semisimple. As an application, we give a positive partial answer to the following question raised by M. Teply: does the torsion-theoretic version of the Osofsky Theorem hold? In other words, if τ is a hereditary torsion theory such that every cyclic module is τ-injective, does it follow that every module is τ-injective? Finally, we show that a ring is semisimple if and only if every cyclic module is τ-injective τ-complemented.

2. Locally finitely generated Grothendieck categories

Definition 2.1. Let $\mathcal{C}$ be a Grothendieck category. Then an object C of $\mathcal{C}$ is called *completely injective* if for every object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an injective object.

Remark. As an immediate consequence of the existence of an injective hull for every object in $\mathcal{C}$, an object C of $\mathcal{C}$ is completely injective if and only if for every injective object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an injective object.

We begin with a property that will be needed later.

Proposition 2.2. *Let $\mathcal{C}$ be a Grothendieck category and $(U_i)_{i\in I}$ a family of completely injective objects of $\mathcal{C}$. Then every finite direct sum of U_i's is completely injective.*

Proof. Consider a finite direct sum of U_i's, say $U_1 \oplus \cdots \oplus U_n$, and let $f : U_1 \oplus \cdots \oplus U_n \to M$ be a morphism in $\mathcal{C}$. We show that $\mathrm{Im}(f)$ is an injective object. We prove it for $n = 2$, the general case following by induction. Let $f : U_1 \oplus U_2 \to M$ be a morphism in $\mathcal{C}$. Denote by $i_1 : U_1 \to U_1 \oplus U_2$ and $i_2 : U_2 \to U_1 \oplus U_2$ the inclusion morphisms. Also, put $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$. Then it is easy to see that $\mathrm{Im}(f) = \mathrm{Im}(f_1) + \mathrm{Im}(f_2)$. Denote $X = \mathrm{Im}(f_1)$, $Y = \mathrm{Im}(f_2)$, and let $g : U_1 \to X/(X \cap Y)$ be the composition of the natural epimorphisms $U_1 \to X$ and $X \to X/(X\cap Y)$. Then $(X+Y)/Y \cong X/(X\cap Y) \cong \mathrm{Im}(g)$ is an injective object by hypothesis. But Y is also injective, and so $\mathrm{Im}(f) = X + Y$ is an injective object. □

Recall that a Grothendieck category $\mathcal{C}$ is called *locally finitely generated* if it has a family of finitely generated generators [12].

Corollary 2.3. *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then every finitely generated object in $\mathcal{C}$ is injective.*

Example 2.4. The conclusion of Proposition 2.2 does not hold for an infinite family. Indeed, let us consider an infinite family of fields $(K_i)_{i\in I}$ and denote $R = \prod_{i\in I} K_i$. Then R is a commutative von Neumann regular ring, that is, a V-ring, and so every simple R-module is injective. Now let $(e_i)_{i\in I}$ be the family of primitive orthogonal idempotents in R. Clearly, each $S_i = Re_i$ is a simple R-module, and so injective. Then each S_i is actually completely injective. Also, we have $\bigoplus_{i\in I} S_i = \mathrm{Soc}(R)$. Clearly, $\bigoplus_{i\in I} S_i$ is not injective, because otherwise this would imply that $R = \mathrm{Soc}(R)$. Now if we take $M = \bigoplus_{i\in I} S_i$ and f to be the identity homomorphism, it follows that $C = M$ is not completely injective.

Example 2.5. If R is a right hereditary ring, then it is clear that the class of completely injective objects in the category Mod-R of right R-modules coincides with the class of injective objects in Mod-R.

In order to be able to state the Osofsky-Smith Theorem, we need the definition of an extending object in a Grothendieck category, which is the same as for modules.

Definition 2.6. Let $\mathcal{C}$ be a Grothendieck category. An object M of $\mathcal{C}$ is called *extending* if every subobject of M is essential in a direct summand of M. Equivalently, M is extending if and only if every essentially closed subobject of M is a direct summand of M.

