Geometry
Problem booklet

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Contents

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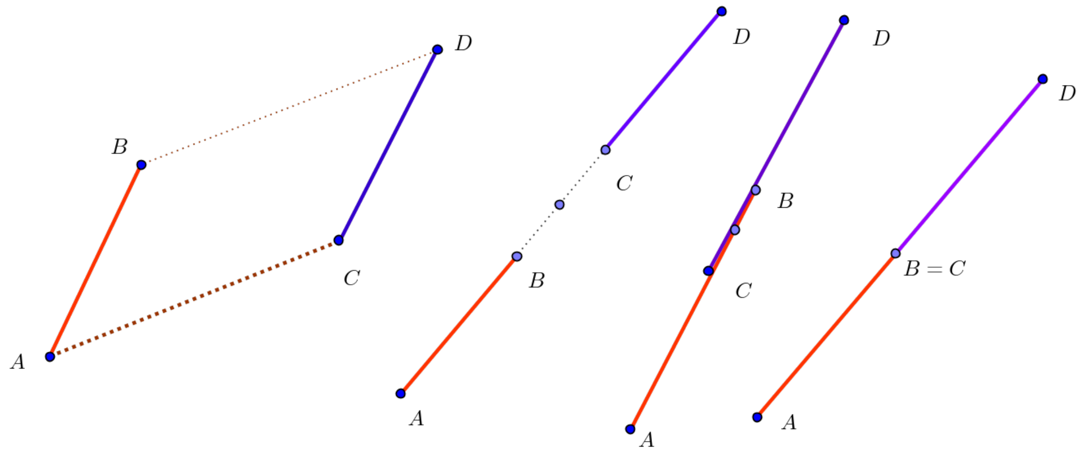
1 Week 1: Vector algebra

1.1 Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1. The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.



Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.1. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.2. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.2. The equivalence classes with respect to the equipollence relation are called *(free) vectors*.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P}/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.2. If two ordered pairs (A, B) and (C, D) are equippllent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

Proposition 1.3. *1.* $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$.

2. $\forall A, B, O \in \mathcal{P}$, $\exists! X \in \mathcal{P}$ *such that* $\overrightarrow{AB} = \overrightarrow{OX}$.

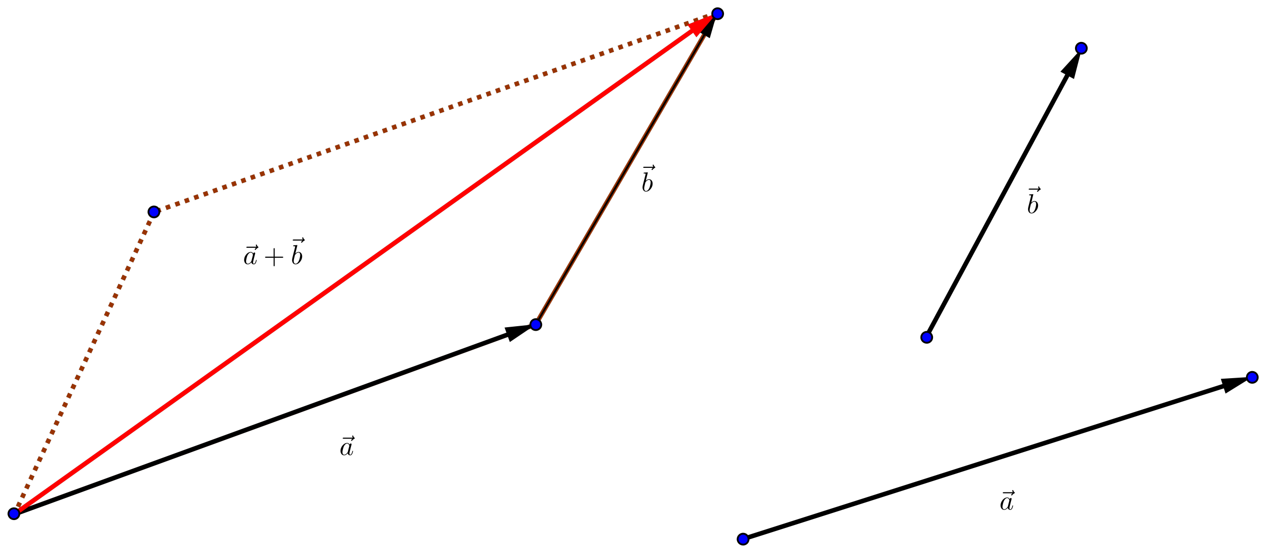
3. $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

Definition 1.3. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\overrightarrow{r}_M$ and is called the *position vector* of M with respect to O.

Corollary 1.4. *The map* $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \overrightarrow{r}_M$ *is one-to-one and onto, i.e. bijective.*

1.1.1 Operations with vectors

• **The addition of vectors** Let $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{a} + \overrightarrow{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \overrightarrow{a}$, $\overrightarrow{A'B'} = \overrightarrow{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.3(3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\overrightarrow{a} + \overrightarrow{b}$ is independent on the choice of the point O.

Proposition 1.5. *The set* $\mathcal{V}$ *endowed to the binary operation* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\overrightarrow{a}, \overrightarrow{b}) \mapsto \overrightarrow{a} + \overrightarrow{b}$, *is an abelian group whose zero element is the vector* $\overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{0}$ *and the opposite of* $\overrightarrow{AB}$, *denoted by* $-\overrightarrow{AB}$, *is the vector* $\overrightarrow{BA}$.

In particular the addition operation is associative and the vector

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$$

is usually denoted by $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$. Moreover the expression

$$((\cdots(\overrightarrow{a}_1 + \overrightarrow{a}_2) + \overrightarrow{a}_3 + \cdots + \overrightarrow{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

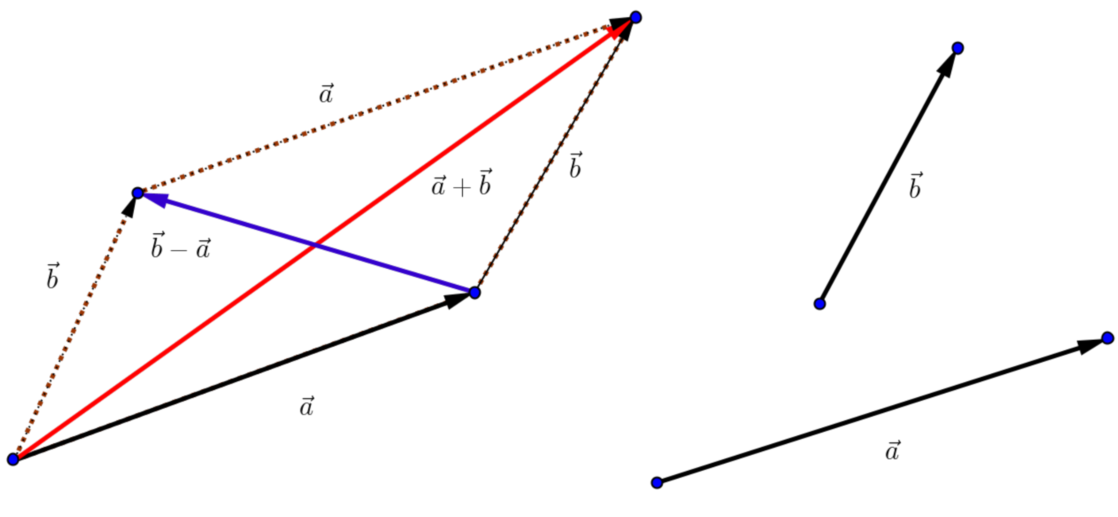
$$\overrightarrow{a}_1 + \overrightarrow{a}_2 + \cdots + \overrightarrow{a}_n .$$

Example 1.1. If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} .$$

This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corollary 1.6. *If $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\overrightarrow{x} \in \mathcal{V}$ such that $\overrightarrow{a} + \overrightarrow{x} = \overrightarrow{b}$. In fact $\overrightarrow{x} = \overrightarrow{b} + (-\overrightarrow{a}) = \overrightarrow{AB}$ and is denoted by $\overrightarrow{b} - \overrightarrow{a}$.*



• **The multiplication of vectors with scalars**

Let $\alpha \in \mathbb{R}$ be a scalar and $\overrightarrow{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \overrightarrow{a}$ as follows: $\alpha \cdot \overrightarrow{a} = \overrightarrow{0}$ if $\alpha = 0$ or $\overrightarrow{a} = \overrightarrow{0}$; if $\overrightarrow{a} \neq \overrightarrow{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \overrightarrow{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \overrightarrow{a} = -(|\alpha| \cdot \overrightarrow{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V}, \ (\alpha, \overrightarrow{a}) \mapsto \alpha \cdot \overrightarrow{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.7. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \overrightarrow{a}=\alpha\cdot \overrightarrow{a}+\beta\cdot \overrightarrow{a}$, $\forall\, \alpha,\beta\in\mathbb{R}$, $\overrightarrow{a}\in\mathcal{V}$.

$(v2)$ $\alpha\cdot(\overrightarrow{a}+\overrightarrow{b})=\alpha\cdot \overrightarrow{a}+\alpha\cdot \overrightarrow{b}$, $\forall\,\alpha\in\mathbb{R}$, $\overrightarrow{a},\overrightarrow{b}\in\mathcal{V}$.

$(v3)$ $\alpha\cdot(\beta\cdot \overrightarrow{a})=(\alpha\beta)\cdot \overrightarrow{a}, \forall\,\alpha,\beta\in\mathbb{R}$.

$(v4)$ $1\cdot \overrightarrow{a}=\overrightarrow{a}$, $\forall\ \overrightarrow{a}\in\mathcal{V}$.

Application 1.1. Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Then:

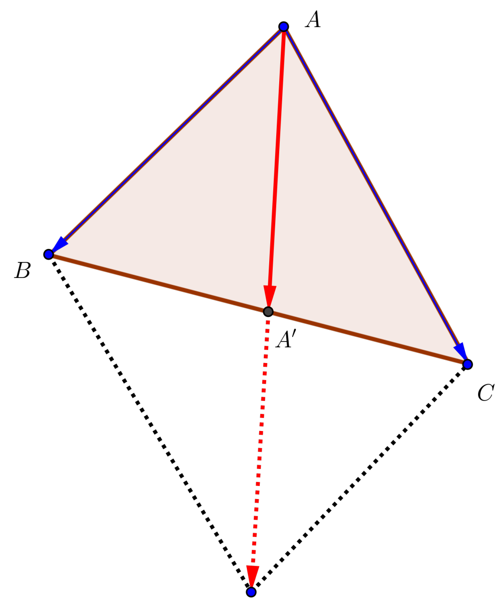
- $2\,\overrightarrow{M_1M_2}=\overrightarrow{A_1A_2}+\overrightarrow{B_1B_2}$ and $2\,\overrightarrow{M_3M_4}=\overrightarrow{A_3A_4}+\overrightarrow{B_3B_4}$.
- M_1, M_2, M_3, M_4 are the vertices of a parallelogram.

1.1.2 The vector structure on the set of vectors

Theorem 1.8. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

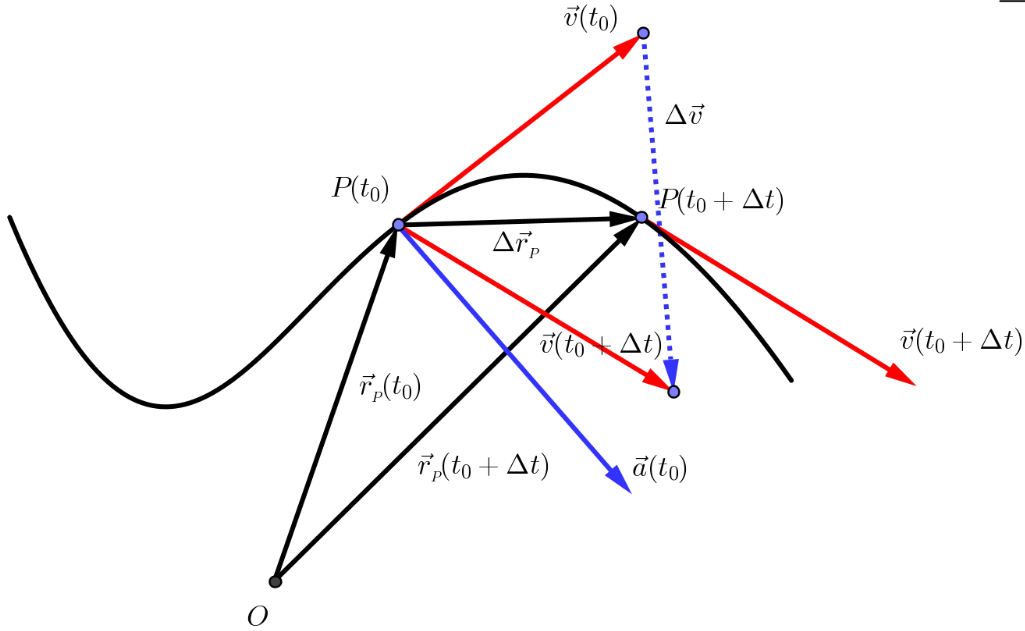
Example 1.2. If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then

$$\overrightarrow{AA'}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC}).$$



A few vector quantities:

1. The force, usually denoted by $\overrightarrow{F}$.
2. The velocity $\frac{d\overrightarrow{r}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{v}_P$ or simply by $\overrightarrow{v}$.
3. The acceleration $\frac{d\overrightarrow{v}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{a}_P$ or simply by $\overrightarrow{a}$.



• **Newton's law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force F is equal to G (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses (m_1 and m_2) and divided by the square of the distance R: $F = G(m_1m_2)/R^2$. (Encyclopdia

Britannica)
• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newtons second law is one of the most important in all of physics. For a body whose mass m is constant, it can be written in the form $F = ma$, where F (force) and a (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

1.2 Problems

1. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:
   (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.
   (b) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.
   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

2. ([4, Problem 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA}=\overrightarrow{a}$ and $\overrightarrow{OE}=\overrightarrow{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE}= 3\,\overrightarrow{OC}$.

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose

diagonals are concurrent at O. Show the equality $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\,\overrightarrow{SO}$.

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.

6. ([4, Problem 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}\,.$$

8. ([4, Problem 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that

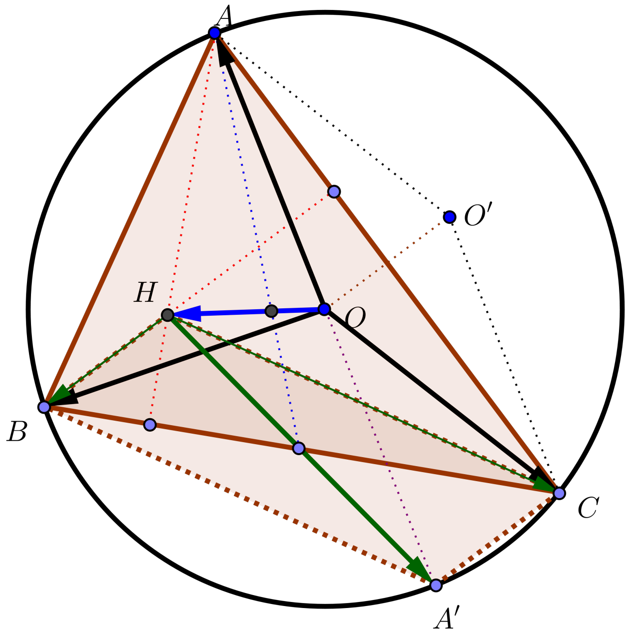
$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

9. ([4, Problem 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.



10. ([4, Problem 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\overrightarrow{HG} = 2\overrightarrow{HO}$.

11. ([4, Problem 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:

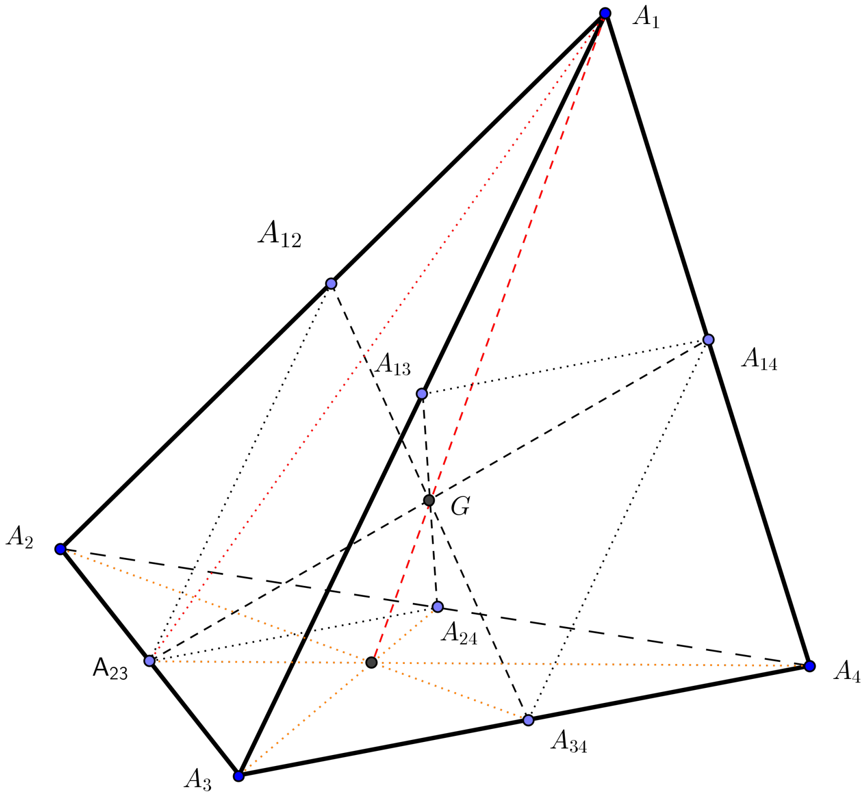
    (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.

    (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.

(c) Determine the ratio in which the point G divides each median.

(d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.

(e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\,\overrightarrow{MG}$.



12. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

13. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{G_1G_2}$.

2 Week 2: Straight lines and planes

2.1 Linear dependence and linear independence of vectors

Definition 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 2.1. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.2. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

Proposition 2.3. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 2.2. The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta}=\overrightarrow{\Delta}'$

Definition 2.2. *We call* director vector *of the straigh line Δ every nonzero vector $\overrightarrow{d}\in\overrightarrow{\Delta}$.*

If $\overrightarrow{d}\in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM}\in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.

Proposition 2.4. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi}= \{\overrightarrow{AM}\in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 2.3. • *The planes π, π' are parallel if and only if $\overrightarrow{\pi}=\overrightarrow{\pi}'$.*

• If $\overrightarrow{d}_1$, $\overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2\rangle$. This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM}\in \langle \overrightarrow{d}_1, \overrightarrow{d}_2\rangle\}.$$

We say that π is the plane which passes through the point A and is parallel to the vectors $\vec{d}_1$ and $\vec{d}_2$.

Remark 2.4. Let $\Delta \subset \mathcal{P}$ be a straight line and $\pi \subset \mathcal{P}$ be given plane.

1. If $A \in \Delta$ is a given point, then $\varphi_O(\Delta) = \vec{r}_A + \vec{\Delta}$.
2. If $B \in \Delta$ is a given point, then $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$.

Generally speaking, a subset X of a vector space is called *linear variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.5. *The bijection φ_O transforms the straight lines and the planes of the affine space $\mathcal{P}$ into the one and two dimnensional linear varieties of the vector space $\mathcal{V}$ respectively.*

2.2 The vector ecuations of the straight lines and planes

Proposition 2.6. *Let Δ be a straight line, let π be a plane, $\{\vec{d}\}$ be a basis of $\vec{\Delta}$ and let $[\vec{d}_1, \vec{d}_2]$ be an ordered basis of $\vec{\pi}$.*

1. *The points $M \in \Delta$ are characterized by the* vector equation of Δ

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d},\ \lambda \in \mathbb{R} \tag{2.1}$$

where $A \in \Delta$ is a given point.

2. *The points $M \in \pi$ are characterized by the* vector equation of π

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.2}$$

where $A \in \pi$ is a given point.

PROOF.

□

Corollary 2.7. *If $A, B \in \mathcal{P}$ are different points, then the vector equation of the line AB is*

$$\vec{r}_M = (1-\lambda)\,\vec{r}_A + \lambda\,\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

PROOF.

□

Corollary 2.8. *If A, B, $C \in \mathcal{P}$ are three noncolinear points, then the vector equation of the plane (ABC) is*

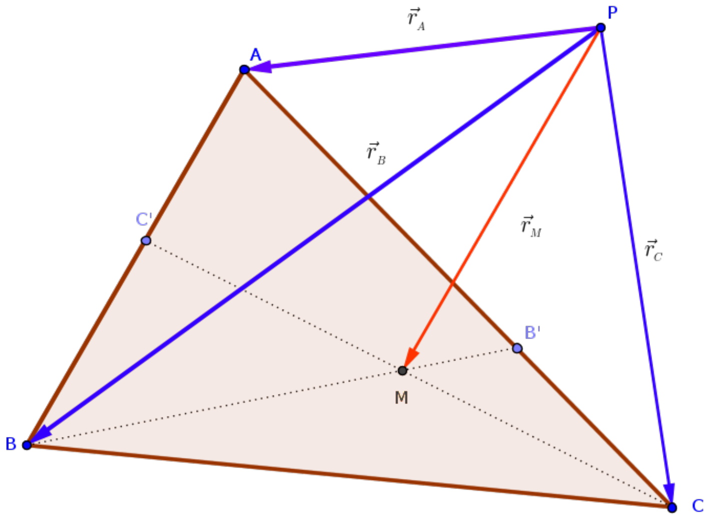
$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

PROOF.

□

Example 2.1. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{2.5}$$



SOLUTION.

□

2.3 Problems

1. ([4, Problem 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$,

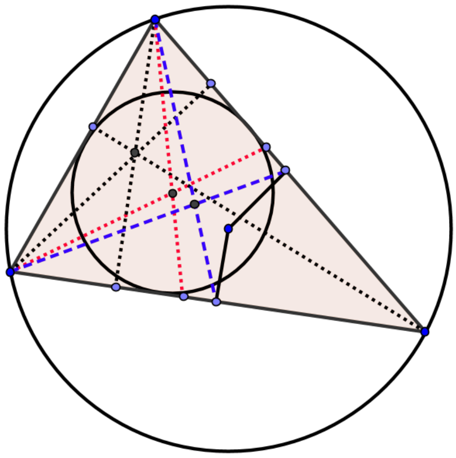
$\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

$$\vec{r}_G := \overrightarrow{PG} = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}.$$

$$\vec{r}_I := \overrightarrow{PI} = \frac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}.$$

$$\vec{r}_H := \overrightarrow{PH} = \frac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}.$$

$$\vec{r}_O := \overrightarrow{PO} = \frac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}.$$



SOLUTION.

2. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

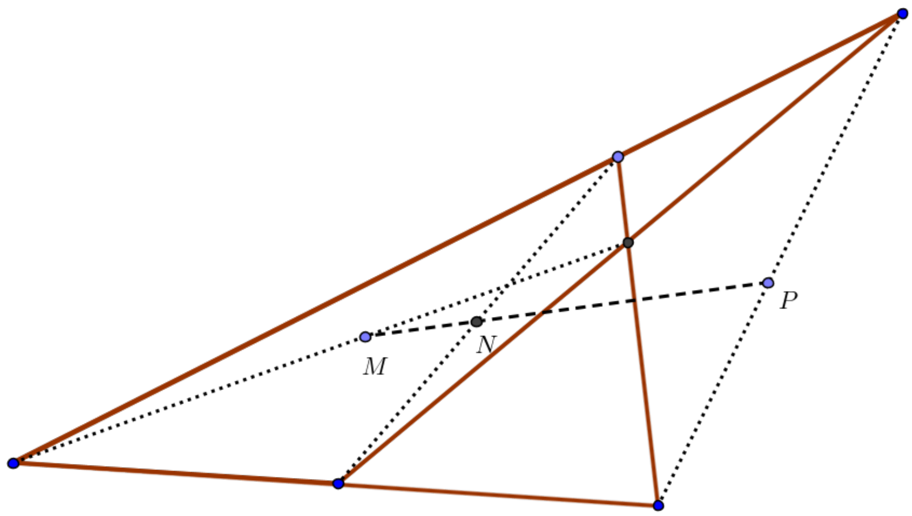
$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} + n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} + n\frac{m-1}{m-n}\,\overrightarrow{OA'}.$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$.

SOLUTION.

3. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



SOLUTION.

4. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

SOLUTION.

5. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   SOLUTION.

6. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \not\parallel A'B'$, $BC \not\parallel B'C'$ and $CA \not\parallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   SOLUTION.

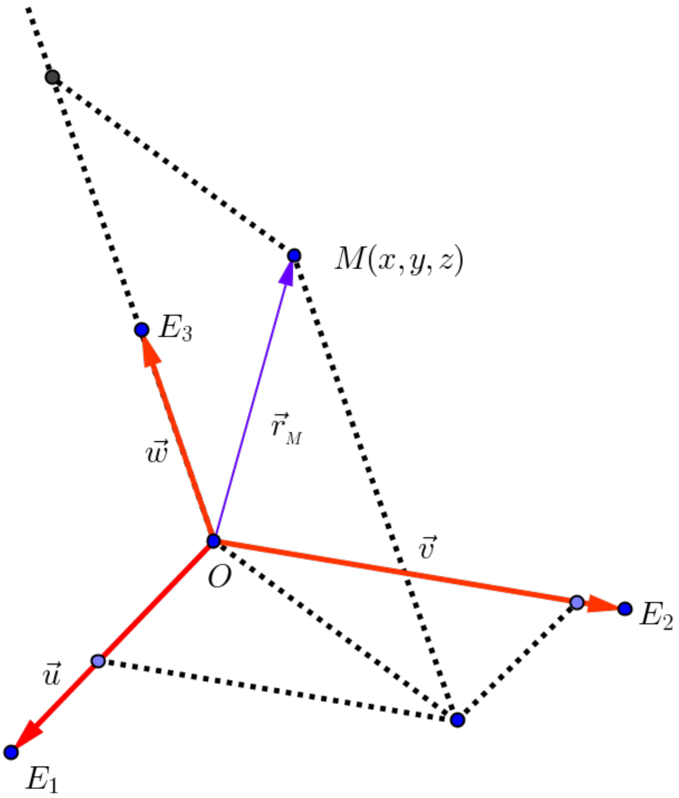
3 Week 3: Cartezian equations of lines and planes

3.1 Cartesian and affine reference systems

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of the coordinates of $\overrightarrow{x}$ with respect to b is denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$. To emphasize the coordinates of $\overrightarrow{x}$ with respect to b, we shall use the notation $\overrightarrow{x}(x_1, x_2, x_3)$.



Definition 3.1. A *cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ of the space $\mathcal{P}$, consists in a point $O \in \mathcal{P}$ called the *origin* of the reference system and an ordered basis $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ of the vector space $\mathcal{V}$.

Denote by E_1, E_2, E_3 the points for which $\vec{u} = \overrightarrow{OE_1}$, $\vec{v} = \overrightarrow{OE_2}$, $\vec{w} = \overrightarrow{OE_3}$.

Definition 3.2. The system of points (O, E_1, E_2, E_3) is called *the affine reference system associated to the cartesian reference system* $R = (O, \vec{u}, \vec{v}, \vec{w})$.

The straight lines OE_i, $i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\vec{r}_M = \overrightarrow{OM}$ with respect to the basis $[\vec{u}, \vec{v}, \vec{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$. Also, for the column matrix of coordinates of the vector $\vec{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\vec{r}_M = x\vec{u} + y\vec{v} + z\vec{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 3.1. If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B\vec{u} + y_B\vec{v} + z_B\vec{w} - (x_A\vec{u} + y_A\vec{v} + z_A\vec{w}) \\ &= (x_B - x_A)\vec{u} + (y_B - y_A)\vec{v} + (z_B - z_A)\vec{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Remark 3.2. If $R = (O, b)$ is a cartesian reference system, where $b = [\vec{u}, \vec{v}, \vec{w}]$ is an ordered basis of $\mathcal{V}$, recall that $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$, $\varphi_O(M) = \overrightarrow{OM}$ is bijective and $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$, $\psi_b(x, y, z) = x\vec{u} + y\vec{v} + z\vec{w}$ is a linear isomorphism. The bijection φ_O defines a unique vector structure over $\mathcal{P}$ such that φ_O becomes an isomorphism. This vector structure depends on the choice of $O \in \mathcal{P}$. Therefore a point $M \in \mathcal{P}$ could be identified either with its position vector $\vec{r}_M = \varphi_O(M)$, or, with the triplet $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$ of its coordinates with respect to the reference system R. If $f : X \longrightarrow \mathbb{R}^3$ is a given application, then $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$ will be denoted by M_f. A similar discussion can be done for a cartesian reference system $R' = (O', b')$ of a plane π, where $b' = [\vec{u}', \vec{v}']$ is an ordered basis of $\vec{\pi}$.

Example 3.1 (Homework). Consider the tetrahedron $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$. Find the coordinates of:

1. the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[1] respectively.
2. the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively.

[1]The centroids of its faces

SOLUTION.

3.2 The cartesian equations of the straight lines

Let Δ be the straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\overrightarrow{d}\,(p, q, r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda \overrightarrow{d},\ \lambda \in \mathbb{R}. \tag{3.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$

Indeed, the vector equation of Δ can be written, in terms of the coordinates of the vectors $\overrightarrow{r}_M$, $\overrightarrow{r}_{A_0}$ and $\overrightarrow{d}$, as follows:

$$\begin{aligned} & x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w} = x_0 \overrightarrow{u} + y_0 \overrightarrow{v} + z_0 \overrightarrow{w} + \lambda(p \overrightarrow{u} + q \overrightarrow{v} + r \overrightarrow{w}) \\ \Longleftrightarrow\ & x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w} = (x_0 + p\lambda) \overrightarrow{u} + (y_0 + q\lambda) \overrightarrow{v} + (z_0 + r\lambda) \overrightarrow{w},\ \lambda \in \mathbb{R} \end{aligned}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{3.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge\ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different points of the line Δ, then

$$\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$$

is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{3.4}$$

Example 3.2. Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[2] respectively. Show that the medians AG_A, BG_B, CG_C and DG_D are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids G_A, G_B, G_C, G_D are $(2/3,1,0)$, $(4/3,1/3,2/3)$, $(1/3,1/3,2/3)$ and $(2/3,1/3,-1/3)$ respectively. The equations of the medians AG_A and BG_B are

$$(AG_A)\ \frac{x-1}{2/3-1}=\frac{y+1}{1-(-1)}=\frac{z-1}{0-1}\Longleftrightarrow\frac{x-1}{-1/3}=\frac{y+1}{2}=\frac{z-1}{-1}$$
$$(BG_B)\ \frac{x+1}{4/3+1}=\frac{y-1}{1/3-1}=\frac{z+1}{2/3+1}\Longleftrightarrow\frac{x+1}{7/3}=\frac{y-1}{-2/3}=\frac{z+1}{5/3}.$$

Thus, the director space of the median AG_A is $\left\langle\left(-\frac{1}{3},2,-1\right)\right\rangle=\langle(-1,6,-3)\rangle$ and the director space of the median BG_B is $\langle(\frac{7}{3},-\frac{2}{3},\frac{5}{3})\rangle=\langle(7,-2,5)\rangle$. Consequently, the parametric equations of the medians AG_A and BG_B are

$$(AG_A)\ \begin{cases} x=1-t \\ y=-1+6t \\ z=1-3t \end{cases},\ t\in\mathbb{R} \text{ and } (BG_B)\ \begin{cases} x=-1+7s \\ y=1-2s \\ z=-1+5s \end{cases},\ s\in\mathbb{R}.$$

Thus, the two medians AG_A and BG_B are concurrent if and only if there exist $s,t\in\mathbb{R}$ such that

$$\begin{cases} 1-t=-1+7s \\ -1+6t=1-2s \\ 1-3t=-1+5s \end{cases}\Longleftrightarrow\begin{cases} 7s+t=2 \\ 2s+6t=2 \\ 5s+3t=2 \end{cases}\Longleftrightarrow\begin{cases} 7s+t=2 \\ s+3t=1 \\ 5s+3t=2. \end{cases}$$

This system is compatible and has the unique solution $s=t=\frac{1}{4}$, which shows that the two medians AG_A and BG_B are concurrent and

$$AG_A\cap BG_B=\left\{G\left(\frac{3}{4},\frac{1}{2},\frac{1}{4}\right)\right\}.$$

One can similarly show that $BG_B\cap CG_C=CG_C\cap AG_A=\left\{G\left(\frac{3}{4},\frac{1}{2},\frac{1}{4}\right)\right\}$.

Example 3.3 (Homework)**.** Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively. Show that the lines MR, PQ and NS are concurrent and find the coordinates of their intersection point.

SOLUTION.

[2] The centroids of its faces

3.3 The cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1\ (p_1, q_1, r_1)$, $\overrightarrow{d}_2\ (p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\operatorname{rank}\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1$ (p_1, q_1, r_1), $\overrightarrow{d}_2\ (p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

Indeed, the vector equation of π can be written, in terms of the coordinates of the vectors $\overrightarrow{r}_M$, $\overrightarrow{r}_{A_0}$, $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$, as follows:

$$x\overrightarrow{u} + y\overrightarrow{v} + z\overrightarrow{w} = x_0\overrightarrow{u} + y_0\overrightarrow{v} + z_0\overrightarrow{w} + \lambda_1(p_1\overrightarrow{u} + q_1\overrightarrow{v} + r_1\overrightarrow{w}) + \lambda_2(p_2\overrightarrow{u} + q_2\overrightarrow{v} + r_2\overrightarrow{w})$$
$$\Longleftrightarrow x\overrightarrow{u} + y\overrightarrow{v} + z\overrightarrow{w} = (x_0 + \lambda_1 p_1 + \lambda_2 p_2)\overrightarrow{u} + (y_0 + \lambda_1 q_1 + \lambda_2 q_2)\overrightarrow{v} + (z_0 + \lambda_1 r_1 + \lambda_2 r_2)\overrightarrow{w},$$
$$\lambda_1, \lambda_2 \in \mathbb{R},$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane π and are called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (3.6) with the unknowns λ_1, λ_2 is a necessary and sufficient condition for the point $M(x, y, z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{3.7}$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane π in terms of the Cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π. On can put the equation (3.7) in the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{3.8}$$

$$Ax+By+Cz+D=0, \tag{3.9}$$

where the coefficients A, B, C satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if $A\neq 0$, then the equation (3.9) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (3.8) in the form

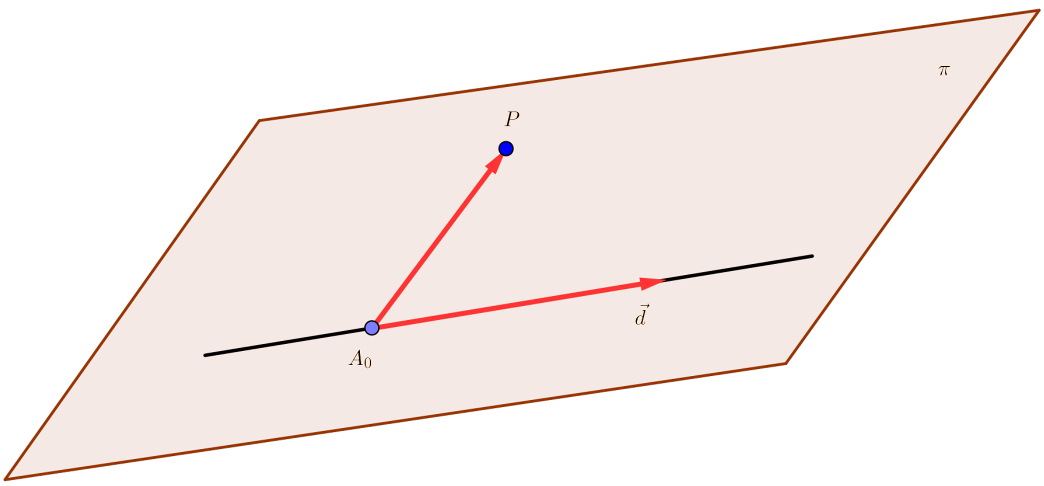
$$AX+BY+CZ=0 \tag{3.10}$$

where $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

Example 3.4. Write the equation of the plane determined by the point $P(-1,1,2)$ and the line $(\Delta)\ \frac{x-1}{3}=\frac{y}{2}=\frac{z+1}{-1}$.

Solution. Note that $P\notin\Delta$, as $\frac{-1-1}{3}\neq\frac{1}{2}\neq -3=\frac{2+1}{-1}$, i.e. the point P and the line Δ determine, indeed, a plane, say π. One can regard π as the plane through the point $A_0(1,0,-1)$ which is parallel to the vectors $\overrightarrow{A_0P}\ (-1-1,1-0,2-(-1))=\overrightarrow{A_0P}\ (-2,1,3)$ and $\vec{d}\ (3,2,-1)$. Thus, the equation of π is

$$\begin{vmatrix} x-1 & y & z+1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 0 \Longleftrightarrow x-y+z=0.$$



Example 3.5 (Homework)**.** Generalize Example 3.4: Write the equation of the plane determined by the line $(\Delta)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$ and the point $M(x_M,y_M,z_M)\notin\Delta$.

SOLUTION.

Remark 3.3. If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.11}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.12}$$

Thus, four points $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and ony if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.13}$$

Example 3.6 (Homework)**.** Write the equation of the plane determined by the points $M_1(3, -2, 1)$, $M_2(5, 4, 1)$ and $M_3(-1, -2, 3)$.

SOLUTION.

Remark 3.4. If $A(a,0,0),\ B(0,b,0),\ C(0,0,c)$ are three points ($abc \neq 0$), then for the equation of the plane (ABC) we have successively:

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\Longleftrightarrow ab(z-c) + bcx + acy = 0 \Longleftrightarrow bcx + acy + abz = abc$$

$$\Longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3.14}$$

The equation (3.14) of the plane (ABC) is said to be in *intercept form* and the x,y,z-intercepts of the plane (ABC) are a,b,c respectively.

Example 3.7 (Homework)**.** Write the equation of the plane (π) $3x - 4y + 6z - 24 = 0$ in intercept form.

SOLUTION.

3.4 Appendix: The Cartesian equations of lines in the two dimensional setting

3.4.1 Cartesian and affine reference systems

If $b = [\vec{e}, \vec{f}]$ is an ordered basis of the director subspace $\vec{\pi}$ of the plane π and $\vec{x} \in \vec{\pi}$, recall that the column vector of $\vec{x}$ with respect to b is being denoted by $[\vec{x}]_b$. In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\vec{x} = x_1 \vec{e} + x_2 \vec{f}$.

Definition 3.3. A *cartesian reference system* of the plane π, is a system $R = (O, \vec{e}, \vec{f})$, where O is a point from π called the *origin* of the reference system and $b = [\vec{e}, \vec{f}]$ is a basis of the vector space $\vec{\pi}$.

Denote by E, F the points for which $\vec{e} = \overrightarrow{OE},\ \vec{f} = \overrightarrow{OF}$.

Definition 3.4. The system of points (O, E, F) is called *the affine reference system associated to the cartesian reference system* $R = (O, \vec{e}, \vec{f})$.

The straight lines OE, OF, oriented from O to E and from O to F respectively, are called *the coordinate axes*. The coordinates x, y of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x\overrightarrow{e} + y\overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark 3.5. If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = x_B\overrightarrow{e} + y_B\overrightarrow{f} - (x_A\overrightarrow{e} + y_A\overrightarrow{f}) \\ &= (x_B - x_A)\overrightarrow{e} + (y_B - y_A)\overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

3.4.2 Parametric and Cartesian equations of Lines

Let Δ be a line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}(p, q)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + t\overrightarrow{d},\ t \in \mathbb{R}. \tag{3.15}$$

If (x, y) are the coordinates of a generic point $M \in \Delta$, then its vector equation (3.15) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases},\ t \in \mathbb{R}. \tag{3.16}$$

The relations are called the *parametric equations* of the line Δ and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \tag{3.17}$$

called the *canonical equation*of Δ. If $q = 0$, then the equation (3.17) becomes $y = y_0$.

If $A(x_A, y_A)$ are two different points of the plane π, then $\overrightarrow{AB}(x_B - x_A, y_B - y_A)$ is a director vector of the line AB and the canonical equation of the line AB is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{3.18}$$

The equation (3.18) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three poins $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{3.19}$$

3.4.3 General Equations of Lines

We can put the equation (3.17) in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + a^2 > 0, \tag{3.20}$$

which means that any line from π is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.20) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through $P_0\left(-\frac{c}{a}, 0\right)$ and parallel to $\overline{v}\left(-\frac{b}{a}, 1\right)$. The equation (3.20) is called *general equation* of the line.

Remark 3.6. The lines

$$(d)\ ax + by + c = 0 \text{ and } (\Delta)\ \frac{x - x_0}{p} = \frac{x - x_0}{q}$$

are parallel if and only if $ap + bq = 0$. Indeed, we have:

$$\begin{aligned} d \| \Delta \quad &\Longleftrightarrow \vec{d} = \vec{\Delta} \Longleftrightarrow \langle \vec{u}\ (p, q) \rangle = \langle \vec{v}\ \left(-\tfrac{b}{a}, 1\right) \rangle \Longleftrightarrow \exists t \in \mathbb{R} \text{ s.t. } \vec{u}\ (p, q) = t\ \vec{v}\ \left(-\tfrac{b}{a}, 1\right) \\ &\Longleftrightarrow \exists t \in \mathbb{R} \text{ s.t. } = -t\tfrac{b}{a} \text{ and } q = t \Longleftrightarrow ap + bq = 0. \end{aligned}$$

3.4.4 Reduced Equations of Lines

Consider a line given by its general equation $Ax + By + C = 0$, where at least one of the coefficients A and B is nonzero. One may suppose that $B \neq 0$, so that the equation can be divided by B. One obtains

$$y = mx + n \tag{3.21}$$

which is said to be the *reduced equation* of the line.

Remark: If $B = 0$, (3.20) becomes $Ax + C = 0$, or $x = -\frac{C}{A}$, a line parallel to Oy. (In the same way, if $A = 0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y = mx + n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two different points on d and φ be the angle determined by d and Ox (see Figure 1); $\varphi \in [0, \pi] \setminus \{\pi/2\}$. The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ belong to d, hence

$$\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n, \end{cases}$$

and $x_2 \neq x_1$, since d is not parallel to Oy. Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \tag{3.22}$$

The number $m = \tan \varphi$ is called the *angular coefficient* of the line d. It is immediate that the equation of the line passing through the point $P_0(x_0, y_0)$ and of the given angular coefficient m is

$$y - y_0 = m(x - x_0). \tag{3.23}$$

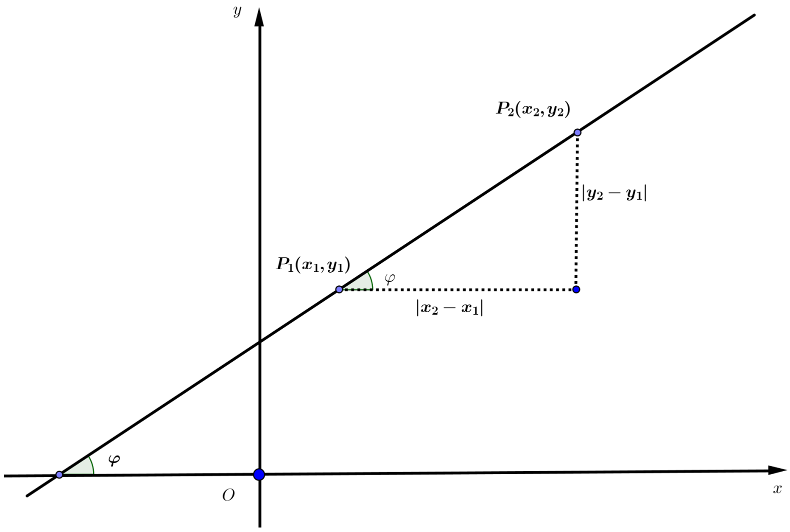


Figure 1:

3.4.5 Intersection of Two Lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical*.

If $d_i : a_ix + b_iy + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{3.24}$$

3.4.6 Bundles of Lines ([1])

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{3.25}$$

Remark: Tthe *reduced bundle* of line through P_0 is,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{3.26}$$

and covers the bundle of lines through P_0, except the line $x = x_0$. Similarly, the family of lines

$$x - x_0 = k(y - y_0), \qquad k \in \mathbb{R}, \tag{3.27}$$

covers the bundle of lines through P_0, exceptthe line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1x + b_1y + c_1 = 0 \\ d_2 : a_2x + b_2y + c_2 = 0 \end{cases},$$

assumed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{3.28}$$

Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

3.4.7 The Angle of Two Lines ([1])

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$ (see Figure 2). One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

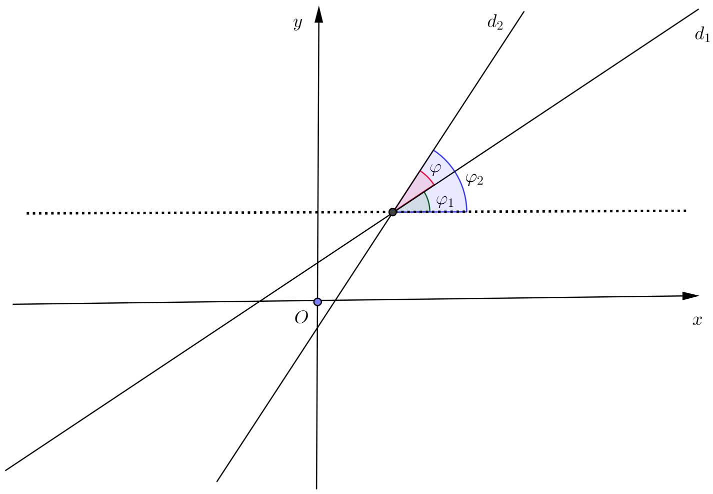


Figure 2:

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_2}{1 + m_1m_2}. \tag{3.29}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3.30}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \tag{3.31}$$

3.5 Problems

1. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\ (1,-1,0)$ and $\overrightarrow{v}_2\ (-3,2,4)$.

   HINT.
$$\begin{vmatrix} x-0 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

2. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to

   (a) The x-axis;

   (b) The line $(d_1)\ \dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\ (1,0,2)$.

   SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

   and is parallel to the line

$$(d_2)\ \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$

HINT.
$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0.$$

4. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a} \text{ where } a \text{ is a constatnt.}$$
Show that the plane (A,B,C) passes through a fixed point.

SOLUTION. The equation of the plane (ABC) can be written in intercept form, namely
$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1.$$
The given relation shows that the point $P(a,a,a) \in (ABC)$ whenever α,β,γ verifies the given relation.

5. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane (π) $3x-y+2z-15=0$ and intersects the line
$$(d)\ \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$

6. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through equal intercepts.

SOLUTION. The general equation of such a plane is $x+y+z=a$. In this particular case $a=1+(-2)+3=2$ and the equation of the required plane is $x+y+z=2$.

7. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines
$$(d_1)\left\{\begin{array}{rcrcrcrl} x & + & 2y & - & z & + & 1 & =0 \\ x & - & y & + & z & - & 1 & =0 \end{array}\right. \qquad (d_2)\left\{\begin{array}{rcrcrl} 2x & - & y & + & z & =1 \\ x & - & y & + & z & =0. \end{array}\right.$$

SOLUTION. We need to find some director parameters of the lines (d_1) and (d_2). In this respect we may solve the two systems. The general solution of the first system is
$$\left\{\begin{array}{l} x=-\frac{1}{3}t+\frac{1}{3} \\ y=\frac{2}{3}t-\frac{2}{3} \\ z=t \end{array}\right. ,\ t\in\mathbb{R}$$
and the general solution of the second system is
$$\left\{\begin{array}{l} x=1 \\ y=t+1 \\ z=t \end{array}\right. ,\ t\in\mathbb{R}$$
and these are the parametric equations of the lines (d_1) and (d_2). Thus, the direction of the line (d_1) is the 1-dimensional subspace
$$\left\langle\left(-\frac{1}{3},\frac{2}{3},1\right)\right\rangle=\langle(-1,2,3)\rangle,$$

and the direction of the line (d_2) is the 1-dimensional subspace $\langle(0,1,1)\rangle$.

Consequently, some director parameters of the line (d_1) are $p_1 = -1, q_1 = 2, r_1 = 3$ and some director parameters of the line (d_2) are $p_2 = 0, q_2 = r_2 = 1$. Finaly, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

The computation of the determinant is left to the reader.

A few questions in the two dimensional setting ([1])

8. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

   SOLUTION.

9. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

   SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \dfrac{2}{3}$.

SOLUTION.

11. Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that $\dfrac{1}{OA} + \dfrac{1}{OB} = k$ (constant). Prove that the lines AB passes through a fixed point.
SOLUTION.

12. Find the equation of the line passing through the intersection point of
$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$
and crossing the positive half axis of Oy at the point A with $OA = 3$.
SOLUTION.

13. Find the parametric equations of the line through P_1 and P_2, when
   (a) $P_1(3, -2)$, $P_2(5, 1)$;
   (b) $P_1(4, 1)$, $P_2(4, 3)$.

SOLUTION.

14. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.
SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.
SOLUTION.

16. Find the vector equation of the line P_1P_2, where
    (a) $P_1(2, -1)$, $P_2(-5, 3)$;
    (b) $P_1(0, 3)$, $P_2(4, 3)$.

SOLUTION.

17. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 through the point $M_0(2, 1)$, in the following situations:

    (a) d_1 is parallel with d;

    (b) d_1 is orthogonal on d;

    (c) the angle determined by d and d_1 is $\varphi = \frac{\pi}{4}$.

    SOLUTION.

18. The vertices of the triangle ΔABC are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

    (a) Find the coordinates of A, B, C.

    (b) Find the equations of the median lines of the triangle.

    (c) Find the equations of the heights of the triangle.

    SOLUTION.

4 Week 4

4.1 Analytic conditions of parallelism and nonparallelism

4.1.1 The parallelism between a line and a plane

Proposition 4.1. *The equation of the director subspace $\overrightarrow{\pi}$, of the plane $\pi : Ax + By + Cz + D = 0$ is $AX + BY + CZ = 0$.*

Proof. We first recall that

$$\overrightarrow{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}, \tag{4.1}$$

where $A_0 \in \pi$ is an arbitrary point, and the representation (4.1) of $\overrightarrow{\pi}$ is independent on the choice of $A_0 \in \pi$. According to equation (3.8), the equation of a plane π can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where $A_0(x_0, y_0, z_0)$ is a point in π. In other words,

$$M(x, y, z) \in \pi \Longleftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which shows that

$$\begin{aligned}
\overrightarrow{\pi} &= \{\overrightarrow{A_0M}\,(x - x_0, y - y_0, z - z_0) \mid M(x, y, z) \in \pi\} \\
&= \{\overrightarrow{A_0M}\,(x - x_0, y - y_0, z - z_0) \mid A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\} \\
&= \{\overrightarrow{v}\,(X, Y, Z) \in \mathcal{V} \mid AX + BY + CZ = 0\}.
\end{aligned}$$

Thus, the equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{v}\,(X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In other words, the *equation of the director subspace* $\overrightarrow{\pi}$ is $AX + BY + CZ = 0$. □

Corollary 4.2. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ if and only if

$$Ap + Bq + Cr = 0 \tag{4.2}$$

Proof. Indeed,

$$\begin{aligned}
\Delta \| \pi &\Longleftrightarrow \overrightarrow{\Delta} \subseteq \overrightarrow{\pi} \Longleftrightarrow \langle (p, q, r) \rangle \subseteq \overrightarrow{\pi} \\
&\Longleftrightarrow \overrightarrow{d}\,(p, q, r) \in \overrightarrow{\pi} \Longleftrightarrow Ap + Bq + Cr = 0.
\end{aligned}$$

□

4.1.2 The intersection point of a straight line and a plane

Proposition 4.3. *Consider a straight line*

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ *which are not parallel to each other, i.e.*

$$Ap + Bq + Cr \neq 0.$$

The coordinates of the intersection point $d \cap \pi$ *are*

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}, \end{cases} \tag{4.3}$$

where $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F(x, y, z) = Ax + By + Cz + D$.