An object M of $\mathcal{C}$ is called *completely extending* if for every object M of $\mathcal{C}$ and every morphism $f : C \to M$, $\mathrm{Im}(f)$ is an extending object.

Let $\mathcal{C}$ be a Grothendieck category. For a class $\mathcal{P}$ of objects of $\mathcal{C}$, by a $\mathcal{P}$-subobject we mean a subobject belonging to $\mathcal{P}$. Let $\mathcal{P}$ be a class of finitely generated objects in $\mathcal{C}$ with the following properties:

(P_1) $\mathcal{P}$ is closed under quotients;

(P_2) If $X \in \mathcal{P}$ and Y is a $\mathcal{P}$-subobject of a quotient object of X, then there is a $\mathcal{P}$-subobject Z of X that projects onto Y.

Some examples of such classes $\mathcal{P}$ in $\mathcal{C}$ are the following: the class of all finitely generated objects, the class of finitely generated semisimple objects, any class of finitely generated objects closed under subobjects and quotients.

Now basically the same proof of the basic theorem for modules (see [7] or [10]) works in our categorical context. This has also been noted in the original paper of B. Osofsky and P.F. Smith [10].

Theorem 2.7. *Let $\mathcal{C}$ be a Grothendieck category. Let $\mathcal{P}$ be a class of finitely generated objects in $\mathcal{C}$ satisfying (P_1) and (P_2) and let $M \in \mathcal{P}$ be such that every $\mathcal{P}$-subobject of M is completely extending. Then M is a finite direct sum of uniform objects.*

The next two corollaries are obtained as [10, Corollaries 1 and 2].

Corollary 2.8. *Let $\mathcal{C}$ be a Grothendieck category such that every finitely generated object is extending. Then every finitely generated object is a finite direct sum of uniform objects.*

Corollary 2.9. *Let $\mathcal{C}$ be a Grothendieck category. Let M be an object of $\mathcal{C}$ such that every quotient of every finitely generated subobject of M is injective. Then M is semisimple.*

Recall that a Grothendieck category $\mathcal{C}$ is called *semisimple* if every object of $\mathcal{C}$ is semisimple [12]. Now Corollaries 2.3 and 2.9 yield the Osofsky-Smith Theorem in locally finitely generated Grothendieck categories, stated as follows.

Theorem 2.10. *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then $\mathcal{C}$ is semisimple.*

By Corollary 2.3, the property of complete injectivity of the finitely generated generators of a locally finitely generated Grothendieck category passes to each finitely generated object. Now we immediately have the following consequences of Theorem 2.10.

Corollary 2.11. [8, Theorem] *Let R be a ring with identity such that every cyclic (finitely generated) module is injective. Then R is semisimple.*

Corollary 2.12. [4, Corollary 7.14] *Let R be a ring with identity, M a module and $\sigma[M]$ the category of M-subgenerated modules. Suppose that every cyclic (finitely generated) module in $\sigma[M]$ is M-injective. Then M is semisimple.*

Corollary 2.13. *Let R be a ring with enough idempotents such that every cyclic (finitely generated) module is injective. Then R is semisimple.*

Recall that a Grothendieck category $\mathcal{C}$ is called *spectral* if every object of $\mathcal{C}$ is injective. It is well known that $\mathcal{C}$ is semisimple if and only if it is locally finitely generated and spectral [12]. This suggests us to raise the following natural question, whose positive answer would generalize the Osofsky-Smith Theorem 2.10.

Question 1. *If $\mathcal{C}$ is a Grothendieck category with a family of completely injective generators, does it follow that $\mathcal{C}$ is spectral?*

3. Applications to torsion theories

Throughout this section, R is a ring with identity, all modules are unitary right R-modules, and M is a module. Also, Mod-R denotes the category of unitary right R-modules, $\sigma[M]$ denotes the full subcategory of Mod-R consisting of M-subgenerated modules, and $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in Mod-R. Recall that a submodule B of a module A is called *τ-dense* (respectively *τ-closed*) in A if A/B is τ-torsion (respectively τ-torsionfree). Also, a module M is called *τ-injective* if for every module B and every τ-dense submodule A of B, every homomorphism $A \to M$ extends to a homomorphism $B \to M$. For further background on torsion theories the reader is referred to [5] or [12].