Proof. The parametric equations of (d) are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{4.4}$$

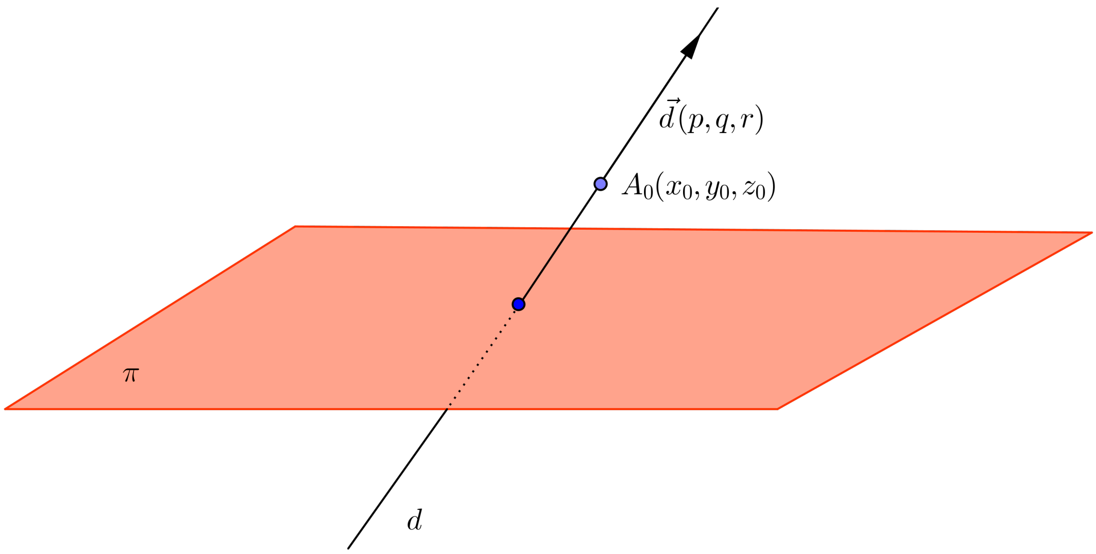
The unique value of $t \in \mathbb{R}$, which corresponds to the intersection point $d \cap \pi$, can be found by solving the equation

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + rt) + D = 0.$$

Its unique solution is

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}$$

and can be used to obtain the required coordinates (4.3) by replacing this value in (4.4). $\square$



Example 4.1 (Homework)**.** Decide whether the line d and the plane π are parallel or concurrent and find the coordinates of the intersection point of of Δ and π whenever $\Delta \not\parallel \pi$:

1. $d : \frac{x+2}{1} = \frac{y-1}{3} = \frac{z-3}{1}$ and $\pi : x - y + 2z = 1$.
2. $d : \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-2}{-1}$ and $\pi : 2x - y + 3z - 1 = 0$.

SOLUTION.

4.1.3 Parallelism of two planes

Proposition 4.4. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

Then $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 2$, *i.e.* $\overrightarrow{\pi}_1 = \overrightarrow{\pi}_2$.
3. *rank* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Remark 4.1. Note that

$$\text{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1 \Leftrightarrow \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0$$
$$\Leftrightarrow A_1B_2 - A_2B_1 = A_1C_2 - A_2C_1 = B_1C_2 - C_2B_1 = 0. \tag{4.5}$$

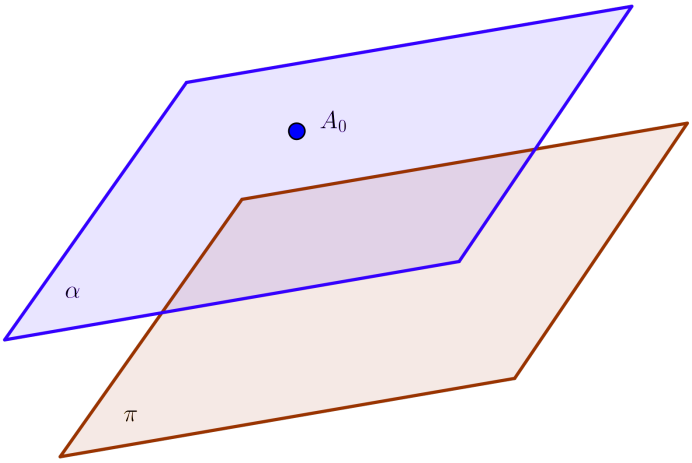
The relations (4.5) are often written in the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \tag{4.6}$$

although at most two of the coefficients A_2, B_2 or C_2 might be zero. In fact relations (4.6) should be understood in terms of linear dependence of the vectors $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$, i.e. $(A_1, B_1, C_1) = k(A_2, B_2, C_2)$, where $k \in \mathbb{R}$ is the common value of those ratios (4.6) which do not involve any zero coefficients. Let us finally mention that the equivalences (4.5) prove the equivalence (3) $\Longleftrightarrow$ (4) of Proposition 4.4.

Example 4.2. The equation of the plane α passing through the point $A_0(x_0, y_0, z_0)$, which is parallel to the plane $\pi: \; Ax + By + Cz + D = 0$ is

$$\alpha: \; A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



4.1.4 Straight lines as intersections of planes

Corollary 4.5. *Consider the planes*

$$(\pi_1) \; A_1x + B_1y + C_1z + D_1 = 0, \; (\pi_2) \; A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1. $\pi_1 \nparallel \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 1$.
3. $rank \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$
4. *The vectors* $(A_1, B_1, C_1), \; (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using the characterization of parallelism between a line and a plane, given by Proposition 4.2, we shall find the direction of a straight line which is given as the intersection of two planes. Consider the planes $(\pi_1) \; A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2) \; A_2x + B_2y + C_2z + D_2 = 0$ such that

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus, $\overrightarrow{\Delta} = \overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\overrightarrow{\Delta}$ are

$$(\overrightarrow{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \tag{4.7}$$

By solving the system (4.7) one can therefore deduce that $\overrightarrow{d}\ (p,q,r) \in \overrightarrow{\Delta} \Leftrightarrow\ \exists \lambda \in \mathbb{R}$ such that

$$(p,q,r) = \lambda \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \tag{4.8}$$

The relation is usually (4.8) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{4.9}$$

Let us finaly mention that we usually choose the values

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \tag{4.10}$$

for the director parameters (p,q,r) of Δ.

Example 4.3. Write the equations of the plane through $P(4,-3,1)$ which is parallel to the lines

$$(\Delta_1) \begin{cases} 2x - z + 1 = 0 \\ 3y + 2z - 2 = 0. \end{cases} \text{ and } (\Delta_2) \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0. \end{cases}$$

SOLUTION. One can see the required plane as the one through $P(4,-3,1)$ which is parallel to the director vectors $\overrightarrow{d}_1\ (p_1,q_1,r_1)$ and $\overrightarrow{d}_2\ (p_2,q_2,r_2)$ of Δ_1 and Δ_2 respectively. One can choose

$$\begin{aligned} p_1 &= \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 & \quad p_2 &= \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \\ q_1 &= \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \quad \text{and} & \quad q_2 &= \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \\ r_1 &= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 & \quad r_2 &= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3. \end{aligned}$$

Thus, the equation of the required plane is

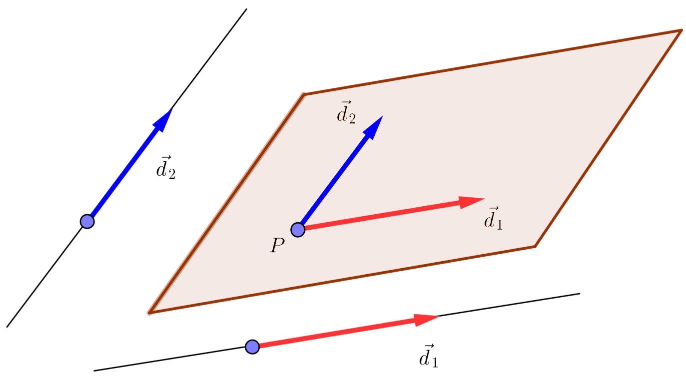


Figure 3:

$$\begin{aligned} \begin{vmatrix} x-4 & y+3 & z-1 \\ 3 & -4 & 6 \\ 4 & -1 & -3 \end{vmatrix} = 0 &\iff 12(x-4) - 3(z-1) + 24(y+3) + 16(z-1) + 6(x-4) + 9(y+3) = 0 \\ &\iff 18(x-4) + 33(y+3) + 13(z-1) = 0 \\ &\iff 18x + 33y + 13z - 72 + 99 - 13 = 0 \\ &\iff 18x + 33y + 13z + 14 = 0. \end{aligned}$$

4.2 Pencils of planes

Definition 4.1. The collection of all planes containing a given straight line

$$(\Delta)\left\{\begin{array}{l}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{array}\right.$$

is called the *pencil* or the *bundle* of planes through Δ.

Proposition 4.6. *The plane π belongs to the pencil of planes through the straight line Δ if and only if the equation of the plane π is*

$$\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0. \tag{4.11}$$

for some $\lambda,\mu\in\mathbb{R}$ such that $\lambda^2+\mu^2>0$.

Proof. Every plane in the family (4.11) obviously contains the line Δ.

Conversely, assume that π is a plane through the line Δ. Consider a point $M\in\pi\setminus\Delta$ and recall that π is completely determined by Δ and M. On the other hand M and Δ are obviously contained in the plane $F_1(x_M,y_M,z_M)F_2(x,y,z)-F_2(x_M,y_M,z_M)F_1(x,y,z)=0$ of the family (4.11), where $F_1,F_2:\mathbb{R}^3\longrightarrow\mathbb{R}$, $F_i(x,y,z)=A_ix+B_iy+C_iz+D_i$, for $i=1,2$. Thus the plane π belongs to the family (4.11) and its equation is

$$F_1(x_M,y_M,z_M)F_2(x,y,z)-F_2(x_M,y_M,z_M)F_1(x,y,z)=0.$$

□

Remark 4.2. The family of planes $A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$, where λ covers the whole real line $\mathbb{R}$, is the so called *reduced pencil of planes* through Δ and it consists in all planes through Δ except the plane of equation $A_2x+B_2y+C_2z+D_2=0$.

Example 4.4. Write the equations of the plane parallel to the line $d:\ x=2y=3z$ passing through the line

$$\Delta:\left\{\begin{array}{l}x+y+z=0\\2x-y+3z=0.\end{array}\right.$$

SOLUTION. Note that none of the planes $x+y+z=0$ and $x-y+3z=0$, passing through (Δ), is parallel to (d), as $1\cdot1+1\cdot\frac{1}{2}+1\cdot\frac{1}{3}\neq0$ and $2\cdot1+(-1)\cdot\frac{1}{2}+3\cdot\frac{1}{3}\neq0$. Thus, the required plane is in a reduced pencil of planes, such as the family $\pi_\lambda: x+y+z+\lambda(2x-y+3z)=0,\ \lambda\in\mathbb{R}$. The parallelism relation between (d) and $\pi_\lambda:(2\lambda+1)x+(1-\lambda)y+(3\lambda+1)z=0$ is

$$(2\lambda+1)\cdot1+(1-\lambda)\cdot\frac{1}{2}+(3\lambda+1)\cdot\frac{1}{3}=0\Longleftrightarrow12\lambda+6+3-3\lambda+6\lambda+2=0\Longleftrightarrow\lambda=-\frac{11}{15}.$$

Thus, the required plane is

$$\pi_{-11/15}:\left(-2\frac{11}{15}+1\right)x+\left(1+\frac{11}{15}\right)y+\left(-3\frac{11}{15}+1\right)z=0\Longleftrightarrow-7x+26y-18z=0.$$

Appendix

4.3 Projections and symmetries

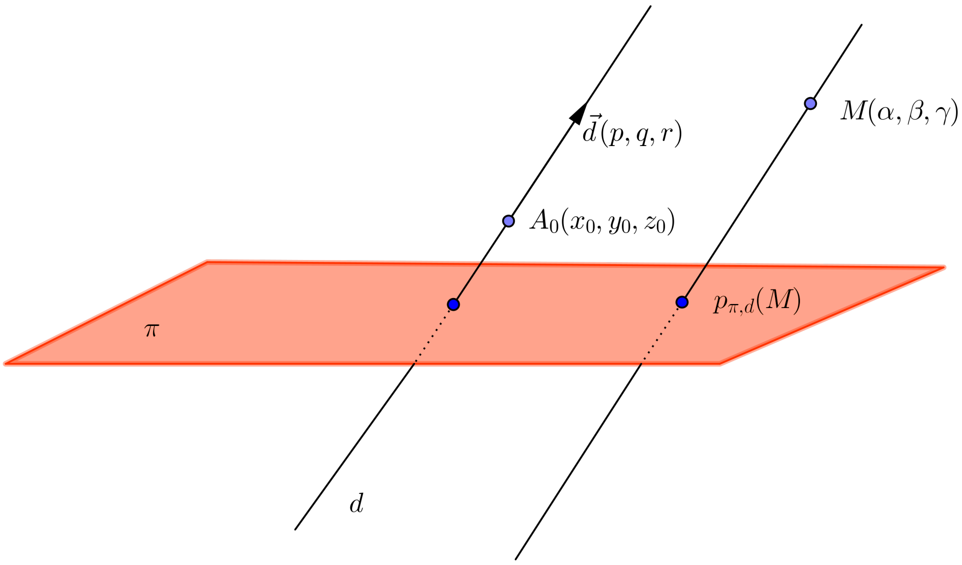
4.3.1 The projection on a plane parallel with a given line

Consider a straight line
$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.
$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on π parallel to d, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d. Due to relations (4.3), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of M, are
$$\begin{cases} x_{_M} - p\dfrac{F(x_{_M}, y_{_M}, z_{_M})}{Ap + Bq + Cr} \\ y_{_M} - q\dfrac{F(x_{_M}, y_{_M}, z_{_M})}{Ap + Bq + Cr} \\ z_{_M} - r\dfrac{F(x_{_M}, y_{_M}, z_{_M})}{Ap + Bq + Cr}, \end{cases} \tag{4.12}$$
where $F(x, y, z) = Ax + By + Cz + D$.

Consequently, the position vector of $p_{\pi,d}(M)$ is
$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \overrightarrow{d} . \tag{4.13}$$

Proposition 4.7. *If $R = (O, b)$ is the Cartesian reference system behind the equations of the line*
$$(d)\ \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$
and the plane $(\pi)\ Ax + By + Cz + D = 0$, concurrent with (d), then
$$[p_{\pi,d}(M)]_{_R} = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} [M]_R - \frac{D}{Ap + Bq + Cr} [\overrightarrow{d}]_b,$$
where $\overrightarrow{d}\ (p, q, r)$ stands for the director vector of the line (d).



4.3.2 The symmetry with respect to a plane parallel with a given line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ *the symmetry of $\mathcal{P}$ with respect to π parallel to d*. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \tag{4.14}$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2\,\overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{Ap+Bq+Cr}\,\overrightarrow{d}\,. \tag{4.15}$$

Proposition 4.8. *If $R = (O, b)$ is the Cartesian reference system behind the equations of the line*

$$(d)\ \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and the plane $(\pi)\ Ax + By + Cz + D = 0$, concurrent with (d), then

$$(Ap+Bq+Cr)[s_{\pi,d}(M)]_R = \begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}[M]_R - 2D[\overrightarrow{d}]_b, \tag{4.16}$$

where $\overrightarrow{d}\,(p,q,r)$ stands for the director vector of the line (d).

4.3.3 The projection on a straight line parallel with a given plane

Consider a straight line

$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{d,\pi} : \mathcal{P} \longrightarrow d$ of $\mathcal{P}$ on d, whose value $p_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π. Due to relations (4.3), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 - p\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ y_0 - q\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ z_0 - r\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.17}$$

where $G_M(x,y,z) = A(x - x_M) + B(y - y_M) + C(z - z_M)$. Consequently, the position vector of $p_{d,\pi}(M)$ is

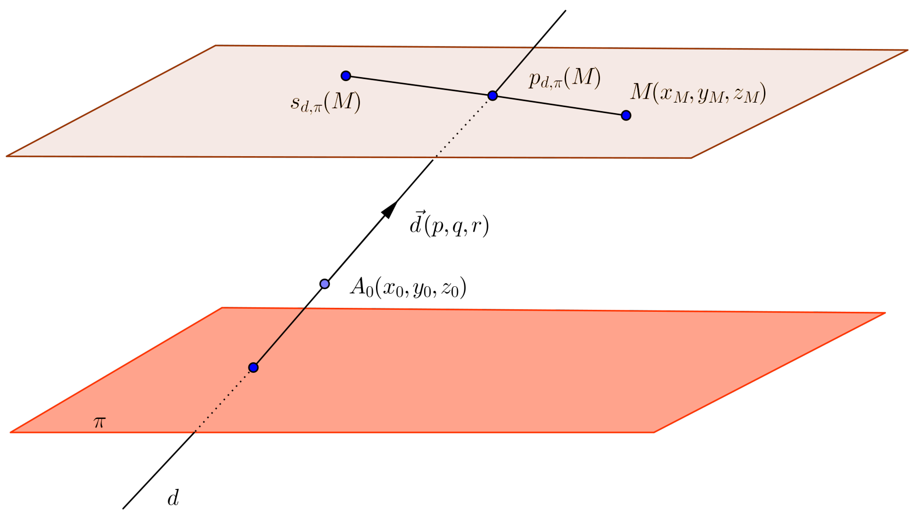
$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} - \frac{G_M(A_0)}{Ap+Bq+Cr}\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.18}$$

Note that $G_M(A_0) = A(x_0 - x_M) + B(y_0 - y_M) + C(z_0 - z_M) = F(A_0) - F(M)$, where $F(x,y,z) = Ax + By + Cz + D$. Consequently the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 + p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0 + q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0 + r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.19}$$

and the position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M)-F(A_0)}{Ap+Bq+Cr}\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.20}$$



4.3.4 The symmetry with respect to a line parallel with a plane

We call the function $s_{d,\pi} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to d parallel to π*. The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{4.21}$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2\,\overrightarrow{OA_0} - \overrightarrow{OM} + 2\frac{F(M) - F(A_0)}{Ap + Bq + Cr}\,\overrightarrow{d}\,. \end{aligned} \tag{4.22}$$

4.4 Problems

1. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

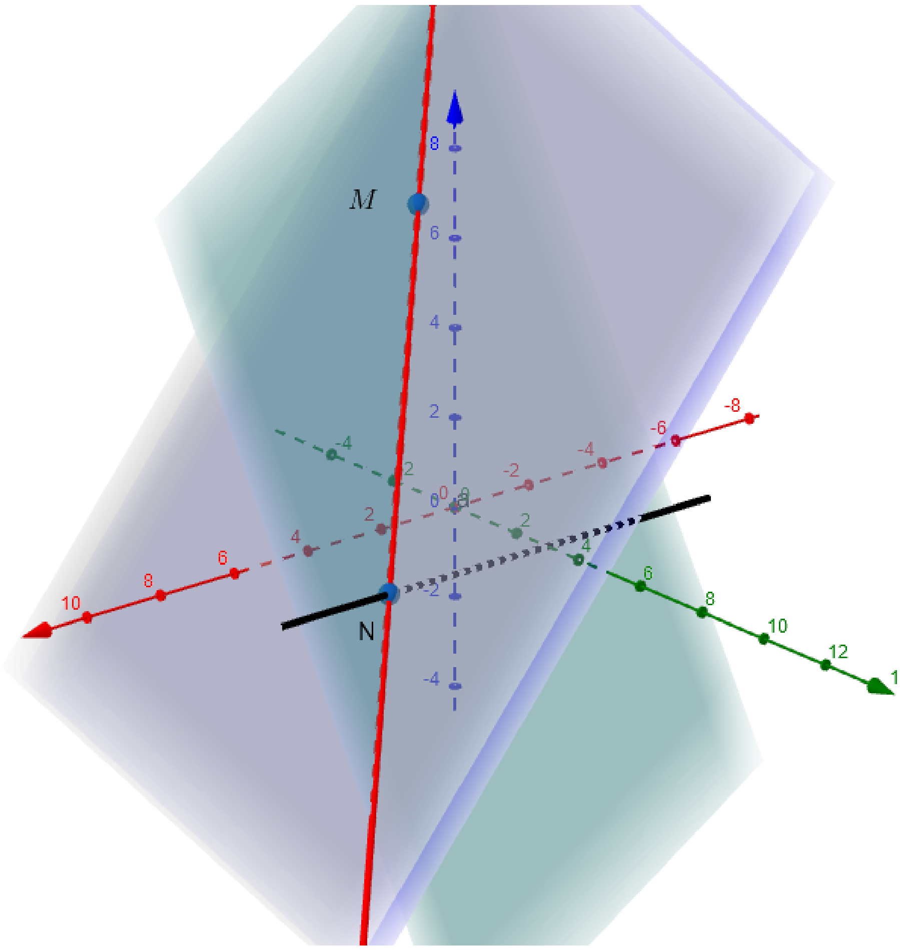
and the point $A(-1,2,6)$.

SOLUTION.

2. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

SOLUTION 1. The equation of the plane α passing through the point $M(1,0,7)$, which is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$, is $(\alpha)\ 3(x-1) - (y-0) + 2(z-7) = 0$, i.e. $(\alpha)\ 3x - y + 2z - 17 = 0$.



The parametric equations of the line d are

$$\begin{cases} x = 1 + 4t \\ y = 3 + 2t \\ z = t \end{cases}, \; t \in \mathbb{R}.$$

The coordinates of the intersection point N between the line (d) and the plane α can be obtained by solving the equation $3((1+4t) - (3+2t) + 2t - 17 = 0$. The required line is MN.

SOLUTION 2. The required line can be equally regarded as the intersection line between the plane α (passing through the point $M(1,0,7)$, which is parallel to the plane (π)) and the plane determined by the given line (d) and the point M. While the equation $3x - y + 2z - 17 = 0$ of α was already used above, the equation of the plane determined by the line (d) and the point M can be determined via the pencil of planes through

$$(d) \begin{cases} \dfrac{x-1}{4} = \dfrac{y-3}{2} \\ \dfrac{y-3}{2} = \dfrac{z}{1} \end{cases} \Leftrightarrow (d) \begin{cases} x - 2y + 5 = 0 \\ y - 2z - 3 = 0. \end{cases}$$

Note that none of the planes $x - 2y + 5 = 0$ or $y - 2z - 3 = 0$ passes through M, which means that the plane determined by d and M is in the reduced pencil of planes

$$(\pi_\lambda)\; x - 2y + 5 = 0 + \lambda(y - 2z - 3) = 0.$$

The plane determined by d and M can be found by imposing on the coordinates of M to verify the equation of π_λ.

3. Write the equations of the projection of the line

$$(d) \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction $\overrightarrow{u}\ (1, 1, -2)$. Write the equations of the symmetry of the line d with respect to the plane π parallel to the direction $\overrightarrow{u}\ (1, 1, -2)$.

SOLUTION.

4. Prove Proposition 4.7

SOLUTION.

5. Prove Proposition 5.6

   SOLUTION.

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0,\ \ d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and $\pi \not\parallel d$.

SOLUTION.

7. Show that two different parallel lines are mapped onto parallel lines by a symmetry $s_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0,\ \ d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and$\pi \not\parallel d$.

SOLUTION.

8. Assume that $R=(O,b)$ ($b=[\overrightarrow{u},\overrightarrow{v},\overrightarrow{w}]$) is the Cartesian reference system behind the equations of a plane $\pi: Ax+By+Cz+D=0$ and a line

$$d: \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}.$$

If $\pi \not\parallel d$, show that

(a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)}= p(\overrightarrow{MN})$, for all $M,N\in\mathcal{V}$, where $p:\mathcal{V}\longrightarrow\mathcal{V}$ is the linear transformation whose matrix representation is

$$[p]_b=\frac{1}{Ap+Bq+Cr}\begin{pmatrix} Bq+Cr & -Bp & -Cp \\ -Aq & Ap+Cr & -Cq \\ -Ar & -Br & Ap+Bq \end{pmatrix}.$$

SOLUTION.

(b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)}= s(\overrightarrow{MN})$, for all $M,N\in\mathcal{V}$, where $s:\mathcal{V}\longrightarrow\mathcal{V}$ is the linear transformation whose matrix representation is

$$[s]_b=\frac{1}{Ap+Bq+Cr}\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}.$$

SOLUTION.

9. Consider a plane $\pi : Ax + By + Cz + D = 0$ and a line
$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$
If $\pi \not\parallel d$, show that
(a) $p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}$.
(b) $s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}$.

SOLUTION.

4.5 Projections and symmetries in the two dimensional setting

4.5.1 The intersection point of two concurrent lines

Consider two lines
$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$
şi $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e.
$$ap + bq \neq 0.$$
The parametric equations of d are:
$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, t \in \mathbb{R} \tag{4.23}$$

The value of $t \in \mathbb{R}$ for which this line (4.23) punctures the line Δ can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt)$$

to verify the equationof the line Δ, namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where $F(x, y) = ax + by + c$.

The coordinates of the intersection point $d \cap \Delta$ are:

$$\begin{aligned} &x_0 - p\frac{F(x_0, y_0)}{ap + bq} \\ &y_0 - q\frac{F(x_0, y_0)}{ap + bq}. \end{aligned} \tag{4.24}$$

4.5.2 The projection on a line parallel with another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e. $ap + bq \neq 0$. For these given data we may define the projection $p_{\Delta,d} : \pi \to \Delta$ of π on Δ parallel cu d, whose value $p_{\Delta,d}$ at $M \in \pi$ is the intersection point between Δ and the line through M which is parallelt to d. Due to relations (4.24), the coordinates of $p_{\Delta,d}(M)$, in terms of the coordinates of M are:

$$\begin{aligned} &x_M - p\frac{F(x_M, y_M)}{ap + bq} \\ &y_M - q\frac{F(x_M, y_M)}{ap + bq}, \end{aligned}$$

where $F(x, y) = ax + by + c$.

Consequently, the position vector of $p_{\Delta,d}(M)$ is

$$\overrightarrow{Op_{\Delta,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq}\overrightarrow{d},$$

where $\overrightarrow{d} = p\overrightarrow{e} + g\overrightarrow{f}$.

Proposition 4.9. *If R is the Cartesian reference system of the plane π behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[p_{\Delta,d}(M)]_R = \frac{1}{ap + bq}\begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix}[M]_R - \frac{c}{ap + bq}[\overrightarrow{d}]_b. \tag{4.25}$$

4.5.3 The symmetry with respect to a line parallel with another line

We call the function $s_{\Delta,d} : \pi \to \pi$, whose value $s_{\Delta,d}$ at $M \in \pi$ is the symmetric point of M with respect to $p_{\Delta,d}(M)$, the *symmetry of π with respect to Δ parallel to d*. The direction of d is equally called the direction of the symmetry and π is called the *axis of the symmetry*. For the position vector of $s_{\Delta,d}(M)$ we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.}$$
$$\overrightarrow{Os_{\Delta,d}(M)} = 2\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap+bq}\overrightarrow{d},$$

where $F(x,y) = ax + by + c$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{ap+bq} \\ y_M - 2q\dfrac{F(x_M, y_M)}{ap+bq}. \end{cases}$$

Proposition 4.10. *If R is the Cartesian reference system of the plane π behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x-x_0}{p} = \frac{y-y_0}{q},$$

then

$$[s_{\Delta,d}(M)]_R = \frac{1}{ap+bq}\begin{pmatrix} -ap+bq & -2bp \\ -2aq & ap-bq \end{pmatrix}[M]_R - \frac{2c}{ap+bq}[\overrightarrow{d}]_b. \tag{4.26}$$

5 Week 5: Products of vectors

5.1 The dot product

Definition 5.1. The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 \text{ if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ || \vec{a} || \cdot || \vec{b} || cos(\widehat{\vec{a}, \vec{b}}) \text{ if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \tag{5.1}$$

is called the *dot product* of the vectors $\vec{a}, \vec{b}$.

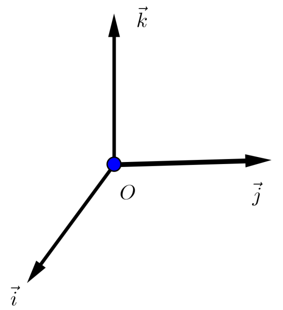
Remark 5.1. 1. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0.$

2. $\vec{a} \cdot \vec{a} = || \vec{a} || \cdot || \vec{a} || \cos 0 = || \vec{a} ||^2.$

Proposition 5.1. *The dot product has the following properties:*

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \ \forall \ \vec{a}, \vec{b} \in \mathcal{V}.$
2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b}), \ \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}.$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, \ \forall \ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$
4. $\vec{a} \cdot \vec{a} \geq 0, \ \forall \ \vec{a} \in \mathcal{V}.$
5. $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}.$

Definition 5.2. A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $|| \vec{i} || = || \vec{j} || = || \vec{k} || = 1$, $\vec{i} \perp \vec{j}, \vec{j} \perp \vec{k}, \vec{k} \perp \vec{i}$ ($\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1, \ \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$). A Cartesian reference system $R = (O, \vec{i}, \vec{j}, \vec{k})$ is said to be *orthonormal* if the basis $[\vec{i}, \vec{j}, \vec{k}]$ is orthonormal.



orthonormal basis

Proposition 5.2. *Let* $[\vec{i}, \vec{j}, \vec{k}]$ *be an orthonormal basis and* $\vec{a}, \vec{b} \in \mathcal{V}$. *If* $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}, \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, *then*

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{5.2}$$

Proof. Indeed,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\ &= a_1 b_1 \vec{i} \cdot \vec{i} + a_1 b_2 \vec{i} \cdot \vec{j} + a_1 b_3 \vec{i} \cdot \vec{k} \\ &+ a_2 b_1 \vec{j} \cdot \vec{i} + a_2 b_2 \vec{j} \cdot \vec{j} + a_2 b_3 \vec{j} \cdot \vec{k} \\ &+ a_3 b_1 \vec{k} \cdot \vec{i} + a_3 b_2 \vec{k} \cdot \vec{j} + a_3 b_3 \vec{k} \cdot \vec{k} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned}$$

□

Remark 5.2. Let $[\vec{i},\vec{j},\vec{k}]$ be an orthonormal basis and $\vec{a},\vec{b}\in\mathcal{V}$. If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, then

1. $\vec{a}\cdot\vec{a}=a_1^2+a_2^2+a_3^2$ and we conclude that $||\vec{a}||=\sqrt{\vec{a}\cdot\vec{a}}=\sqrt{a_1^2+a_2^2+a_3^2}$.

2.
$$\begin{aligned}\cos(\widehat{\vec{a},\vec{b}}) &= \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\cdot||\vec{b}||}\\ &= \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}.\end{aligned} \tag{5.3}$$

In particular

$$\begin{aligned}\cos(\widehat{\vec{a},\vec{i}}) &= \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{j}}) &= \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{k}}) &= \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.\end{aligned}$$

3. $\vec{a}\perp\vec{b}\Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0$

5.1.1 Applications of the dot product

$\Diamond$ **The two dimensional setting**

- **The distance between two points** Consider two points $A(x_A,y_A),\ B(x_B,y_B)\in\pi$. The norm of the vector $\overrightarrow{AB}$ $(x_B-x_A,y_B-y_A$ is
$$||\overrightarrow{AB}||=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}.$$

- **The equation of the circle**
Recall that the circle $\mathcal{C}(O,r)$ is the locus of points M in the plane such that $\mathrm{dist}(O,M)=r\Longleftrightarrow \|\overrightarrow{OM}\|=r$. If (a,b) are the coordinates of O and (x,y) are the coordinates of M, then
$$\begin{aligned}\|\overrightarrow{OM}\|=r &\Longleftrightarrow \sqrt{(x-a)^2+(y-b)^2}=r\Longleftrightarrow (x-a)^2+(y-b)^2=r^2\\ &\Longleftrightarrow x^2+y^2-2ax-2by+c=0,\end{aligned} \tag{5.4}$$
where $c=a^2+b^2-r^2$. Conversely, every equation of the form $x^2+y^2+2ex+2fy+g=0$ is the equation of the circle centered at $(-e,-f)$ and having the radius $r=\sqrt{e^2+f^2-g}$, whenever $e^2+f^2\geq g$. One can find the equation of the circle circumscribed to the triangle ABC by imposing the requirement on the coordinates (x_A,y_A), (x_B,y_B) and (x_C,y_C) of its vertices A,B,C to verify the equation $x^2+y^2+2ex+2fy+g=0$. A point $M(x,y)$ belongs to this circumcircle if and only if
$$\begin{cases} x^2+y^2+2ex+2fy+g=0\\ x_A^2+y_A^2+2ex_A+2fy_A+g=0\\ x_B^2+y_B^2+2ex_B+2fy_B+g=0\\ x_C^2+y_C^2+2ex_C+2fy_C+g=0\end{cases} \tag{5.5}$$

On can regard the system (5.5) as linear with the unknowns e, g, f, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_B^2+y_B^2 & x_B & y_B & 1 \\ x_C^2+y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0,$$

which is the equation of the circumcircle of the triangle ABC.

- **The normal vector of a line** If $R = (O, b)$ is the orthonormal Cartesian reference sytem behind the equation of a line $(d)\ ax + by + c = 0$, then $\vec{n}\ (a, b)$ is a normal vector to the direction $\vec{d}$ of d. Indeed, every vector of the direction $\vec{d}$ of d has the form $\overrightarrow{PM}$, where $P(x_P, y_P)$ and $M(x, y)$ are two points on the line d. Thus, $ax_P + by_P + c = 0 = ax_M + by_M + c$, which shows that

$$a(x_M - x_P) + b(y_M - y_P) = 0,$$

namely

$$\vec{n} \cdot \overrightarrow{PM} = 0 \Longleftrightarrow \vec{n} \perp \overrightarrow{PM}\ .$$

- **The distance from a point to a line** If $(d)\ ax + by + c = 0$ is a line and $M(x_M, y_M) \in \pi$ a given point, then the distance from M to d is

$$\delta(M, d) = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \tag{5.6}$$

Indeed, $\delta(M, d) = |\delta|$, where δ is the real scalar with the property $\overrightarrow{PM} = \delta \frac{\vec{n}}{\|\vec{n}\|}$ and $P(x_P, y_P)$ is the orthogonal projection of $M(x_M, y_M)$ on d. Thus $\overrightarrow{PM}\ (x_M - x_P, y_M - y_P)$ andul

$$\begin{aligned} \delta(M, d) &= |\delta| = \left| \overrightarrow{PM} \cdot \frac{\vec{n}}{\| \vec{n} \|} \right| = \frac{|\overrightarrow{PM} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_M - x_P) + b(y_M - y_P)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_M + by_M - ax_P - by_P)|}{\sqrt{a^2 + b^2}} = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

$\diamondsuit$ **The three dimensional setting**

- **The distance between two points** Consider two points $A(x_A, y_A, z_A),\ B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A, z_B - z_A)$ is

$$\| \overrightarrow{AB} \| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

- **The equation of the sphere**

Recall that the sphere $\mathcal{S}(O, r)$ is the locus of points M in space such that $\text{dist}(O, M) = r \Longleftrightarrow \| \overrightarrow{OM} \| = r$. If (a, b, c) are the coordinates of O and (x, y, z) are the coordinates of M, then

$$\begin{aligned} \| \overrightarrow{OM} \| = r \Longleftrightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r \Longleftrightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \\ \Longleftrightarrow x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \end{aligned}$$

where $d = a^2 + b^2 + c^2 - r^2$. Conversely, every equation of the form

$$x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0$$

is the equation of the sphere centered at $(-e,-g,-f)$ and having the radius $r=\sqrt{e^2+f^2+g^2-h}$, whenever $e^2+f^2+g^2\geq h$. One can find the equation of the sphere circumscribed to the tetrahedron $ABCD$ by imposing the requirement on the coordinates (x_A,y_A,z_A), (x_B,y_B,z_B) and (x_C,y_C,z_C) and (x_D,y_D,z_D) of its vertices A,B,C,D to verify the equation $x^2+y^2+z^2-2ax-2by-2cz+d=0$. A point $M(x,y,z)$ belongs to this circumcircle if and only if

$$\begin{cases} x^2+y^2+z^2+2ex+2fy+2gz+h=0\\ x_A^2+y_A^2+z_A^2+2ex_A+2fy_A+2gz_A+h=0\\ x_B^2+y_B^2+z_B^2+2ex_B+2fy_B+2gz_B+h=0\\ x_C^2+y_C^2+z_C^2+2ex_C+2fy_C+2gz_C+h=0\\ x_D^2+y_D^2+z_D^2+2ex_D+2fy_D+2gz_A+h=0 \end{cases} \tag{5.7}$$

On can regard the system (5.7) as linear with the unknowns e,g,f,h, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1\\ x_A^2+y_A^2+z_A^2 & x_A & y_A & z_A & 1\\ x_B^2+y_B^2+z_B^2 & x_B & y_B & z_B & 1\\ x_C^2+y_C^2+z_C^2 & x_C & y_C & z_C & 1\\ x_D^2+y_D^2+z_D^2 & x_D & y_D & z_A & 1 \end{vmatrix}=0,$$

which is the equation of the circumsphere of the tetrahedron $ABCD$.

- **The normal vector of a plane.** Consider the plane $\pi : Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of π becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{5.8}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (5.8) tells us that $\overrightarrow{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\overrightarrow{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\overrightarrow{n}\perp\overrightarrow{\pi}$, where $\overrightarrow{n}(A,B,C)$. This is the reason to call $\overrightarrow{n}(A,B,C)$ *the normal vector* of the plane π.

- **The distance from a point to a plane.** Consider the plane $\pi : Ax+By+Cz+D=0$, a point $P(x_P,y_P,z_P)\in\mathcal{P}$ and M the orthogonal projection of P on π. The real number δ given by $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$ is called the *oriented distance* from P to the plane π, where $\overrightarrow{n}_0=\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}(A,B,C)$. Since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, it follows that $\delta(P,M)=\|\overrightarrow{MP}\|=|\delta|$, where $\delta(P,M)$ stands for the distance from P to π. We shall show that

$$\delta=\frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, we get successively:

$$\begin{aligned}\delta &= \overrightarrow{n}_0\cdot\overrightarrow{MP}=\left(\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}\right)\cdot\overrightarrow{MP}=\frac{\overrightarrow{n}\cdot\overrightarrow{MP}}{\|\overrightarrow{n}\|}\\ &= \frac{A(x_P-x_M)+B(y_P-y_M)+C(z_P-z_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P-(Ax_M+By_M+Cz_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

Consequently, the distance from P to the plane π is

$$\delta(P,\pi)=\|\overrightarrow{MP}\|=|\delta|=\frac{|Ax_P+By_P+Cz_P+D|}{\sqrt{A^2+B^2+C^2}}.$$

Example 5.1. Compute the distance from the point $A(3,1,-1)$ to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

SOLUTION.

$$\delta(A,\pi) = \frac{|22 \cdot 3 + 4 \cdot 1 - 20 \cdot (-1) - 45|}{\sqrt{22^2 + 4^2 + (-20)^2}} = \frac{45}{\sqrt{900}} = \frac{45}{30} = \frac{3}{2}.$$

5.2 Appendix: Orthogonal projections and reflections

5.2.1 The two dimensional setting

Asssume that $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian system of a plane π behind the equation of the line $\Delta : ax + by + c = 0$.

• **The orthogonal projection of a point on a line.** We define the projection of the ambient plane $p_\Delta : \pi \to \Delta$ on Δ, whose value p_Δ at $M \in \pi$ is the intersection point between Δ and the line through M perpendicular to Δ. Due to relations (4.24), the coordinates of $p_\Delta(M)$, in terms of the coordinates of M are:

$$\begin{aligned} &x_M - p\frac{F(x_M, y_M)}{a^2 + b^2} \\ &y_M - q\frac{F(x_M, y_M)}{a^2 + b^2}, \end{aligned}$$

where $F(x,y) = ax + by + c$. Consequently, the position vector of $p_\Delta(M)$ is

$$\overrightarrow{Op_\Delta(M)} = \overrightarrow{OM} - \frac{F(M)}{a^2 + b^2}\overrightarrow{n}_\Delta,$$

where $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$.

Proposition 5.3. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[p_\Delta(M)]_R = \frac{1}{a^2 + b^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix}[M]_R - \frac{c}{a^2 + b^2}[\overrightarrow{n}_\Delta]_b, \tag{5.9}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

• **The reflection of the plane about a line.** We call the function $r_\Delta : \pi \to \pi$, whose value r_Δ at $M \in \pi$ is the symmetric point of M with respect to $p_\Delta(M)$, the *reflection of π about Δ*. For the position vector of $r_\Delta(M)$ we have

$$\overrightarrow{Op_\Delta(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Or_\Delta(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Or_\Delta(M)} = 2\overrightarrow{Op_\Delta(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{a^2 + b^2}\overrightarrow{n}_\Delta,$$

where $F(x,y) = ax + by + c$ and $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{a^2 + b^2} \\ y_M - 2q\dfrac{F(x_M, y_M)}{a^2 + b^2}. \end{cases}$$

Proposition 5.4. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[r_\Delta(M)]_R = \frac{1}{a^2+b^2}\begin{pmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}[M]_R - \frac{2c}{a^2+b^2}[\overrightarrow{n}_\Delta]_b, \tag{5.10}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

Example 5.2. Find the coordinates of the reflected point of $P(-5, 13)$ with respect to the line

$$d : 2x - 3y - 3 = 0,$$

knowing that the Cartesian reference system R behind the coordinates of A and the equation of (d) is orthonormal.

HINT. According to 5.11 it follows that

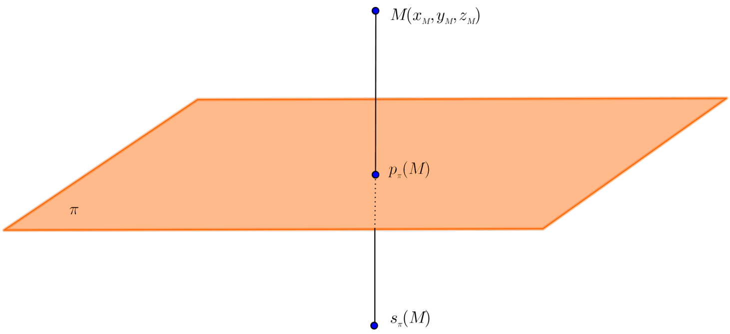
$$[r_d(P)]_R = \frac{1}{2^2+(-3)^2}\begin{pmatrix} -2^2+(-3)^2 & -2\cdot 2\cdot(-3) \\ -2\cdot 2\cdot(-3) & 2^2-(-3)^2 \end{pmatrix}\begin{bmatrix} -5 \\ 13 \end{bmatrix} - \frac{2\cdot(-3)}{2^2+(-3)^2}\begin{bmatrix} 2 \\ -3 \end{bmatrix}. \tag{5.11}$$

5.2.2 The three dimensional setting

• **The orthogonal projection of a point on a plane.** For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on π which passes through M.



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \ t \in \mathbb{R}. \tag{5.12}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (5.12) and the value of $t \in \mathbb{R}$ for which this orthogonal line (5.12) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \overrightarrow{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\overrightarrow{n}_\pi = A \overrightarrow{i} + B \overrightarrow{j} + C \overrightarrow{k}$ is the normal vector of the plasne π.

• **The orthogonal projection of the space on a plane.**

The coordinates of the orthogonal projection $p_\pi(M)$ of M on th eplane π are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi . \tag{5.13}$$

Proposition 5.5. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane (π) $Ax + By + Cz + D = 0$, then*

$$(A^2 + B^2 + C^2)[p_\pi(M)]_R = \begin{pmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{pmatrix} [M]_R - D[\overrightarrow{n}_\pi]_b. \tag{5.14}$$

Remark 5.3. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of (5.13). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \| \\ &= | - \tfrac{F(M)}{\| \overrightarrow{n}_\pi \|^2} | \cdot \| \overrightarrow{n}_\pi \| = \frac{|F(M)|}{\| \overrightarrow{n}_\pi \|}. \end{aligned}$$

• **The reflection of the space about a plane.** In order to find the position vector of the orthogonally symmetric point $r_\pi(M)$ of M w.r.t. π, we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Or_\pi(M)} \right),$$

namely

$$\overrightarrow{Or_\pi(M)} = 2\, \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi .$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. π, is called the *reflection* in the plane π and is denoted by r_π.

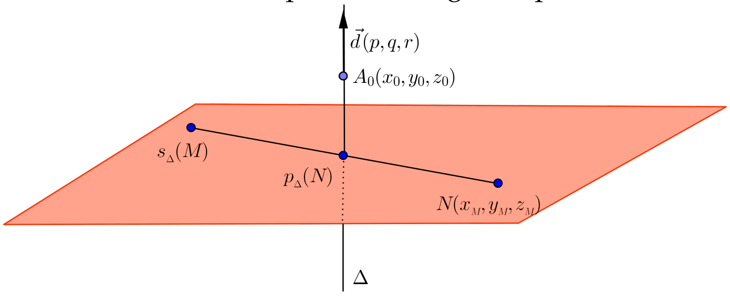
Proposition 5.6. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane (π) $Ax + By + Cz + D = 0$, then*

$$(A^2+B^2+C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2+B^2+C^2 & -2AB & -2AC \\ -2AB & A^2-B^2+C^2 & -2BC \\ -2AC & -2BC & A^2+B^2-C^2 \end{pmatrix} [M]_R - 2D[\overrightarrow{n}_\pi]_b. \tag{5.15}$$

• **The orthogonal projection of the space on a line.** For a given line

$$\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ, as well as the coordinates of the orthogonally symmetric point M with respect to Δ. The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the plane $p(x-x_N)+q(y-y_N)+r(z-z_N)=0$ orthogonal on the line Δ which passes through the point N.



The parametric equations of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \; t \in \mathbb{R}. \tag{5.16}$$

The orthogonal projection of N on the line Δ is at its intersection point with the plane

$$p(x-x_N)+q(y-y_N)+r(z-z_N)=0,$$

and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x-x_N)+q(y-y_N)+r(z-z_N)=0$ can be found by imposing the condition on the point of coordinate (x_0+pt, y_0+qt, z_0+rt) to verify the equation of the plane, namely $p(x_0+pt-x_N)+q(y_0+qt-y_N)+r(z_0+rt-z_N)=0$. Thus

$$t = -\frac{p(x_0-x_N)+q(y_0-y_N)+r(z_0-z_N)}{p^2+q^2+r^2} = -\frac{G(x_0,y_0,z_0)}{\| \vec{d}_\Delta \|^2},$$

where $G(x,y,z) = p(x-x_N)+q(y-y_N)+r(z-z_N)$ and $\vec{d}_\pi = p\vec{i} + q\vec{j} + r\vec{k}$ is the director vectoir of the line Δ. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ y_0 - q\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ z_0 - r\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\| \vec{d}_\Delta \|^2} \vec{d}_\Delta, \tag{5.17}$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The reflection of the space about a line.** In order to find the position vector of the orthogonally symmetric point $r_\Delta(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Or_\Delta(N)}\right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2\,\overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2\,\overrightarrow{OA_0} - 2\frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2}\,\overrightarrow{d}_{\Delta} - \overrightarrow{ON}\,.$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. δ, is called the *reflection* in the line δ and is denoted by r_{δ}.

5.3 Problems

1. (**2p**) Consider the triangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = 4\,\overrightarrow{AB}\cdot\overrightarrow{AC}\,.$$

2. (**2p**) Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that

   (a) $\overrightarrow{MA}\cdot\overrightarrow{MC} = \overrightarrow{MB}\cdot\overrightarrow{MD}$.

   (b) $\overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2$.

3. (**3p**) Find the angle between:

   (a) the straight lines

$$(d_1)\left\{\begin{array}{rrrrrrrrl} x & + & 2y & + & z & - & 1 & = 0 \\ x & - & 2y & + & z & + & 1 & = 0 \end{array}\right. \quad (d_2)\left\{\begin{array}{rrrrrrrrl} x & - & y & - & z & - & 1 & = 0 \\ x & - & y & + & 2z & + & 1 & = 0. \end{array}\right.$$

   (b) the planes

$$\pi_1:\ x+3y+2z+1=0 \text{ and } \pi_2: 3x+2y-z=6.$$

(c) the plane xOy and the straight line M_1M_2, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

4. (**3p**) Consider the noncoplanar vectors $\overrightarrow{OA}\ (1,-1,-2)$, $\overrightarrow{OB}\ (1,0,-1)$, $\overrightarrow{OC}\ (2,2,-1)$ related to an orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.

5. (**2p**) Find the points on the z-axis which are equidistant with respect to the planes

$$\pi_1:\ 12x+9y-20z-19=0 \text{ and } \pi_2:\ 16x+12y+15z-9=0.$$

6. (**2p**) Consider two planes

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$$
$$(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$$

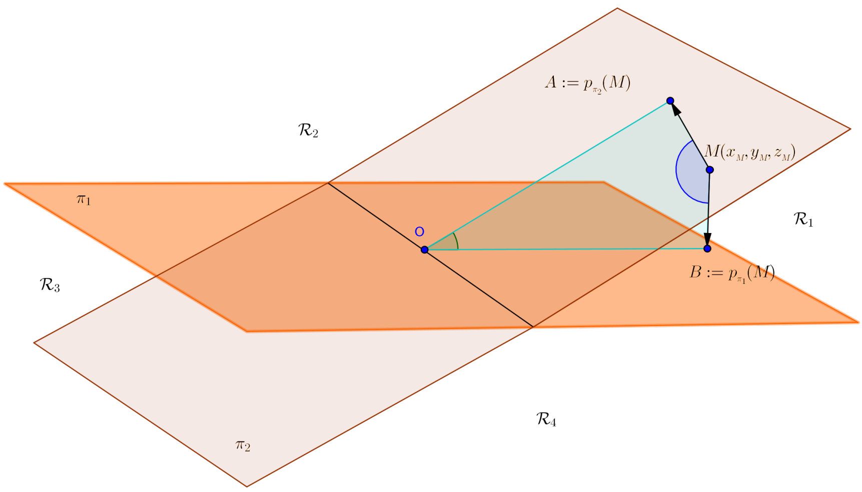
which are not parallel and not perpendicular as well. The two planes π_1, π_2 devide the space into four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes. Show that $M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3$, if and only if

$$F_1(x,y,z) \cdot F_2(x,y,z)(A_1A_2 + B_1B_2 + C_1C_2) < 0,$$

where $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$ and $F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$.