Now we have the following consequence of the categorical Osofsky-Smith theorem for torsion theories.

Corollary 3.1. *Suppose that every cyclic τ-torsion module is τ-injective. Then every τ-torsion module is τ-injective.*

Proof. Note that $\mathcal{T}$ is generated by the modules of the form R/I for the τ-dense right ideals I of R. Each factor of such an R/I is cyclic τ-torsion, hence τ-torsion τ-injective by hypothesis, and so injective in $\mathcal{T}$. Thus, each such generator R/I is completely injective in $\mathcal{T}$. Now by Theorem 2.10, $\mathcal{T}$ is semisimple, and so spectral. Then every τ-torsion module is injective in $\mathcal{T}$, that is, every τ-torsion module is τ-injective. $\square$

A related question is the following one, which was raised by M. Teply:

Question 2. *If every cyclic module is τ-injective, does it follow that every module is τ-injective?*

Remark. Note that, by Corollary 3.1, if every cyclic τ-torsion module is τ-injective, then every τ-torsion module is τ-injective, and so every τ-torsion module is semisimple by [5, Proposition 8.15]. Hence Question 2 reduces to the case of a specialization of the Dickson torsion theory [3]. Recall that the Dickson torsion theory is the hereditary torsion theory generated by all simple modules. Its torsion class consists of all semiartinian modules, whereas its torsionfree class consists of all modules with zero socle.

In what follows we shall obtain a positive answer in case τ is of finite type. Recall that a torsion theory is called *of finite type* if its Gabriel filter contains a cofinal subset of finitely generated left ideals. A module is called *τ-finitely generated* if it has a finitely generated τ-dense submodule. We need the following lemma.

Lemma 3.2. *Suppose that every cyclic module is τ-injective. Then every τ-finitely generated module is τ-injective.*

Proof. First we show that every finitely generated module is τ-injective. Let M be a finitely generated module, say $M = Rx_1 + \cdots + Rx_n$. Use induction on n. For $n = 1$ it is clear. Suppose that every module generated by $n-1$ elements is τ-injective. Then $M/(Rx_1 + \cdots + Rx_{n-1}) \cong Rx_n/((Rx_1+\cdots+Rx_{n-1}) \cap Rx_n)$ is cyclic, and so τ-injective. But $Rx_1 + \cdots + Rx_{n-1}$ is also τ-injective, so that M is τ-injective.

Now let M be a τ-finitely generated module, hence M has some τ-dense finitely generated submodule N. Then N is τ-injective by the argument given in the previous paragraph. Clearly M/N is τ-torsion, and hence τ-injective by Corollary 3.1. Thus it follows that M is τ-injective. $\square$

Theorem 3.3. *Let τ be of finite type and suppose that every cyclic module is τ-injective. Then every module is τ-injective.*

Proof. Let I be a τ-dense left ideal of R. Then there exists a finitely generated left ideal $J \subseteq I$ and we have I/J τ-torsion. Then J is τ-injective by Lemma 3.2, hence it is a direct summand of R, and so a direct summand of I, say $I = J \oplus J'$. But $J' \cong I/J$ is τ-torsion, hence τ-injective. It follows that I is τ-injective, hence I is a direct summand of R. Therefore, every module is τ-injective by [5, Proposition 8.10]. □

There are situations when the condition that every cyclic τ-torsion module is τ-injective assures that every module is τ-injective. We present one based on the recent result stating that every Baer module over a commutative domain is projective [1, Theorem 3.4]. Recall that a module M is called *τ-projective* if $\mathrm{Ext}^1_R(M,T) = 0$ for every τ-torsion module T. If R is a commutative domain and τ is the usual torsion theory in Mod-R, then a τ-projective module is called *Baer*. We need the following easy lemma.