Hint. The non-parallellism relation between the two planes is equivalent with the condition

$$\operatorname{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$



The point M belongs to the union $\mathcal{R}_1 \cup \mathcal{R}_3$ if and only if the angle of the vectors $\overrightarrow{Mp_{\pi_1}(M)}$ and $\overrightarrow{Mp_{\pi_2}(M)}$ is at leat $90°$, as the quadrilateral $OAMB$ is inscriptible. More formally

$$M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 \Leftrightarrow m(\widehat{\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}}) > 90°$$
$$\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0,$$

where $p_{\pi_1}(M)$, $p_{\pi_2}(M)$ are the orthogonal projections of M on the planes π_1 and π_2 respectively.

7. **(3p)** Consider the planes $(\pi_1)\ 2x + y - 3z - 5 = 0$, $(\pi_2)\ x + 3y + 2z + 1 = 0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

8. **(3p)** Let a, b be two real numbers such that $a^2 \neq b^2$. Consider the planes:
$$(\alpha_1) ax + by - (a+b)z = 0$$
$$(\alpha_2) ax - by - (a-b)z = 0$$
and the quadric $(\mathcal{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$. If $a^2 < b^2$, show that the quadric $\mathcal{C}$ is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary, $a^2 > b^2$, show that the quadric $\mathcal{C}$ is contained in the obtuse regions of the dihedral angles formed by the two planes.

9. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that

   (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).

   (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.

   (c) The heights of the tetrahedron are concurrent.
   (Such a tetrahedron is said to be orthocentric)

Solution. Denote by $\overrightarrow{AB}=\vec{b}$, $\overrightarrow{AC}=\vec{c}$ and $\overrightarrow{AD}=\vec{d}$.

(a) $AB \perp CD \implies \vec{b}\,(\vec{d} - \vec{c}) = 0 \implies \vec{b}\vec{d}=\vec{b}\vec{c}= k$
$AD \perp BC \implies \vec{d}\,(\vec{c} - \vec{b}) = 0 \implies \vec{c}\vec{d}=\vec{b}\vec{d}= k,$
deci
$\vec{c}\vec{b}=\vec{c}\vec{d} \implies \vec{c}\,(\vec{b} - \vec{d}) = 0 \implies AC \perp BD.$

(b) $AB^2 + CD^2 = \vec{b}^2 + (\vec{d} - \vec{c})^2 = \vec{b}^2 + \vec{d}^2 + \vec{c}^2 - 2k;$
$AC^2 + BD^2 = \vec{c}^2 + (\vec{d} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k;$
$BC^2 + AD^2 = \vec{d}^2 + (\vec{c} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k.$

(c) We shall show that there exists a point H such that $AH \perp (DBC)$, $BH \perp (ACD)$, $CH \perp (ABD)$, $DH \perp (ABC)$. Let $\overrightarrow{h} = \overrightarrow{AH} = m\ \overrightarrow{a} + n\ \vec{b} + p\ \vec{c}$. Writing the conditions $\overrightarrow{AH} \perp \overrightarrow{BC}, \overrightarrow{CD}$; $\overrightarrow{BH} \perp \overrightarrow{AC}, \overrightarrow{AD}$; $\overrightarrow{CH} \perp \overrightarrow{AB}, \overrightarrow{AD}$; $\overrightarrow{DH} \perp \overrightarrow{AB}, \overrightarrow{AC}$ we obtain a consistent system with one single solution:

$$\begin{cases} b^2 m + kn + kp = k \\ km + c^2 n + kp = k \\ km + kn + d^2 p = k. \end{cases} \tag{5.18}$$

Indeed the the matrix of the system is

$$A = \begin{pmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{pmatrix}$$

and for its determinant we have successively

$$\begin{aligned} \det(A) &= \begin{vmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{vmatrix} = \begin{vmatrix} b \cdot b & b \cdot c & b \cdot c \\ c \cdot b & c \cdot c & c \cdot d \\ d \cdot b & d \cdot c & d \cdot d \end{vmatrix} \\ &= \begin{vmatrix} b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 \\ c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 & c_1d_1 + c_2d_2 + c_3d_3 \\ d_1b_1 + d_2b_2 + d_3b_3 & d_1c_1 + d_2c_2 + d_3c_3 & d_1^2 + d_2^2 + d_3^2 \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_1 & c_2 & d_2 \\ b_1 & c_3 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d}) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d})^2.$$

The linear independence of the vectors $\vec{b}, \vec{c}, \vec{d}$ ensure that $(\vec{b}, \vec{c}, \vec{d}) \neq 0$ and shows that the linear system (5.18) is consistent and has one single solution.

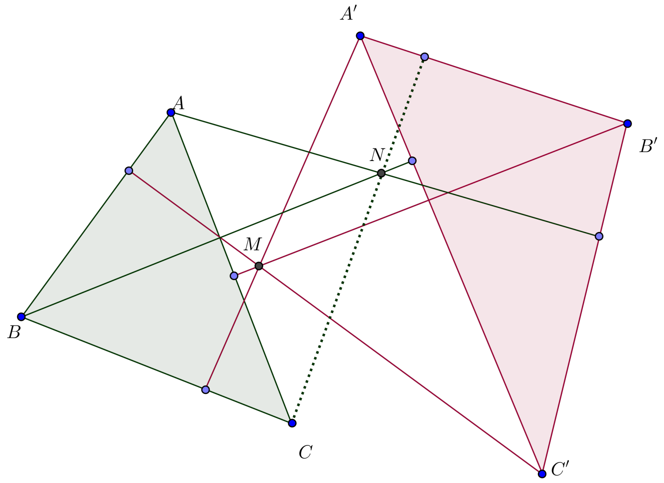
10. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show

that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

Solution Due to the given hypothesis, we have

$$\overrightarrow{MA'} \cdot \overrightarrow{BC} = \overrightarrow{MB'} \cdot \overrightarrow{CA} = \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \tag{5.19}$$

We now consider the perpendicular lines from the vertices A and B on the edges $B'C'$ and $C'A'$ and denote by N their intersection point.



Thus

$$\overrightarrow{NA} \cdot \overrightarrow{B'C'} = \overrightarrow{NB} \cdot \overrightarrow{C'A'} = 0.$$

By using the relations (5.19) we obtain

$$\begin{aligned}
& \overrightarrow{MA'} \cdot \overrightarrow{BC} + \overrightarrow{MB'} \cdot \overrightarrow{CA} + \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \\
\Leftrightarrow\ & \overrightarrow{MA'} \cdot (\overrightarrow{NC} - \overrightarrow{NB}) + \overrightarrow{MB'} \cdot (\overrightarrow{NA} - \overrightarrow{NC}) + \overrightarrow{MC'} \cdot (\overrightarrow{NB} - \overrightarrow{NA}) = 0 \\
\Leftrightarrow\ & (\overrightarrow{MB'} - \overrightarrow{MC'}) \cdot \overrightarrow{NA} + (\overrightarrow{MC'} - \overrightarrow{MA'}) \cdot \overrightarrow{NB} + (\overrightarrow{MA'} - \overrightarrow{MB'}) \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{C'B'} \cdot \overrightarrow{NA} + \overrightarrow{A'C'} \cdot \overrightarrow{NB} + \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \Leftrightarrow NC \perp A'B'.
\end{aligned}$$

11. (**2p**) Find the orthogonal projection
    (a) of the point $A(1,2,1)$ on the plane $\pi: \ x + y + 3z + 5 = 0$.
    (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3} = \dfrac{y-1}{2} = \dfrac{z-3}{4}$.

A few questions in the two dimensional setting

12. (**3p**) Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$ for which the sum $AP + PB$ is minimum, when $A(-7, 1)$ and $B(-5, 5)$.

13. (**2p**) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x - y + 2 = 0$, $x - 4y - 8 = 0$ and $x + 4y - 8 = 0$.

14. (**3p**) Given the bundle of lines of equations $(1 - t)x + (2 - t)y + t - 3 = 0$, $t \in \mathbb{R}$ and $x + y - 1 = 0$, find:

    (a) the coordinates of the vertex of the bundle;

(b) the equation of the line in the bundle which cuts Ox and Oy in M respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$.

15. **(2p)** Let $\mathcal{B}$ be the bundle of lines of vertex $M_0(5,0)$. An arbitrary line from $\mathcal{B}$ intersects the lines $d_1 : y - 2 = 0$ and $d_2 : y - 3 = 0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

16. **(3p)** The vertices of the quadrilateral $ABCD$ are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$ and $D((-3,-1)$.

    (a) Find the coordinates of the intersection points $\{E\} = AB \cap CD$ and $\{F\} = BC \cap AD$;

    (b) Prove that the midpoints of the segments $[AC]$, $[BD]$ and $[EF]$ are collinear.

17. (**3p**) Let M be a point whose coordinates satisfy

$$\frac{4x+2y+8}{3x-y+1}=\frac{5}{2}.$$

    (a) Prove that M belongs to a fixed line (d);
    (b) Find the minimum of x^2+y^2, when $M \in d \setminus \{M_0(-1,-2)\}$.

18. (**3p**) Find the locus of the points whose distances to two orthogonal lines have a constant ratio.

6 Week 6:

6.1 The vector product

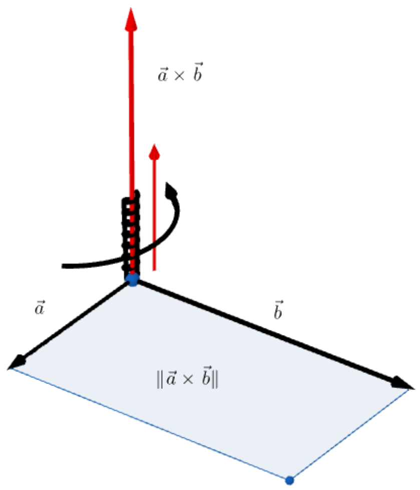
Definition 6.1. The *vector product* or the *cross product* of the vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ is a vector, denoted by $\overrightarrow{a} \times \overrightarrow{b}$, which is defined to be zero if $\overrightarrow{a}, \overrightarrow{b}$ are linearly dependent (collinear), and if $\overrightarrow{a}, \overrightarrow{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\overrightarrow{a} \times \overrightarrow{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ of $\mathcal{V}$;

2. if $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$, then the sense of $\overrightarrow{a} \times \overrightarrow{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, advances when it is being rotated simultaneously with the vector $\overrightarrow{a}$ from $\overrightarrow{a}$ towards $\overrightarrow{b}$ within the vector subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ and the support half line of $\overrightarrow{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\overrightarrow{a} \times \overrightarrow{b}$ is defined by

$$|| \overrightarrow{a} \times \overrightarrow{b} || = || \overrightarrow{a} || \cdot || \overrightarrow{b} || \sin(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$



Remark 6.1. 1. *The* norm *(*magnitude *or* length*) of the vector $\overrightarrow{a} \times \overrightarrow{b}$ is actually the area of the parallelogram constructed on the vectors $\overrightarrow{a}, \overrightarrow{b}$.*

2. *The vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.*

Proposition 6.1. *The vector product has the following properties:*

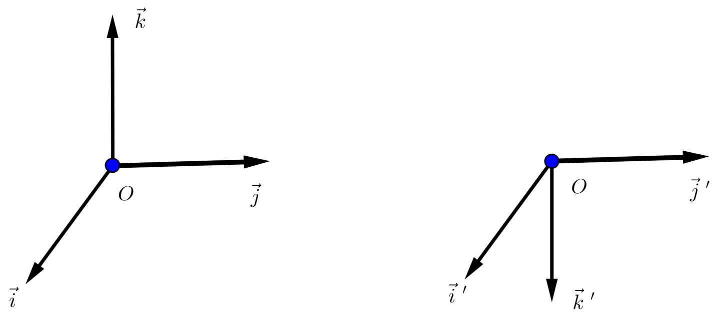
1. $\overrightarrow{a} \times \overrightarrow{b} = - \overrightarrow{b} \times \overrightarrow{a}, \forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$;

2. $(\lambda \overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (\lambda \overrightarrow{b}) = \lambda(\overrightarrow{a} \times \overrightarrow{b}), \forall \lambda \in \mathbb{R}, \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$;

3. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}, \forall \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}$.

6.2 The vector product in terms of coordinates

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*.



direct orthonormal basis   inverse orthonormal basis

Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 6.2. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{6.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{6.2}$$

Proof. Indeed,

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\
&= a_1b_1 \vec{i} \times \vec{i} + a_1b_2 \vec{i} \times \vec{j} + a_1b_3 \vec{i} \times \vec{k} \\
&+ a_2b_1 \vec{j} \times \vec{i} + a_2b_2 \vec{j} \times \vec{j} + a_2b_3 \vec{i} \times \vec{k} \\
&+ a_3b_1 \vec{k} \times \vec{i} + a_3b_2 \vec{k} \times \vec{j} + a_3b_3 \vec{k} \times \vec{k} \\
&= a_1b_2 \vec{k} - a_1b_3 \vec{j} - a_2b_1 \vec{k} + a_2b_3 \vec{i} + a_3b_1 \vec{j} - a_3b_2 \vec{i} \\
&= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}
\end{aligned}$$

□

One can rewrite formula (6.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{6.3}$$

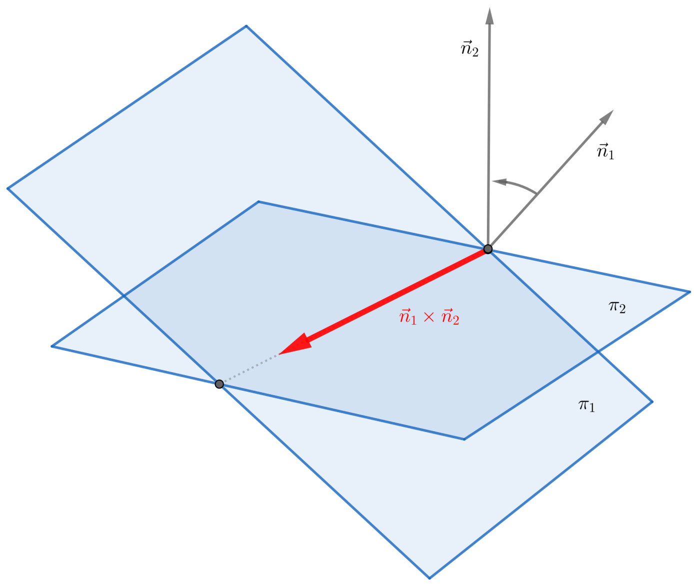
the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

Remark 6.2. If $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is the direct Cartesian orthonormal reference system behind the equations of the line

$$(\Delta)\ \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0, \end{cases}$$

then we can recover the director parameters (4.10) of Δ, in this particular case of orthonormal Cartesian reference systems, by observing that $\overrightarrow{n}_1 \times \overrightarrow{n}_2$ is a director vector of Δ, where

$$\overrightarrow{n}_1 = A_1 \overrightarrow{i} + B_1 \overrightarrow{j} + C_1 \overrightarrow{k}$$
$$\overrightarrow{n}_2 = A_2 \overrightarrow{i} + B_2 \overrightarrow{j} + C_2 \overrightarrow{k}.$$



Recall that

$$\overrightarrow{n}_1 \times \overrightarrow{n}_2 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \overrightarrow{k}.$$

Note however that the director parameters were obtained before for arbitrary Cartesian reference systems (See (4.10)).

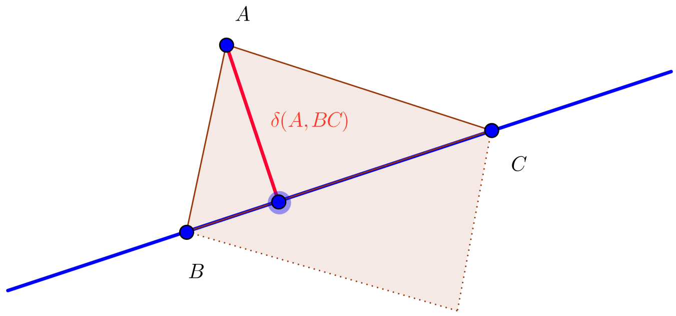
6.3 Applications of the vector product

- **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} || \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||$. On the other hand

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix},$$

as the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively. Thus,

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$



- **The distance from one point to a straight line.**

  (a) *The distance $\delta(A,BC)$ from the point $A(x_A,y_A,z_A)$ to the straight line BC*, where $B(x_B,y_B,z_B)$ and $C(x_C,y_C,z_C)$. Since

  $$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A,BC)}{2}$$

  it follos that

  $$\delta^2(A,BC) = \frac{4S^2_{ABC}}{\| \overrightarrow{BC} \|^2}.$$

  Thus, we obtain

  $$\delta^2(A,BC) = \frac{\begin{vmatrix} y_B-y_A & z_B-z_A \\ y_C-y_A & z_C-z_A \end{vmatrix}^2 + \begin{vmatrix} z_B-z_A & x_B-x_A \\ z_C-z_A & x_C-x_A \end{vmatrix}^2 + \begin{vmatrix} x_B-x_A & y_B-y_A \\ x_C-x_A & y_C-y_A \end{vmatrix}^2}{(x_C-x_B)^2+(y_C-y_B)^2+(z_C-z_B)^2}.$$

  (b) The distance from $\delta(A,d)$ from one point $A(A_A,y_A,z_A)$ to the straight line

  $$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}.$$

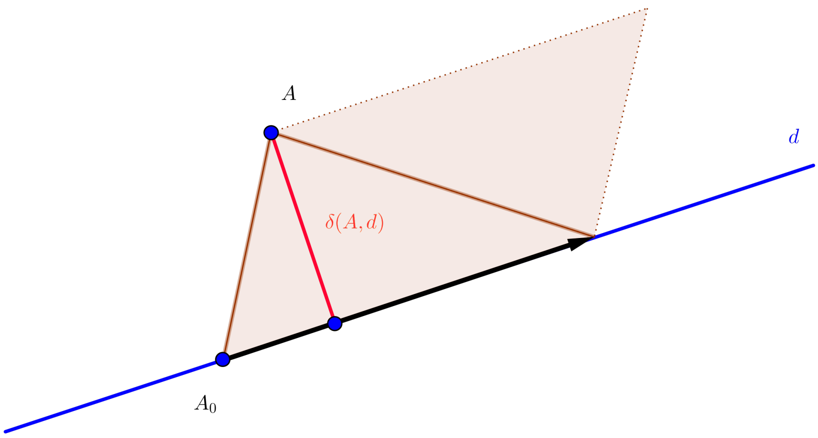
  $$\delta(A,d) = \frac{\| \vec{d} \times \overrightarrow{A_0A} \|}{\|\vec{d}\|}, \tag{6.4}$$

  where $A_0(x_0,y_0,z_0) \in \delta$.
  Since

  $$\begin{aligned} \vec{d} \times \overrightarrow{A_0A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A-y_0 & z_A-z_0 \end{vmatrix}\vec{i} + \begin{vmatrix} r & p \\ z_A-z_0 & x_A-x_0 \end{vmatrix}\vec{j} + \begin{vmatrix} p & q \\ x_A-x_0 & y_A-y_0 \end{vmatrix}\vec{k} \end{aligned}$$

  it follows that

  $$\delta(A,d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A-y_0 & z_A-z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A-z_0 & x_A-x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A-x_0 & y_A-y_0 \end{vmatrix}^2}}{\sqrt{p^2+q^2+r^2}}.$$



6.4 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$

Proposition 6.3.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}, \quad \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}. \tag{6.5}$$

Proof. (Sketch) If the vectors $\vec{b}$ and $\vec{c}$ are linearly dependent, then both sides are obviously zero. Otherwise one can choose an orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$, related to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, such that

$$\vec{b} = b_1\, \vec{i}, \vec{c} = c_1\, \vec{i} + c_2\, \vec{j}, \vec{a} = a_1\, \vec{i} + a_2\, \vec{j} + a_3\, \vec{k}\,.$$

For example one can choose $\vec{i}$ to be $\vec{b} \,/ \parallel \vec{b} \parallel$ and $\vec{j}$ a unit vector in the subspace $\langle \vec{b}, \vec{c} \rangle$ which is perpendicular on $\vec{b}$. Finally, one can choose $\vec{k} = \vec{i} \times \vec{j}$. By computing the two sides of the equality 6.5, in terms of coordinates and the vectors $\vec{i}, \vec{j}, \vec{k}$, one gets the same result. □

Corollary 6.4. *1.* $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{a} \end{vmatrix}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V};$

2. $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ *(Jacobi's identity).*

Proof. While the first identity follows immediately via 6.5, for the Jacobi's identity we get successively:

$$\begin{aligned} & \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} + (\vec{b} \cdot \vec{a})\, \vec{c} - (\vec{b} \cdot \vec{c})\, \vec{a} + (\vec{c} \cdot \vec{b})\, \vec{a} - (\vec{c} \cdot \vec{a})\, \vec{b} = \vec{0}\,. \end{aligned}$$

□

6.5 Problems

1. **(2p)** Show that $\parallel \vec{a} \times \vec{b} \parallel \leq \parallel \vec{a} \parallel \cdot \parallel \vec{b} \parallel, \forall\, \vec{a}, \vec{b}, \in \mathcal{V}$.

   Solution.

2. **(3p)** Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ be pairwise noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\overrightarrow{a}$, $\overrightarrow{CA}=\overrightarrow{b}$, $\overrightarrow{AB}=\overrightarrow{c}$ is

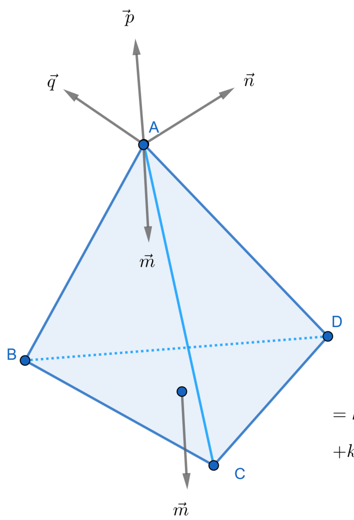
$$\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{c} \times \overrightarrow{a}.$$

From the equalities of the norms deduce the low of sines.

Solution.

3. **(3p)** Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

Solution.



The proportionality of $\vec{m}, \vec{n}, \vec{p}, \vec{q}$ with the areas of the corresponding faces of the tetrahedron show that

$\vec{m} = k\overrightarrow{BD} \times \overrightarrow{BC},\ \vec{n} = k\overrightarrow{AC} \times \overrightarrow{AD}$

$\vec{p} = k\overrightarrow{AD} \times \overrightarrow{AB},\ \vec{q} = k\overrightarrow{AB} \times \overrightarrow{AC}$

Thus, $\vec{m} + \vec{n} + \vec{p} + \vec{q}$

$= k\overrightarrow{BD} \times \overrightarrow{BC} + k\overrightarrow{AC} \times \overrightarrow{AD} +$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC}$

$= k(\overrightarrow{AD} - \overrightarrow{AB}) \times (\overrightarrow{AC} - \overrightarrow{AB}) + k\overrightarrow{AC} \times \overrightarrow{AD}$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} =$

$= k\overrightarrow{AD} \times \overrightarrow{AC} - k\overrightarrow{AD} \times \overrightarrow{AB} - k\overrightarrow{AB} \times \overrightarrow{AC} + k\overrightarrow{AB} \times \overrightarrow{AB} =$

$+ k\overrightarrow{AC} \times \overrightarrow{AD} + k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}.$

4. **(2p)** Find the distance from the point $P(1, 2, -1)$ to the straight line $(d)\ x = y = z$.

Solution.

5. **(3p)** Find the area of the triangle ABC and the lengths of its hights, where $A(-1,1,2)$, $B(2,-1,1)$ and $C(2,-3,-2)$.

6. **(3p)** Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

   Solution. A director vector of the common perpendicular d_{ij} is $\vec{d}_i \times \vec{d}_j$, where $\vec{d}_r$ stands for a director vector of d_r. Therefore we have successively:

$$\begin{aligned} d_{12} \perp d_{34} &\Leftrightarrow \vec{d}_1 \times \vec{d}_2 \perp \vec{d}_3 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_3 \times \vec{d}_4) = 0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_3 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_2 \cdot \vec{d}_3 & \vec{d}_2 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3). \end{aligned}$$

   Similalry

$$\begin{aligned} d_{13} \perp d_{24} &\Leftrightarrow \vec{d}_1 \times \vec{d}_3 \perp \vec{d}_2 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_3) \cdot (\vec{d}_2 \times \vec{d}_4) = 0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_3 \cdot \vec{d}_2 & \vec{d}_3 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_3 \cdot \vec{d}_2). \end{aligned}$$

Therefore we have

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3) = (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4),$$

which shows that

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) - (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = 0 \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_3 \\ \vec{d}_4 \cdot \vec{d}_2 & \vec{d}_4 \cdot \vec{d}_3 \end{vmatrix} = 0 \Leftrightarrow d_{14} \perp d_{23}.$$

7 Week 7: The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 7.1. *If* $[\vec{i}, \vec{j}, \vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned}\vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}\\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}\\ \vec{c} &= c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}\end{aligned}$$

then

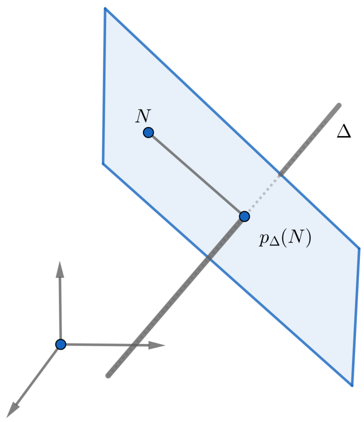
$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{7.1}$$

Proof. Indeed, we have successively:

$$\begin{aligned}(\vec{a}, \vec{b}, \vec{c}) &= (\vec{a} \times \vec{b}) \cdot \vec{c}\\ &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right) \cdot (c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k})\\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.\end{aligned}$$

□

Remark 7.1. Taking into account the formula (7.2) for the distance $\delta(N, \Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ as well as Proposition 6.3 we deduce that



$$\begin{aligned}\delta(N, \Delta) &= \| \overrightarrow{Np_\Delta(N)} \| \\ &= \| \overrightarrow{NO} + \overrightarrow{Op_\Delta(N)} \| = \left\| \overrightarrow{NA_0} - \frac{\vec{d}_\Delta \cdot \overrightarrow{NA_0}}{\| \vec{d}_\Delta \|^2} \vec{d}_\Delta \right\|\end{aligned} \tag{7.2}$$

$$= \frac{\| \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{d}_{\Delta}\right) \overrightarrow{NA_0} - \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}\right) \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|^2}$$
$$= \frac{\| \overrightarrow{d}_{\Delta} \times (\overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta}) \|}{\| \overrightarrow{d}_{\Delta} \|^2} = \frac{\| \overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|}.$$

Thus, we recovered the distance formula from one point to one straight line (see formula 6.4) by using different arguments.

Corollary 7.2. 1. *The free vectors* $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ *are linearly dependent (collinear) iff* $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = 0$

2. *The free vectors* $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ *are linearly independent (noncollinear) if and only if* $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) \neq 0$

3. *The free vectors* $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ *form a basis of the space* $\mathcal{V}$ *if and only if* $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) \neq 0$.

4. *The correspondence* $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ *is trilinear and skew-symmetric, i.e.*

$$\begin{array}{l} (\alpha \overrightarrow{a} + \alpha' \overrightarrow{a}', \overrightarrow{b}, \overrightarrow{c}) = \alpha(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) + \alpha'(\overrightarrow{a}', \overrightarrow{b}, \overrightarrow{c}) \\ (\overrightarrow{a}, \beta \overrightarrow{b} + \beta' \overrightarrow{b}', \overrightarrow{c}) = \beta(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) + \beta'(\overrightarrow{a}, \overrightarrow{b}', \overrightarrow{c}) \\ (\overrightarrow{a}, \overrightarrow{b}, \gamma \overrightarrow{c} + \gamma' \overrightarrow{c}') = \gamma(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) + \gamma'(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}') \end{array} \tag{7.3}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall\ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a}', \overrightarrow{b}', \overrightarrow{c}' \in \mathcal{V}$ *şi*

$$(\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3) = sgn(\sigma)(\overrightarrow{a}_{\sigma(1)}, \overrightarrow{a}_{\sigma(2)}, \overrightarrow{a}_{\sigma(3)}),\ \forall\ \overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3 \in \mathcal{V} \text{ şi } \forall\, \sigma \in S_3 \tag{7.4}$$

Remark 7.2. *One can rewrite the relations (7.4) as follows:*

$$\begin{aligned} (\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3) &= (\overrightarrow{a}_2, \overrightarrow{a}_3, \overrightarrow{a}_1) = (\overrightarrow{a}_3, \overrightarrow{a}_1, \overrightarrow{a}_2) \\ &= -(\overrightarrow{a}_2, \overrightarrow{a}_1, \overrightarrow{a}_3) = -(\overrightarrow{a}_1, \overrightarrow{a}_3, \overrightarrow{a}_2) = -(\overrightarrow{a}_3, \overrightarrow{a}_2, \overrightarrow{a}_1), \end{aligned}$$

$\forall\ \overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3 \in \mathcal{V}$

Corollary 7.3. 1. $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c} = \overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})\ \forall\ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}$.

2. *For every* $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d} \in \mathcal{V}$ *the* Laplace formula holds:

$$(\overrightarrow{a} \times \overrightarrow{b}) \cdot (\overrightarrow{c} \times \overrightarrow{d}) = \begin{vmatrix} \overrightarrow{a} \cdot \overrightarrow{c} & \overrightarrow{a} \cdot \overrightarrow{d} \\ \overrightarrow{b} \cdot \overrightarrow{c} & \overrightarrow{b} \cdot \overrightarrow{d} \end{vmatrix}.$$

Proof. While the first identity is obvious, for the Laplace formula we have successively:

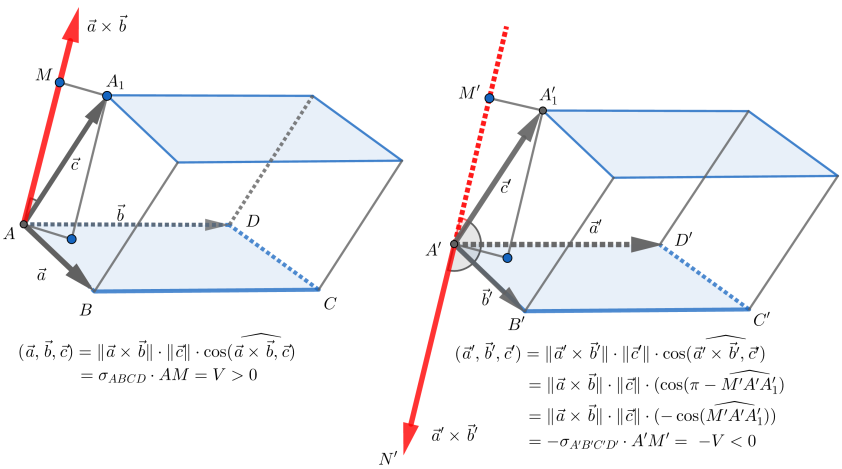
$$\begin{aligned} (\overrightarrow{a} \times \overrightarrow{b}) \cdot (\overrightarrow{c} \times \overrightarrow{d}) &= \left(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \times \overrightarrow{d}\right) = \left(\overrightarrow{c} \times \overrightarrow{d}, \overrightarrow{a}, \overrightarrow{b}\right) \\ &= \left[(\overrightarrow{c} \times \overrightarrow{d}) \times \overrightarrow{a}\right] \cdot \overrightarrow{b} = -\left[(\overrightarrow{a} \cdot \overrightarrow{d})\, \overrightarrow{c} - (\overrightarrow{a} \cdot \overrightarrow{c})\, \overrightarrow{d}\right] \cdot \overrightarrow{b} \\ &= -(\overrightarrow{a} \cdot \overrightarrow{d})(\overrightarrow{c} \cdot \overrightarrow{b}) + (\overrightarrow{a} \cdot \overrightarrow{c})(\overrightarrow{d} \cdot \overrightarrow{b}) = \begin{vmatrix} \overrightarrow{a} \cdot \overrightarrow{c} & \overrightarrow{a} \cdot \overrightarrow{d} \\ \overrightarrow{b} \cdot \overrightarrow{c} & \overrightarrow{b} \cdot \overrightarrow{d} \end{vmatrix}. \end{aligned}$$

□

Definition 7.1. The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 7.2. The *oriented* volume of the parallelepiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Proposition 7.4. *The triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ *of the noncoplanar vectors* $\vec{a}, \vec{b}, \vec{c}$ *is equal with the oriented volume of the parallelepiped constructed on these vectors.*



7.1 Applications of the triple scalar product

7.1.1 The distance between two straight lines

If $d_1,\ d_2$ are two straight lines, then the distance between them, denoted by $\delta(d_1, d_2)$, is being defined as

$$\min\{||\ \overrightarrow{M_1M_2}\ ||\ |\ M_1 \in d_1,\ M_2 \in d_2\}.$$

1. If $d_1 \cap d_2 \neq \emptyset$, then $\delta(d_1, d_2) = 0$.

2. If $d_1 || d_2$, then $\delta(d_1, d_2) = ||\ \overrightarrow{MN}\ ||$ where $\{M\} = d \cap d_1,\ \ \{N\} = d \cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $||\ \overrightarrow{MN}\ ||$ is independent on the choice of the line d.

3. We now assume that the straight lines d_1, d_2 are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d \perp d_1, d_2$ and $d \cap d_1 = \{M_1\},\ d \cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines d_1, d_2 and obviously $\delta(d_1, d_2) = ||\ \overrightarrow{M_1M_2}\ ||$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and their vectors şi au vectorii directori $\vec{d}_1\ (p_1, q_1, r_1)\ \vec{d}_2\ (p_2, q_2, r_2)$, that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line d_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \vec{k}$$

it follows that the equations of the common perpendicular are

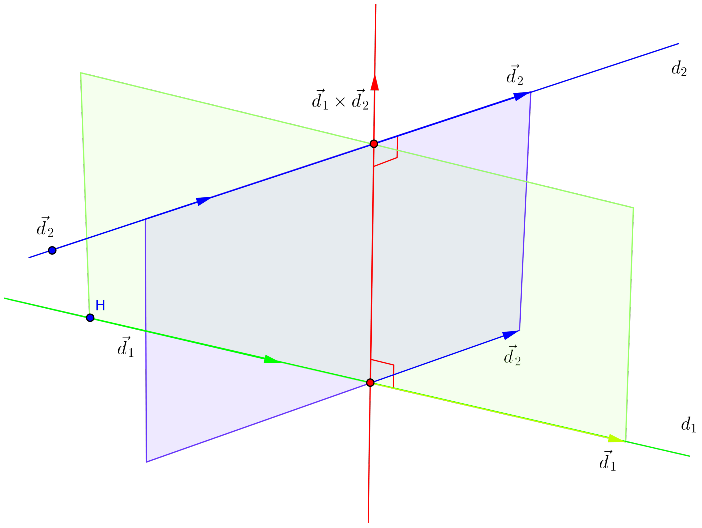


Figure 4: Perpendiculara comună a dreptelor d_1 şi d_2

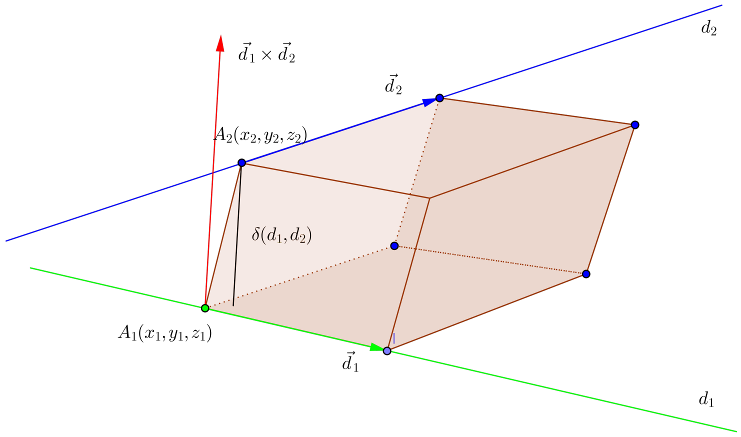
$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{7.5}$$

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2)|}{|| \vec{d}_1 \times \vec{d}_2 ||}. \tag{7.6}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{7.7}$$



7.1.2 The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{7.8}$$

7.2 Problems

1. **(2p)** Show that

   (a) $|(\vec{a}, \vec{b}, \vec{c})| \leq \| \vec{a} \| \cdot \| \vec{b} \| \cdot \| \vec{c} \|$;
   Solution.

   (b) **(2p)** $(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}) = 2(\vec{a}, \vec{b}, \vec{c})$.
   Solution.

2. **(3p)** Prove the following identity:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d})\, \vec{b} - (\vec{b}, \vec{c}, \vec{d})\, \vec{a} = (\vec{a}, \vec{b}, \vec{d})\, \vec{c} - (\vec{a}, \vec{b}, \vec{c})\, \vec{d}\ .$$

Solution. By using the identity $\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{u}\cdot\vec{v})\,\vec{v}$ for $\vec{u}=\vec{a}\times\vec{b}$, $\vec{v}=\vec{c}$ and $\vec{w}=\vec{d}$ we obtain

$$\begin{aligned}(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) &= \vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{u}\cdot\vec{v})\,\vec{v}\\ &=\left[(\vec{a}\times\vec{b})\cdot\vec{d}\right]\vec{c}-\left[(\vec{a}\times\vec{b})\cdot\vec{c}\right]\vec{d}\\ &=(\vec{a},\vec{b},\vec{d})\,\vec{c}-(\vec{a},\vec{b},\vec{c})\,\vec{d}\,.\end{aligned}$$

By using the identity $(\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{v}\cdot\vec{w})\,\vec{u}$ for $\vec{u}=\vec{a}$, $\vec{v}=\vec{b}$ and $\vec{w}=\vec{c}\times\vec{d}$ we obtain

$$\begin{aligned}(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) &= (\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{v}\cdot\vec{w})\,\vec{u}\\ &=\left[\vec{a}\cdot(\vec{c}\times\vec{d})\right]\vec{b}-\left[\vec{b}\cdot(\vec{c}\times\vec{d})\right]\vec{a}\\ &=(\vec{a},\vec{c},\vec{d})\,\vec{b}-(\vec{b},\vec{c},\vec{d})\,\vec{a}\,.\end{aligned}$$

3. **(3p)** Prove the following identity: $(\vec{u}\times\vec{v},\vec{v}\times\vec{w},\vec{w}\times\vec{u})=(\vec{u},\vec{v},\vec{w})^2$.

   Solution. We have successively:

$$\begin{aligned}(\vec{u}\times\vec{v},\vec{v}\times\vec{w},\vec{w}\times\vec{u}) &= \left[(\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w})\right]\cdot(\vec{w}\times\vec{u})\\ &=\left[(\vec{u},\vec{v},\vec{w})\,\vec{v}-(\vec{u},\vec{v},\vec{v})\,\vec{w}\right]\cdot(\vec{w}\times\vec{u})\\ &=(\vec{u},\vec{v},\vec{w})\left[\vec{v}\cdot(\vec{w}\times\vec{u})\right]=(\vec{u},\vec{v},\vec{w})(\vec{v},\vec{w},\vec{u})=(\vec{u},\vec{v},\vec{w})^2.\end{aligned}$$

4. **(3p)** The *reciprocal vectors* of the noncoplanar vectors $\vec{u},\vec{v},\vec{w}$ are defined by

$$\vec{u}'=\frac{\vec{v}\times\vec{w}}{(\vec{u},\vec{v},\vec{w})},\ \vec{v}'=\frac{\vec{w}\times\vec{u}}{(\vec{u},\vec{v},\vec{w})},\ \vec{w}'=\frac{\vec{u}\times\vec{v}}{(\vec{u},\vec{v},\vec{w})}.$$

   Show that:

   (a)

$$\begin{aligned}\vec{a} &= (\vec{a}\cdot\vec{u}')\,\vec{u}+(\vec{a}\cdot\vec{v}')\,\vec{v}+(\vec{a}\cdot\vec{w}')\,\vec{w}\\ &=\frac{(\vec{a},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{u}+\frac{(\vec{u},\vec{a},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{v}+\frac{(\vec{u},\vec{v},\vec{a})}{(\vec{u},\vec{v},\vec{w})}\,\vec{w}\,.\end{aligned}$$

   (b) the reciprocal vectors of $\vec{u}',\vec{v}',\vec{w}'$ are the vectors $\vec{u},\vec{v},\vec{w}$.

   Solution. (4a) Obviously $\vec{a}=\alpha\,\vec{u}+\beta\,\vec{v}+\gamma\,\vec{c}$, as $\vec{u},\vec{v},\vec{w}$ are three linearly independent vectors of the three dimensional vector space $\mathcal{V}$, i.e. $\vec{u},\vec{v},\vec{w}$ form a basis of $\mathcal{V}$. Moreover we have

$$\begin{aligned}\vec{a}\cdot\vec{u}' &= \frac{\vec{a}\cdot(\vec{v}\times\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\frac{(\vec{a},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\frac{(\alpha\,\vec{u}+\beta\,\vec{v}+\gamma\,\vec{c},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\\ &=\frac{\alpha(\vec{u},\vec{v},\vec{w})+\beta(\vec{v},\vec{v},\vec{w})+\gamma(\vec{w},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\alpha.\end{aligned}$$

   One can similalry show that

$$\vec{a}\cdot\vec{v}'=\frac{(\vec{u},\vec{a},\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\beta \text{ and } \vec{a}\cdot\vec{w}'=\frac{(\vec{u},\vec{v},\vec{a})}{(\vec{u},\vec{v},\vec{w})}=\gamma.$$

(4b) Let us first observe that

$$(\vec{u}', \vec{v}', \vec{w}') = (\vec{w}', \vec{u}', \vec{v}') = \frac{(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u})}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{(\vec{u}, \vec{v}, \vec{w})^2}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{1}{(\vec{u}, \vec{v}, \vec{w})}.$$

On the other hand we have:

$$\frac{\vec{v}' \times \vec{w}'}{(\vec{u}', \vec{v}', \vec{w}')} = (\vec{u}, \vec{v}, \vec{w})(\vec{v}' \times \vec{w}') = (\vec{u}, \vec{v}, \vec{w})\frac{(\vec{w} \times \vec{u}) \times (\vec{u} \times \vec{v})}{(\vec{u}, \vec{v}, \vec{w})^2} = \frac{(\vec{w}, \vec{u}, \vec{v})\, \vec{u} - (\vec{w}, \vec{u}, \vec{u})\, \vec{v}}{(\vec{u}, \vec{v}, \vec{w})} = \vec{u}\,.$$

One can similarly show that

$$\frac{\vec{w}' \times \vec{u}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{v} \ \text{ and } \ \frac{\vec{u}' \times \vec{v}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{w}\,.$$

5. **(2p)** Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

   Solution.

6. **(2p)** Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

   are coplanar. Find the coordinates of their intersection point in that case.

   Solution.

7. **(2p)** Find the distance between the straight lines

$$(d_1)\ \frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{1},\ (d_2)\ \frac{x+1}{3}=\frac{y}{4}=\frac{z-1}{3}$$

as well as the equations of the common perpendicular.

Solution.

8. **(2p)** Find the distance between the straight lines M_1M_2 and d, where $M_1(-1,0,1)$, $M_2(-2,1,0)$ and

$$(d)\ \begin{cases} x + y + z = 1 \\ 2x - y - 5z = 0. \end{cases}$$

as well as the equations of the common perpendicular.

Solution.

8 Week 8: Curves and surfaces

8.1 Regular curves

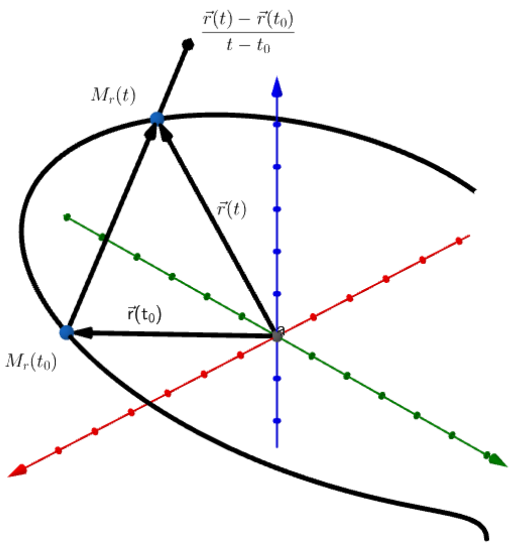
Definition 8.1. A subset C of $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be a *regular curve* if for every $p \in C$ there exists a neighbourhood V of p in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively and a *parametrized differentiable curve* $r : I \to U \cap C$, where $I \subseteq \mathbb{R}$ is an open set, such that

1. r is smooth;
2. $r : I \to U \cap C$ is a homeomorphism;
3. r is regular, i.e. $\overrightarrow{r}\,'(t) \neq \overrightarrow{0}$, $\forall\, t \in I$.

The parametrized differentiable curve $r : I \to V \cap C$ is called *local parametrization* or *local system of coordinates* at p and $V \cap C$ is called *coordinate neighbourhood* at p. Recall that the the tangent line of the local parametrization $r : I \to U \cap C$ at $r(t_0)$, for some $t_0 \in I$, is defined as the limit position of the line $M_r(t_0)M_r(t)$ as $t \longrightarrow t_0$. This tangent line is denoted by $(Tr)(t_0)$. A director vector of the line $M_r(t_0)M_r(t)$ is obviously

$$\frac{\overrightarrow{r}(t) - \overrightarrow{r}(t_0)}{t - t_0},$$

which shows that $\overrightarrow{r}\,'(t_0)$ is a director vector of $(Tr)(t_0)$ and the direction of $(Tr)(t_0)$ is therefore $(d\overrightarrow{r})_{t_0}(\mathbb{R})$.



If $r_1 : I_1 \to U_1 \cap C$ and $r_2 : I_2 \to U_2 \cap C$ are two local parametrizations of of C at $p \in C$, then $r_1(t_1) = r_2(t_2) = p$ for some $t_1 \in I_1$ and $t_2 \in I_2$ and one can easily show that $(d\overrightarrow{r}_1)_{t_1}(\mathbb{R}) = (d\overrightarrow{r}_2)_{t_2}(\mathbb{R})$. This shows that r_1 and r_2 have the same tangent line at $r_1(t_1) = r_2(t_2) = p$.

Proposition 8.1. *The equation of the paremetrized differentiable curve $r : I \longrightarrow \mathbb{R}^2$, $r(t) = (x(t), y(t))$ at $r(t_0)$, for some* regular *point $t_0 \in I$, i.e. $\overrightarrow{r}\,'(t_0) \neq \overrightarrow{0}$ is*

$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)}. \tag{8.1}$$

The equation of the normal line to r at $r(t_0)$, i.e. the line through $M_r(t_0)$ which is perpendicular to $(Tr)(t_0)$ is

$$(Nr)(t_0)\ x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0. \tag{8.2}$$

Proposition 8.2. *The equation of the paremetrized differentiable curve $r : I \longrightarrow \mathbb{R}^3$, $r(t) = (x(t), y(t), z(t))$ at $r(t_0)$, for some* regular *point $t_0 \in I$, i.e. $\overrightarrow{r}'(t_0) \neq \overrightarrow{0}$ is*

$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}. \tag{8.3}$$

The equation of the normal plane to r at $r(t_0)$, i.e. the plane through $M_r(t_0)$ which is perpendicular to $(Tr)(t_0)$ is

$$(Nr)(t_0)\ x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0. \tag{8.4}$$

Remark 8.1.

1. The requirement (3) of definition (8.1), is equivalent with $(dr)_t \neq 0,\ \forall\, t \in \mathbb{R}$;
2. $V \cap C$ is the image of a regular one-to-one parametrized differentiable curve. On the other hand, there are regular one-to-one parametrized differentiable curves whose images are not parts of regular curves;
3. The role of requirement (2) in definition (8.1) is to prevent the self-intersections of the regular curves, which is not the case with the images of regular parametrized differentiable curves.
4. The requirement (3) combined with (2) ensure the existence of a unique tangent line at every point of a regular curve. The tangent line $T_p(C)$ of C at $p \in C$ is defined as the tangent line at p of a local parametrization $r : I \to U \cap C$ of C at p. The tangent line $T_p(C)$ is well-defined as the tangent at p of a local parametrization $r : I \to U \cap C$ at p is independent of r.

Definition 8.2. If $U \subseteq \mathbb{R}^2$ is an open set, $f : U \longrightarrow \mathbb{R}$ is a C^1-smooth function, then the value $a \in \mathrm{Im}(f)$ of f is said to be *regular* if $(\nabla f)(x, y) \neq 0,\ \forall (x, y) \in f^{-1}(a)$, i.e. $(df)_{(x,y)} \neq 0,\ \forall (x, y) \in f^{-1}(a)$.

Theorem 8.3. *(The preimage theorem) If $U \subseteq \mathbb{R}^2$ is an open set, $f : U \longrightarrow \mathbb{R}$ is a C^1-smooth function and $a \in Imf$ is a regular value of f, then the inverse image of a through f,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

is a planar regular curve called the regular curve of implicit cartezian equation $f(x, y) = a$.

Definition 8.3. Let $U \subset \mathbb{R}^2$ be an open set such that $tx \in U$ for every $t \in \mathbb{R}^*_+$ and avery $x \in U$. The function $f : U \to \mathbb{R}$ is said to be *homogeneouos of order* $p \in \mathbb{R}$ whenever $f(tx) = t^p f(x),\ \forall t \in \mathbb{R}^*_+, x \in U$.

For example a homogeneous polynomial function of degree $n \in \mathbb{N}$ is a a homogeneous function of order p.

Example 8.1. If $f : U \to \mathbb{R}$ is a C^1-smooth homoheneous function of order $p \in \mathbb{R}^*$ and $c \in Im\, f \backslash \{0\}$, then $f^{-1}(c)$ is a regular curve.

Indeed, it is enough to show that c is a regular value of f. By differentiating the relation $f(tx) = t^p f(x)$ with respect to t we obtain:

$$(df)_{tx}(x) = pt^{p-1} f(x),\ \forall t \in \mathbb{R}^*_+, x \in U,$$

and the Euler's relation

$$(df)_x(x) = pf(x),\ \forall x \in U. \tag{8.5}$$

follow for $t = 1$. But for $x \in C(f)$ we have $(df)_x = 0$ and thus $(df)_x(x) = 0$, namely $f(x) = 0$. We therefore showed that $B(f) = f(C(f)) \subset \{0\}$, or, equivalently, $\mathbb{R}^* \subset \mathbb{R} \backslash B(f)$, where $C(f) \subseteq U$ stands for the closed set of critical points of f, i.e. $C(f) := \{(x, y) \in U | (df)_{(x,y)} = 0\}$. But since $c \in Im\, f \backslash \{0\}$ we deduce that c is a regular value of f and $f^{-1}(c)$ is a regular curve therefore.