Lemma 3.4. *Every τ-torsion module is τ-injective if and only if every τ-torsion module is τ-projective.*

Corollary 3.5. *Let R be a commutative domain and τ the usual torsion theory in* Mod-R. *The following are equivalent:*

(i) Every cyclic τ-torsion module is injective;
(ii) Every τ-torsion module is injective;
(iii) Every τ-torsion module is Baer;
(iv) Every module is injective;
(v) R is a field.

Proof. Recall that a module is τ-torsion if and only if every non-zero element $x \in M$ is annihilated by a non-zero ideal. Since R/I is τ-torsion for every non-zero ideal of R, τ-injectivity coincides with usual injectivity.

(i)⇒(ii) By Corollary 3.1.

(ii)⇒(iii) By Lemma 3.4.

(iii)⇒(iv) By Lemma 3.4, every τ-torsion module is Baer, and so projective by [1, Theorem 3.4]. Then every module is τ-injective [5, Proposition 8.10], and so injective.

(iv)⇒(v) In this case R is semisimple, and so R must be a field.

(v)⇒(i) Clear. □

In what follows, we establish a characterization of semisimple modules using certain relative injective modules. Let τ be a hereditary torsion theory in the category $\sigma[M]$. Recall that a module $N \in \sigma[M]$ is called *(M,τ)-injective* if N is injective with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with L/K τ-torsion. We consider the following notion which generalizes that of complemented module with respect to a hereditary torsion theory in Mod-R from [11]. A module $N \in \sigma[M]$ is called *(M,τ)-complemented* if every submodule of N is τ-dense in a direct summand of N.

Theorem 3.6. *The following are equivalent:*
(i) M is semisimple;
(ii) Every module in $\sigma[M]$ is (M, τ)-injective (M, τ)-complemented;
(iii) Every cyclic module in $\sigma[M]$ is (M, τ)-injective (M, τ)-complemented;
(iv) Every cyclic module in $\sigma[M]$ is injective in $\sigma[M]$.

Proof. $(i) \Rightarrow (ii)$ Suppose that M is semisimple. Then every module in $\sigma[M]$ is injective in $\sigma[M]$ [14, 20.3], hence (M, τ)-injective. Also, every module in $\sigma[M]$ is semisimple in $\sigma[M]$ [14, 20.3], hence (M, τ)-complemented.

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (iv)$ Let $\mathcal{C}$ be the smallest closed subcategory of $\sigma[M]$ containing the (M, τ)-complemented modules. Then $\mathcal{C} = \sigma[N]$ for some module $N \in \sigma[M]$, and a family of finitely generated generators for $\mathcal{C}$ consists of the modules R/I with $R/I \in \sigma[N]$. Each such R/I is (M, τ)-complemented, and so an object of $\mathcal{C}$. Thus $\mathcal{C} = \sigma[M]$. By an easy adaptation of [13, Lemma 2] in $\sigma[M]$, it follows that τ is a generalization of the Goldie torsion theory, hence (M, τ)-injectivity coincides with injectivity.

(iv)⇒(i) By Corollary 2.12. □

Now we have the following characterization of semisimple rings.

Corollary 3.7. *R is semisimple if and only if every cyclic module is τ-injective τ-complemented.*

The classical Osofsky theorem is obtained by taking $\tau = \tau_G$, i.e. the Goldie torsion theory, or $\tau = \chi$, i.e. the torsion theory with all modules torsion. Note that a module is τ_G-injective τ_G-complemented if and only if it is injective. Also, every module is χ-complemented.

In [2] it has been shown that the class of τ-injective τ-complemented modules is strictly contained in the class of quasi-injective modules. Now recall the following result.

Theorem 3.8. [7, Theorem 6.83] *The following are equivalent:*
(i) R is semisimple;
(ii) Every module is quasi-injective;
(iii) Every finitely generated module is quasi-injective.

The condition that every cyclic module is quasi-injective is, in general, weaker than those in the previous theorem. For instance, $R = \mathbb{Q}[x]/(x^2)$ is self-injective, and every cyclic module is quasi-injective, but R is not semisimple [7]. Hence Corollary 3.7 may be seen as a refinement of Theorem 3.8 for cyclic modules.

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