Proposition 8.4. *The equation of the tangent line $T_{(x_0,y_0)}(C)$ of the planar regular curve C of implicit cartezian equation $f(x,y) = a$ at the point $p = (x_0, y_0) \in C$, is*

$$T_{(x_0,y_0)}(C) : f'_x(p)(x - x_0) + f'_y(p)(y - y_0) = 0,$$

and the equation of the normal line $N_{(x_0,y_0)}(C)$ of C at p is

$$N_{(x_0,y_0)}(C) : \frac{x - x_0}{f'_x(p)} = \frac{y - y_0}{f'_y(p)}.$$

Example 8.2. The tangent line of the general conic

$$C : a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0$$

at some of its regular point $(x_0, y_0) \in C$ is

$$a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0 \tag{8.6}$$

and can be obtained by polarizing the conic's equation, i.e. by replacing:

1. x^2 with x_0x
2. y^2 with y_0y
3. $2x$ with $x + x_0$
4. $2y$ with $y + y_0$
5. $2xy$ with $x_0y + xy_0$.

Indeed, $C = f^{-1}(0)$, where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a second degree polynomial function given by $f(x,y) = a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2$. Since

$$f_x = 2a_{10} + 2_{11}x + 2a_{12}y \text{ and } f_y = 2a_{20} + 2a_{12}x + 2a_{22}y,$$

it follws that

$$\begin{aligned}
T_{(x_0,y_0)}(C) \quad &: \quad (2a_{10} + 2_{11}x + 2a_{12}y)(x - x_0) + (2a_{20} + 2a_{12}x + 2a_{22}y)(y - y_0) = 0 \\
&\iff a_{10}x + a_{11}x_0x + a_{12}y_0x + a_{20}y + a_{12}x_0y + a_22y_0y = a_{10}x_0 + a_{11}x_0^2 + a_{12}y_0x_0 + a_{20}y_0 + a_{12}x_0y_0 + a_{22}y_0^2 \\
&\iff a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 2a_{10}x + 0 + 2a_{20}y_0 + a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 \\
&\iff a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0.
\end{aligned}$$

8.2 Parametrized differentiable surfaces

Definition 8.4. Let $U \subseteq \mathbb{R}^2$ be an open set. A smooth map $r : U \to \mathbb{R}^3$ is said to be a *parametrized differentiable surface*. The set $r(U)$ is called the *trace*, the *support*, or *the image* of r. If the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for $q \in U$, then the parametrized differentiable surface r is said to be *regular* at q. If the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $q \in U$, then the parametrized differentiable surface r is said to be *regular*.

Remark 8.2. Let $U \subseteq \mathbb{R}^2$ be an open set and $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ be a parametrized differentiable surface. Then r is regular at $q \in U$ if and only if

$$\overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q) \neq \overrightarrow{0}.$$

Indeed,

$$\begin{aligned}
r \text{ is regular at } q \in U \quad &\iff (dr)_q : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is one-to-one} \\
&\iff (dr)_q(e_1),\ (dr)_q(e_2) \text{ are linearly independent } (e_1 = (1,0),\ e_2 = (0,1)) \\
&\iff \overrightarrow{r_u}(q) = (d\overrightarrow{r})_q(e_1),\ \overrightarrow{r_v}(q) = (d\overrightarrow{r})_q(e_2) \text{ are linearly independent} \\
&\iff \overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q) \neq \overrightarrow{0},
\end{aligned}$$

where $\overrightarrow{r} : U \longrightarrow \mathcal{V}$, $\overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$.

The image of a parametrized differentiable surface might have self-intersections.

8.2.1 The tangent plane and the normal line to a parametrized surface

Definition 8.5. Let $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ be a regular parametrized differentiable surface and $q = (u_0, v_0) \in U$. The plane $(Tr)(q)$ through $M_r(u_0, v_0)$, whose direction is $(d\overrightarrow{r})_q(\mathbb{R}^2)$, is called the *tangent plane* to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters. The perpendicular line $(Nr)(q)$ on $(Tr)(q)$ at $M_r(q)$ is called *the normal line* to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters.

Remark 8.3. If $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ is a regular parametrized differentiable surface and $q = (u_0, v_0) \in U$, then the vectors $\overrightarrow{r_u}(q) = (d\overrightarrow{r})_q(1,0)$, $\overrightarrow{r_v}(q) = (d\overrightarrow{r})_q(0,1)$ form a basis of the two dimensional vector subspace $(d\overrightarrow{r})(\mathbb{R}^2)$ of $\mathcal{V}$ şi and $\overrightarrow{\nu}(q) = \overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q)$ is therefore a director vector of the normal line to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters.

$$\begin{aligned}
\overrightarrow{\nu}(q) &= \overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} \\
&= \frac{\partial(y,z)}{\partial(u,v)}(q)\,\overrightarrow{i} + \frac{\partial(z,x)}{\partial(u,v)}(q)\,\overrightarrow{j} + \frac{\partial(x,y)}{\partial(u,v)}(q)\,\overrightarrow{k},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)}(q) &= \begin{vmatrix} x_u(q) & y_u(q) \\ x_v(q) & y_v(q) \end{vmatrix}, \\
\frac{\partial(z,x)}{\partial(u,v)}(q) &= \begin{vmatrix} z_u(q) & x_u(q) \\ z_v(q) & x_v(q) \end{vmatrix}, \\
\frac{\partial(y,z)}{\partial(u,v)}(q) &= \begin{vmatrix} y_u(q) & z_u(q) \\ y_v(q) & z_v(q) \end{vmatrix}.
\end{aligned}$$

Proposition 8.5. *If $r : U \to \mathbb{R}^3$ $r(u,v) = (x(u,v), y(u,v), z(u,v))$ regular parametrized differentiable surface and $q = (u_0, v_0) \in U$, then the equation of the tangent plane to r at $M_r(q)$, corresponding to the pair (u_0, v_0) of the parameters, is*

$$\begin{vmatrix} x - x(q) & y - y(q) & z - z(q) \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} = 0,$$

i.e.

$$\frac{\partial(y,z)}{\partial(u,v)}(q)(x - x(q)) + \frac{\partial(z,x)}{\partial(u,v)}(q)(y - y(q)) + \frac{\partial(x,y)}{\partial(u,v)}(q)(z - z(q)) = 0 \tag{8.7}$$

Also, the equation of the normal line to r at $M_r(q)$, corresponding to the pair (u_0, v_0) of the parameters, is:

$$\frac{x - x(q)}{\frac{\partial(y,z)}{\partial(u,v)}(q)} = \frac{y - y(q)}{\frac{\partial(z,x)}{\partial(u,v)}(q)} = \frac{z - z(q)}{\frac{\partial(x,y)}{\partial(u,v)}(q)} \tag{8.8}$$

8.3 Regular surfaces

Definition 8.6. A subset $S \subseteq \mathbb{R}^3$ is called *regular surface* if, for every point $p \in S$, there exists a neighbourhood V of p, in $\mathbb{R}^3$, and a mapping $r : U \to V \cap S$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$, where $U \subseteq \mathbb{R}^2$ is an open set, with the following properties:

1. r is smooth, i.e. its coordinate functions x, y, z have arbitrary high continuous partial derivatives;
2. r is a homeomorphism;
3. For every $q \in U$, the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The function $r : U \to V \cap S$ is called *local parametrization* at p or *local chart* at p or *local coordinate system* at p. The neighbourhood $V \cap S$ of p in S is called *coordinate neighbourhood*. The equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \quad (u,v) \in U, \\ z = z(u,v) \end{cases}$$

are called the *parametric equations of* of the coordinate neighbourhood $V \cap S$. The equation

$$\overrightarrow{r} = \overrightarrow{r}(u,v) \text{ where } \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$$

is called the *vector equation* of the coordinate neighbourhood $V \cap S$.

Remark 8.4.

1. Every open subset O of a regular surface $S \subseteq \mathbb{R}^3$ is a regular surface. Indeed every local parametriation $r : U \longrightarrow S \cap V$ of S at some point $p \in O$ produces a local parametrization

$$U \cap r^{-1}(O) \longrightarrow S \cap C \cap V, \; q \mapsto r(q)$$

of O at p.

2. Every regular surface can be covered by the traces of some families of local charts. Such a family of lacal charts is called an *atlas* of the surface. If the regular surface is compact, then it obviously admits finite atlases. For example the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \; : \; x^2 + y^2 + z^2 = 1\}$$

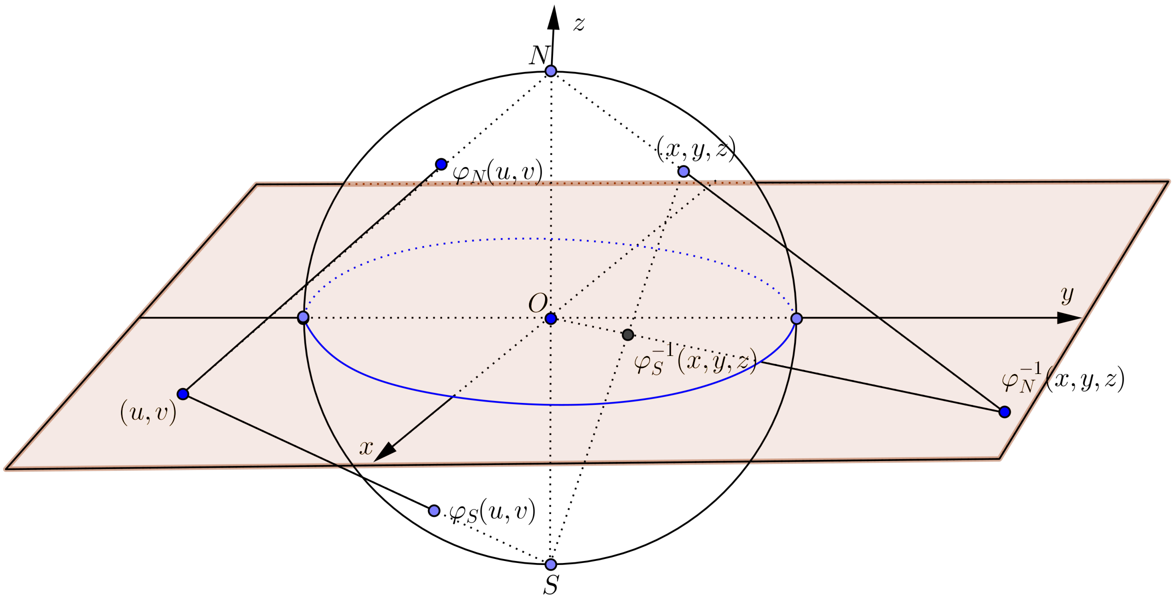
admits an atlas with two local charts $\mathcal{A} = \{(\varphi_S, \varphi_N\}$, where

$$\varphi_S : \mathbb{R}^2 \longrightarrow S^2 \setminus \{S\}, \; \varphi_S(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)$$
$$\varphi_N : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\}, \; \varphi_N(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2} \right)$$

and $S = (0,0,-1)$, $N = (0,0,1)$ are the south and north poles of S^2.

Note that the inverses of φ_S and φ_N are the stereogrphic projections

$$\varphi_S^{-1} : S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2, \; \varphi_S^{-1}(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right)$$
$$\varphi_N^{-1} : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2, \; \varphi_N^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

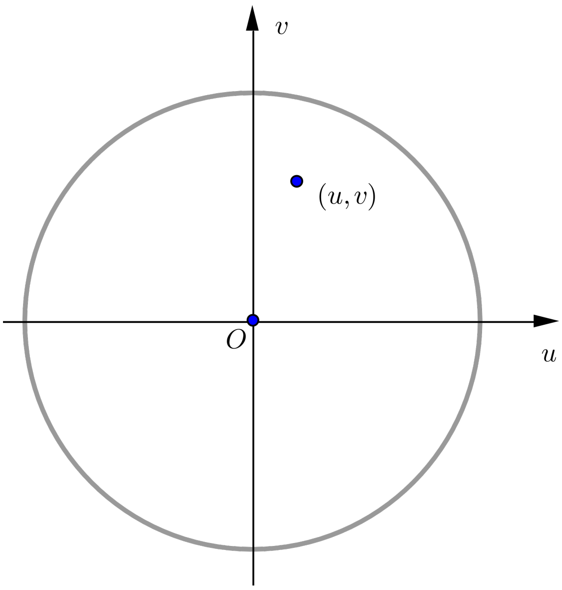


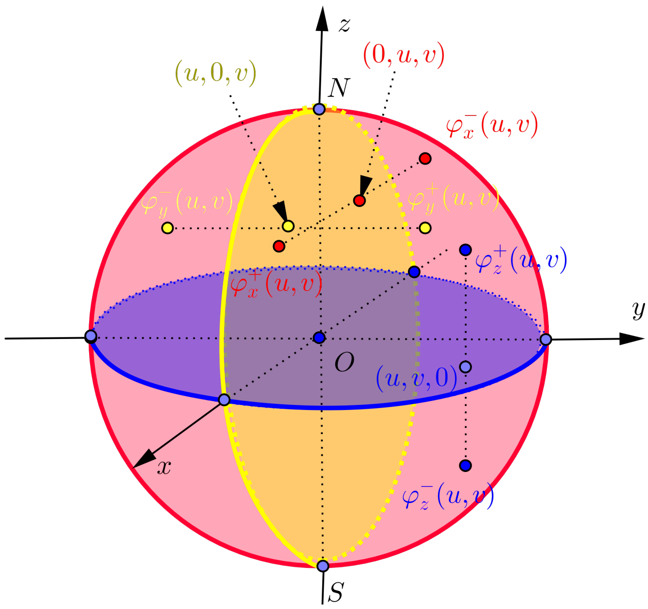
Another atlas of the sphere S^2 has 6 local charts, namely

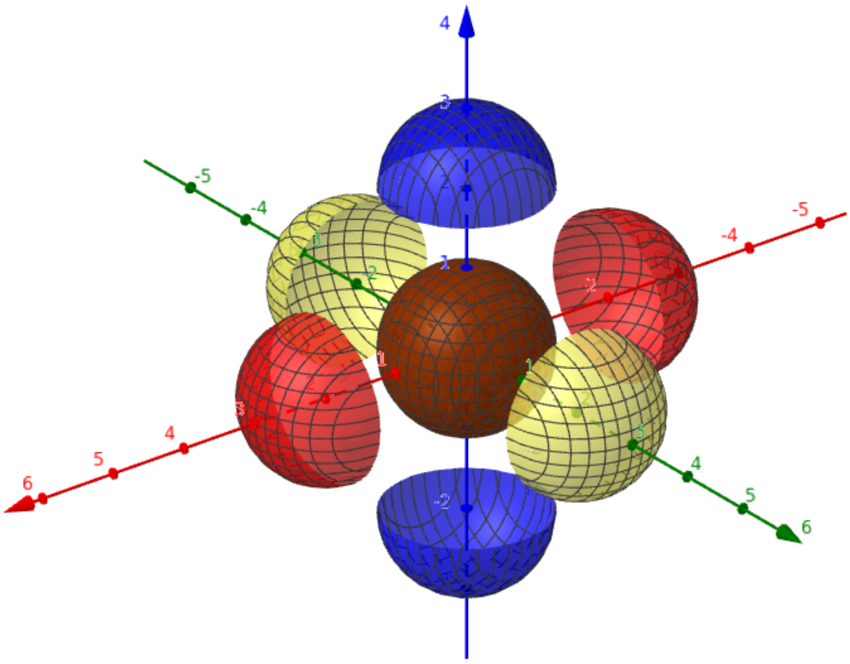
$$\mathcal{A}_1 = \{\varphi_x^{\pm}, \varphi_y^{\pm}, \varphi_z^{\pm} : B(0,1) \longrightarrow S^2\},$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^2$ centered at the origin $0 \in \mathbb{R}^2$ and

$$\varphi_x^{\pm}(u,v) = (\pm\sqrt{1-u^2-v^2}, u, v),$$
$$\varphi_y^{\pm}(u,v) = (u, \pm\sqrt{1-u^2-v^2}, v),$$
$$\varphi_z^{\pm}(u,v) = (u, v, \pm\sqrt{1-u^2-v^2}).$$







Proposition 8.6. *If $U \subseteq \mathbb{R}^2$ is an open set and $f : U \to \mathbb{R}$ is a smooth function, then its graph $G_f = \{(x, y, f(x, y)) \mid (x, y) \in U\}$ is a regular surface.*

For example

1. The elliptic paraboloid $P_e : \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z,\ p, q > 0$ is a regular surface, as P_e is the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = \dfrac{1}{2}\left(\dfrac{x^2}{p} + \dfrac{y^2}{p}\right)$.

2. The hyperbolic paraboloid $P_h : \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z,\ p, q > 0$ is a regular surface, as P_h is the graph of the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(x, y) = \dfrac{1}{2}\left(\dfrac{x^2}{p} - \dfrac{y^2}{q}\right)$.

Theorem 8.7. *(The third preimage theorem). If $U \subseteq \mathbb{R}^3$ is an open set, $f : U \to \mathbb{R}$ is a smooth functionand $a \in Im\, f$ is a regualr value of f, then*

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}$$

is a regular surface in $\mathbb{R}^3$ called the regular surface of implicite Cartesian equation $f(x, y, z) = a$.

Proposition 8.8. *Let $U \subset \mathbb{R}^3$ be an open set such that $tx \in U$ for every $t \in \mathbb{R}_+^*$ and every $x \in U$. A function $f : U \to \mathbb{R}$ is said to be* homogeneous of order $p \in \mathbb{R}$ *if $f(tx) = t^p f(x)$, $\forall t \in \mathbb{R}_+^*, x \in U$. If $f : U \to \mathbb{R}$ is a differentiable and homogeneous function of order $p \in \mathbb{R}^*$ and $c \in Im\, f \setminus \{0\}$, then $f^{-1}(c)$ is a regular surface.*

Proof. Indeed, it is enough to show that c is a regular value of f. Differentiating with respect to t, the relation $f(tx) = t^p f(x)$ we obtain

$$(df)_{tx}(x) = pt^{p-1}f(x),\ \forall t \in \mathbb{R}_+^*,\ \forall x \in U,$$

which shows, by taking $t = 1$, the Euler relation

$$(df)_x(x) = pf(x),\ \forall x \in U. \tag{8.9}$$

But for $x \in C(f)$ we have $(df)_x = 0$ and thus $(df)_x(x) = 0$, which shows that $f(x) = 0$. We therefore showed that $B(f) = f(C(f)) \subset \{0\}$, or, equivalently, $\mathbb{R}^* \subset \mathbb{R}\backslash B(f)$. But since $c \in Im\, f\backslash\{0\}$ we deduce that c is a taken regular value of f, which shows that $f^{-1}(c)$ is a regular surface. □

In particular,

1. the ellipsoid $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$,
2. the hyperboloid of one sheet $H_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$,
3. the hyperboloid of two seets $H_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$.

are all regular surfaces. Let us finally observe that the cone $C : x^2 + y^2 - z^2 = 0$ is not a regular surface.

8.4 The tangent vector space

Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. A *tangent vector* to S at p is the tangent vector $\overrightarrow{\alpha}'(0)$ of a parametrized diferentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$

Proposition 8.9. *Let $U \subseteq \mathbb{R}^2$ be an open set, let $q \in U$ and let $r : U \to S$ be a local parametrization of S. The 2-dimensional subspace $(d\overrightarrow{r})_q(\mathbb{R}^2) \subseteq \mathcal{V}$ coincides with the set of all tangent vectors to S at $r(q)$.*

Definition 8.7. The plane through a point p of a regular surface S, whose direction is the tangent space to S at p, $\overrightarrow{T}_p(S)$, is called the *tangent plane* to S at p and is denoted by $T_p(S)$. The perpendicular line on the tangent plane of the surface S at p is called the *normal line* to the surface S at p.

Proposition 8.10. *If $V \subseteq \mathbb{R}^3$ is an open set, $f : V \to \mathbb{R}$ is a smooth function, $a \in Im\, f$ is a regular value of f and $p \in f^{-1}(a)$, then the equation of the tangent plane to the regular surface $S = f^{-1}(a)$, of implicit equation $f(x, y, z) = a$, at some point $p \in S$ is:*

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) + f_z(p)(z - z_0) = 0. \tag{8.10}$$

and the equation of the normal line to S at p is:

$$\frac{x - x_0}{f_x(p)} = \frac{y - y_0}{f_y(p)} = \frac{z - z_0}{f_z(p)} \tag{8.11}$$

For example the tanget plane of the quadric

$$(Q)\ a_{00} + 2a_{10}x + 2a_{20}y + 2a_{30}z + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + a_{11}x^2 + a_{22}y^2 + a_{33}z^2 = 0$$

at some of its point $A_0(x_0, y_0, z_0) \in Q$ is

$$T_{A_0}(Q)\ a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{30}(z + z_0) + a_{12}(x_0y + xy_0) + a_{13}(z_0x + zx_0) + 2a_{23}(y_0z + yz_0) \\ + a_{11}x_0x + a_{22}y_0y + a_{33}z_0z = 0.$$

and can be obtained by polarizing the quadric's equation, i.e. by replacing

1. x^2 with $x_0 x$
2. y^2 with $y_0 y$
3. z^2 with $z_0 z$
4. $2x$ with $x + x_0$
5. $2y$ with $y + y_0$
6. $2z$ with $z + z_0$
7. $2xy$ with $x_0 y + x y_0$
8. $2yz$ with $y_0 z + y z_0$
9. $2zx$ with $z_0 x + z x_0$.

8.5 Problems

1. **(2p)** Show that the angle between the tangent of the circular helix

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$$

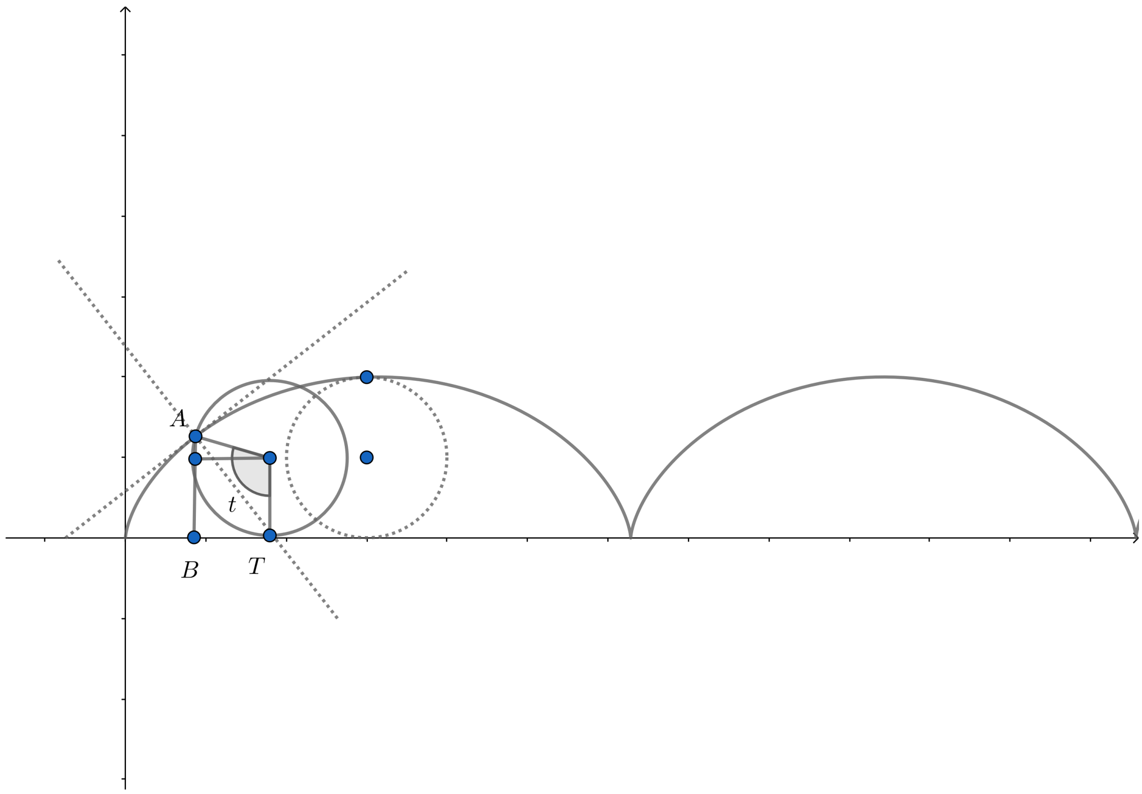
and the z-axis is constant.

Solution.

2. **(3p)** A *cycloid* is the curve traced by a chosen point on the circumference of a circle which rools along a stright line without slipping. Show that the parametric equations of the are:

$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases}, t \in \mathbb{R}.$$

Solution.



3. **(2p)** Show that the normal line to the cycloid at a certain point passes through the tangency point between the generating circle and the line alnog which the generating circle rools on.

   Solution.

4. An *epicycloid* is a plane curve traced by a chosen point on the circumference of a circle which rools without slipping around a fixed circle. Find the equations of the epicicloid.

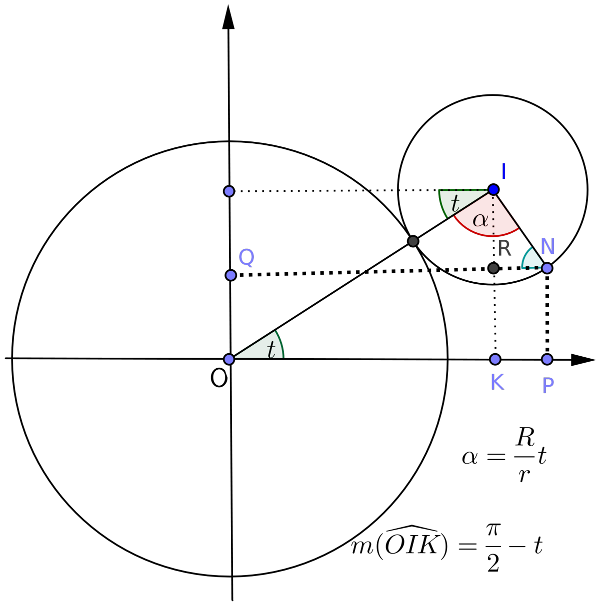
   The equations of the epicicloid are

$$\begin{cases} x = (R+r)\cos t - r\cos\left(\dfrac{R+r}{r}t\right) \\ y = (R+r)\sin t - r\sin\left(\dfrac{R+r}{r}t\right) \end{cases}, t \in \mathbb{R},$$

or

$$\begin{cases} x = r(k+1)\cos t - r\cos(k+1)t \\ y = r(k+1)\sin t - r\sin(k+1)t \end{cases}, t \in \mathbb{R},$$

where $k = \dfrac{R}{r}$. If k is an integer, then the epicycloid is a closed curve.



$$\alpha = \frac{R}{r}t$$

$$m(\widehat{OIK}) = \frac{\pi}{2} - t$$

$$m(\widehat{NIR}) = \alpha - m(\widehat{OIK}) = -\frac{\pi}{2} + \left(\frac{R}{r} + 1\right)t$$

$$m(\widehat{INR}) = \frac{\pi}{2} - m(\widehat{NIR}) = \pi - \frac{R+r}{r}t$$

$$IR = r\sin\frac{R+r}{r}t, RN = -r\cos\frac{R+r}{2}t$$

5. A *hypocycloid* is a plane curve traced by a chosen point on a small circle that rools without slipping within a larger circle. Find the equations of the hypocycloid.

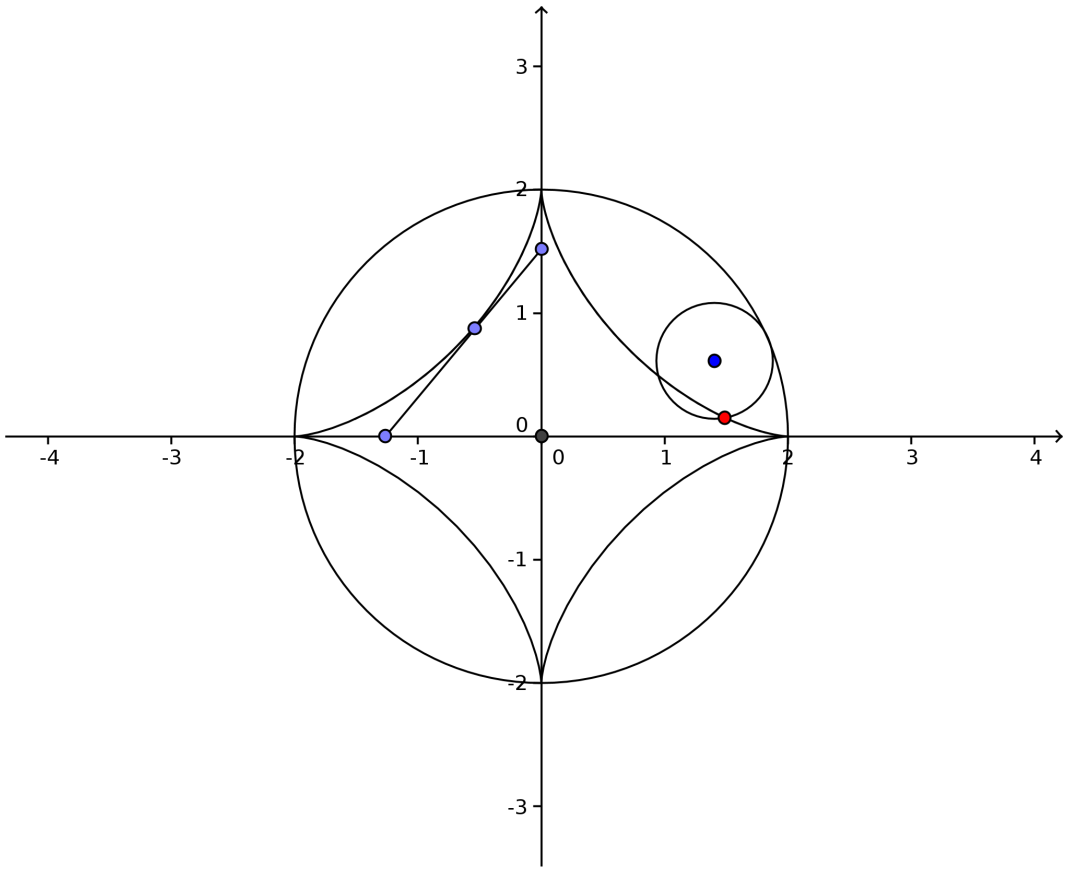
   Answer: The equations of the hypocycloid are:

$$\begin{cases} x = (R-r)\cos t + r\cos\left(\dfrac{R-r}{r}t\right) \\ y = (R-r)\sin t - r\sin\left(\dfrac{R-r}{r}t\right) \end{cases}, t \in \mathbb{R},$$

or

$$\begin{cases} x = r(k-1)\cos t + r\cos(k-1)t \\ y = r(k-1)\sin t - r\sin(k-1)t \end{cases}, t \in \mathbb{R},$$

where $k = \dfrac{R}{r}$. If k is an integer, then the hypocycloid is a closed curve. In particular, for $k = 4$ the hypocicloid is called *astroid*

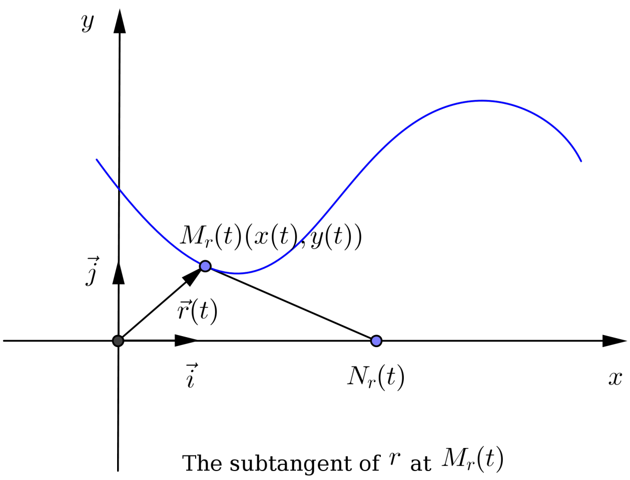


6. The *subtangent* of a planar parametrized differentiable curve is the segment which unify the tangency point between the tangent and the curve with the intersection point between the tangent and the x-axis. Show that the length of the subtangent of the planar parametrized differentiable curve

$$r:(0,\pi)\to\mathbb{R}^2,\ r(t)=a\big(\ln\tan(t/2)+\cos t,\sin t\big),$$

called the *tractrix* is constant and equal to a.

Solution.



The subtangent of r at $M_r(t)$
is the segment $[M_r(t)N_r(t)]$

The parametric equations of the tractrix are

$$\begin{cases} x=a\log\tan(t/2)+a\cos t \\ y=a\sin t \end{cases},\ t\in(0,\pi)$$

and its vector equation is

$$\overrightarrow{r}(t) = (a\ln\tan(t/2) + a\cos t)\overrightarrow{i} + (a\sin t)\overrightarrow{j}.$$

and its tangent vector

$$\begin{aligned}
\overrightarrow{r}'(t) &= \left(a\frac{1}{\tan(t/2)}\,\frac{1}{\cos^2(t/2)}\,\frac{1}{2} - a\sin t\right)\overrightarrow{i} + (a\cos t)\overrightarrow{j} \\
&= \left(\frac{a}{\sin t} - a\sin t\right)\overrightarrow{i} + (a\cos t)\overrightarrow{j} \\
&= \frac{a\cos^2 t}{\sin t}\overrightarrow{i} + (a\cos t)\overrightarrow{j} = a\cos t(\cot t\,\overrightarrow{i} + \overrightarrow{j}).
\end{aligned}$$

Thus, the equation of the tangent line to the tractrix at the regular points $M_r(t)$, i.e. $t \in (0,\pi) \setminus \{0\}$ is

$$(T_r)(t) : \frac{X - x(t)}{x'(t)} = \frac{Y - y(t)}{y'(t)} \iff \frac{X - a\log\tan(t/2) - a\cos t}{a\cos t\cot t} = \frac{Y - a\sin}{a\cos t}. \tag{8.12}$$

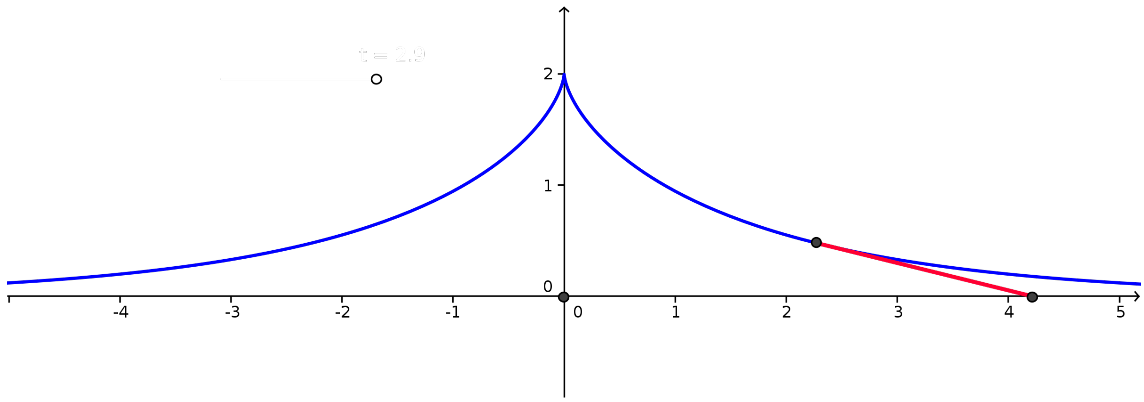
The coordinates of the intersection point $N_r(t)$ of the tangent $T_r(t)$ to the tractrix at $M_r(t)$ with the x-axis can be obtained by taking $Y = 0$ in (8.12), which implies $X = a\log\tan(t/2)$, i.e. $N_r(t)(a\log\tan(t/2), 0)$. The distance between

$$M_r(t)\ (a\log\tan(t/2) + a\cos t, a\sin t) \text{ and } N_r(t)(a\log\tan(t/2), 0)$$

is

$$\sqrt{(a\log\tan(t/2) + a\cos t - a\cos t)^2 + (a\sin t - 0)^2} = \sqrt{a^2} = |a| = a.$$

Note that $t = \pi/2$ is the only singular point of $\overrightarrow{r}$. Since $\overrightarrow{r}''(\pi/2) = a\overrightarrow{j}$, it follows that $t = \pi/2$ is a singular point of order two for $\overrightarrow{r}$, i.e. $\overrightarrow{r}''(\pi/2)$ is a director vector of the tangent line of r at $t = \pi/2$. In other words the y-axis is the tangent line to r at $t = \pi/2$. Note that $M_r(\pi/2)(0,a)$ and $N_r(\pi/2)$ is the origin $O(0,0)$. Thus the distance between $M_r(\pi/2)(0,a)$ and $N_r(\pi/2)$ is a.

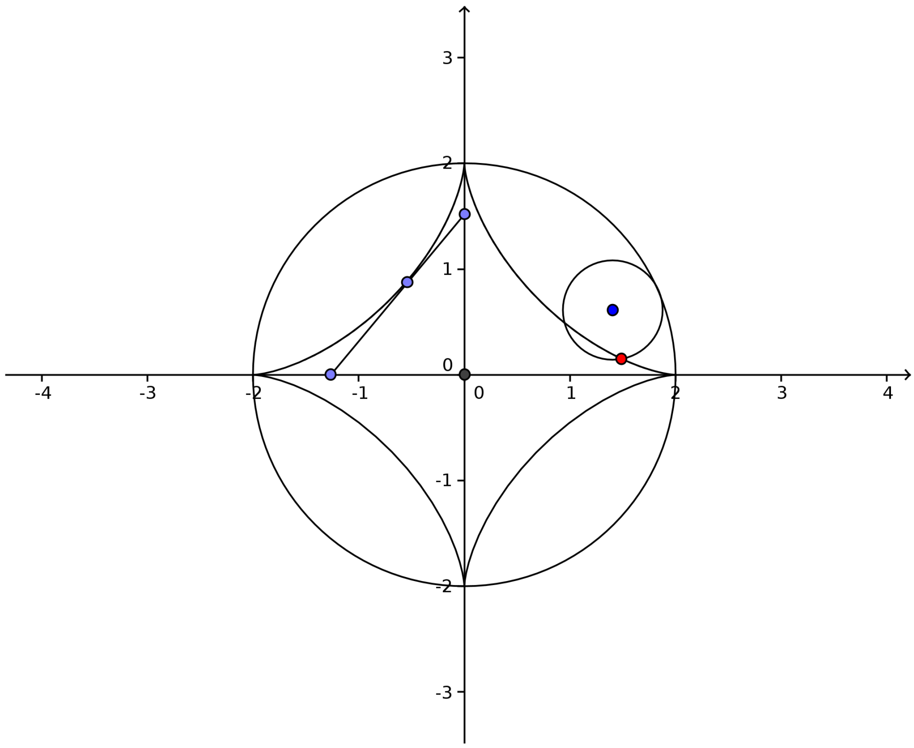


7. **(2p)** Show that the tangents of the astroid

$$\begin{cases} x = r\cos^3 t \\ y = r\sin^3 t \end{cases}$$

determines on the coordinate axes segments of constant length.

Solution.



8. Write the equations of the tangent line and the normal plane for the following curves, whenever these associated objects are well-determined:

   (a) **(2p)**

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = 0 \text{ of the parameter}$$

   (b) **(2p)**

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = \frac{\pi}{4} \text{ of the parameter}$$

Solution.

9. **(2p)** Write the equations of the tangent planes of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ at the points of the form $(x_0, y_0, 0)$ and show that these are parallel to the z-axis.

   Solution.

10. **(2p)** Show that the trace of the parametrized differentiable curve $\alpha : \mathbb{R} \to \mathbb{R}^3,\ \alpha(t) = (e^t \cos t, e^t \sin t, 2t)$ is contained in the regular surface of equation $z = \ln(x^2 + y^2)$ and write the equation of the tangent plane of the surface at the points $\alpha(t)$, $t \in \mathbb{R}$.

    Solution.

11. **(3p)** Show that the tangent planes of the surface of equation $z = xf(\frac{y}{x})$, where f is a differentiable function, are passing through the origin.

    Solution.

12. **(3p)** Show that the set $S = \{(x, y, z) \in \mathbf{R}^3 \,|\, xyz = a^3\},\ a \neq 0$ is a regular surface and the its tangent plane at an arbitrary point $p \in S$ determines on the coordinate axes three points which form, together with the origin a tetrahedron of constant volume (independent of p).

    Solution.

9 Week 9:Conics

9.1 The Ellipse

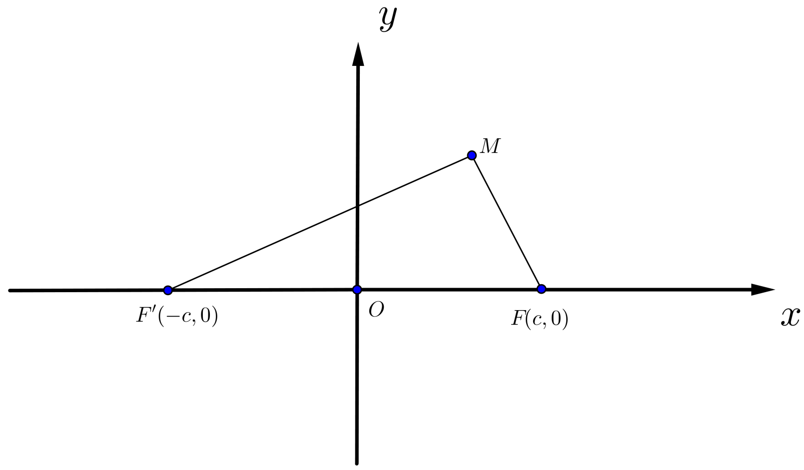
Definition 9.1. An *ellipse* is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called *foci* is constant.

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.



Remark 9.1. In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.

Thus, for the generic point $M(x, y)$ of the ellipse we have succesively:

$$\begin{aligned}
|MF| + |MF'| = 2a \quad \Leftrightarrow \quad & \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \\
& \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2} \\
& x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
& a\sqrt{(x+c)^2 + y^2} = cx + a^2 \\
& a^2(x^2 + 2xc + c^2) + a^2y^2 = c^2x^2 + 2a^2cx + a^2 \\
& (a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0.
\end{aligned}$$

Denote $a^2 - c^2$ by b^2, as $(a > c)$. Thus $b^2x^2 + a^2y^2 - a^2b^2 = 0$, i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{9.1}$$

Remark 9.2. The ellipse

$$(E)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is a regular curve and the equation of its tangent line $T_{P_0}(E)$ at some point $P_0(x_0, y_0) \in E$ is

$$T_{P_0}(E)\ \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1. \tag{9.2}$$

Remark 9.3. The equation (9.1) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its *center*.

Remark 9.4. In order to sketch the graph of the ellipse, observe that it is enough to represent the function

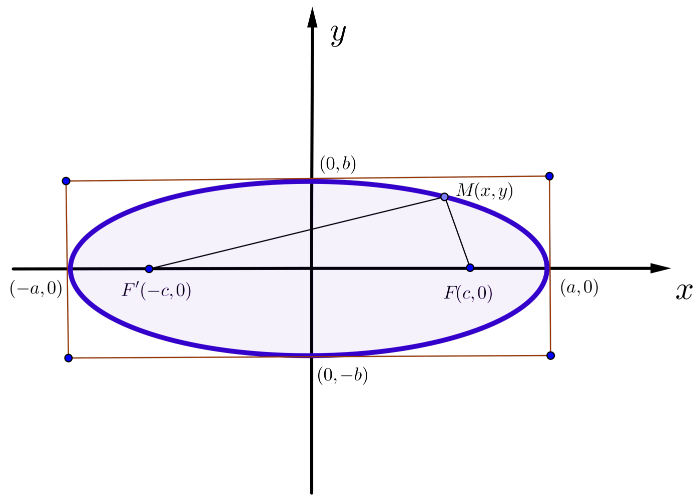
$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+ + +$	0	$- - -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$- - -$	$-$	$- - -$	$\vert$



9.2 The Hyperbola

Definition 9.2. The *hyperbola* is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F' is constant.

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origin at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.

Remark 9.5. In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{array}{l} x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2} \\ c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \\ (c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0. \end{array}$$

By using the notation $c^2 - a^2 = b^2$ $(c > a)$ we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{9.3}$$

Remark 9.6. The hyperbola

$$(H)\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a regular curve and the equation of its tangent line $T_{P_0}(H)$ at some point $P_0(x_0, y_0) \in H$ is

$$T_{P_0}(H)\ \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1. \tag{9.4}$$

The equation (9.3) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 9.7. To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

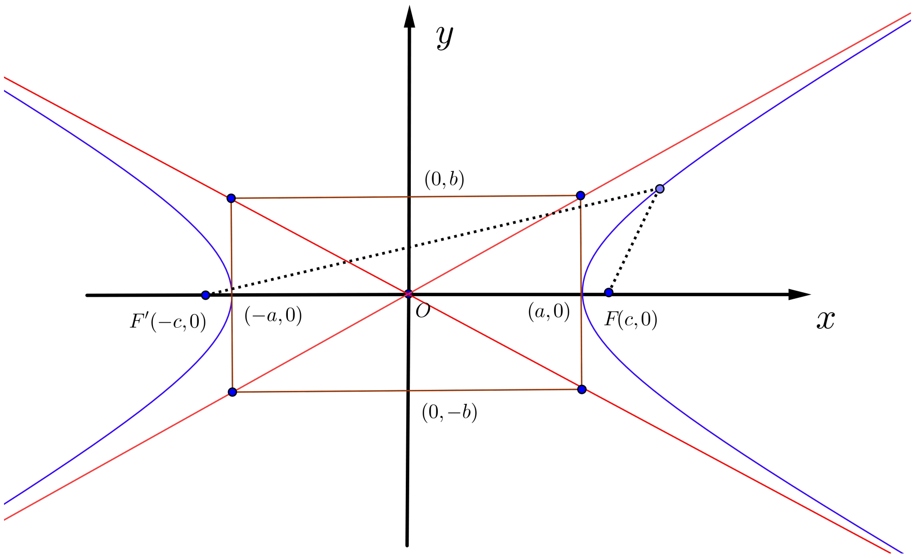
by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \dfrac{b}{a}$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = -\dfrac{b}{a}$, it follows that $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$ are asymptotes of f.

One has, also

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$+ + +$	$+$
$f(x)$	∞	$\searrow$	$0\vert$	$/ / /$	$\vert 0$	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$- - -$	$-$



9.3 The Parabola

Definition 9.3. The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2}, 0\right)$ and $d : x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on Oy.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x + \frac{p}{2}\right)^2 + 0} = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

$$\left(x + \frac{p}{2}\right)^2 = \left(x - \frac{p}{2}\right)^2 = y^2$$

$$x^2 + px + \frac{p^2}{4} = x^2 - px + \frac{p^2}{4} + y^2,$$

and the equation of the parabola is

$$y^2 = 2px. \tag{9.5}$$

Remark 9.8. The parabola

$$(P)\, y^2 = 2px$$

is a regular curve and the equation of its tangent line $T_{P_0}(P)$ at some point $Q_0(x_0, y_0) \in P$ is

$$T_{Q_0}(P)\; y_0 y = p(x + x_0). \tag{9.6}$$

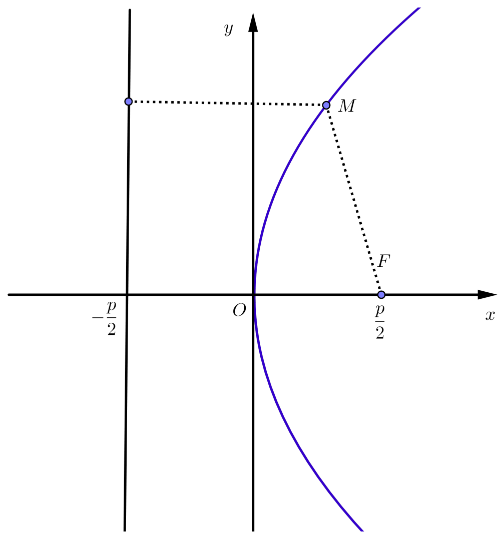
Remark 9.9. The equation (9.5) is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.

Representing the graph of the function $f:[0,\infty)\to[0,\infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.

One has

$$f'(x)=\frac{p}{\sqrt{2px_0}};\qquad f''(x)=-\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+\;+\;+$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$-\;-\;-$	$-$



9.4 Problems

1. Find the equations of the tangent lines to the ellipse $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m\in\mathbb{R}$. (see [1, p. 110]).

   Solution. We are looking for the lines $d:y=mx+n$, which are tangent to the ellipse, i.e. each of them has one single common point with the ellipse. Their intersection is given by the solutions of the system of equations

$$\begin{cases}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0\\ y=mx+n\end{cases},$$

   or, by replacing y in the equation of the ellipse,

$$(a^2m^2+b^2)x^2+2a^2mnx+a^2(n^2-b^2)=0.$$

   The discriminant Δ of the last equation is given by

$$\Delta=4[a^4m^2n^2-a^2(a^2m^2+b^2)(n^2-b^2)]$$

   and the line (d) and the ellipse (E) have one single common point if and only if $a^4m^2n^2-a^2(a^2m^2+b^2)(n^2-b^2)=0$, i.e. $n=\pm\sqrt{a^2m^2+b^2}$. The equations of the tangent lines of direction m are therefore

$$y=mx\pm\sqrt{a^2m^2+b^2}. \tag{9.7}$$

2. **(2p)** Find the equations of the tangent lines to the ellipse $\mathcal{E}: \ x^2+4y^2-20=0$ which are orthogonal to the line $d: \ 2x-2y-13=0$.
Solution.

3. **(2p)** Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{25}+\dfrac{y^2}{16}-1=0$, passing through $P_0(10,-8)$.
Solution.

4. If $M(x,y)$ is a point of the tangent line $T_{M_0}(E)$ of the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at one of its points $M_0(x_0,y_0)\in\mathcal{E}$, show that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\geq 1$.

Solution. Every director vector of the tangent line $T_{M_0}(E) \ : \ \dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1$ is orthogonal to the normal vector $\vec{n}\left(\dfrac{x_0}{a^2},\dfrac{y_0}{b^2}\right)$ of the tangent line $T_{M_0}(E)$. Such an orthogonal vector is $\vec{v}\left(\dfrac{y_0}{b^2},-\dfrac{x_0}{b^2}\right)$. Thus, the parametric equations of the tangent line are

$$T_{M_0}(E) \ : \ \begin{cases} x=x_0+\dfrac{y_0}{b^2}t \\ y=y_0-\dfrac{x_0}{b^2}t \end{cases}, \ t\in\mathbb{R},$$

i.e. the coordinates of M are of this form. In order to completely solve the question , we only need to show that $\varphi\geq 1$, where $\varphi:\mathbb{R}\longrightarrow\mathbb{R}$, $\varphi(t)=\dfrac{\left(x_0+\dfrac{y_0}{b^2}t\right)^2}{a^2}+\dfrac{\left(y_0-\dfrac{x_0}{b^2}t\right)^2}{b^2}$. This is actually the case as

$$\begin{aligned}\varphi(t) &= \frac{x_0^2 + 2\frac{x_0y_0}{b^2}t + \frac{y_0^2}{b^4}t^2}{a^2} + \frac{y_0^2 - 2\frac{x_0y_0}{b^2}t + \frac{x_0^2}{a^2}t^2}{b^2} \\ &= \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{1}{a^2b^2}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right)t^2 = 1 + \frac{t^2}{a^2b^2} \geq 1, \forall t \in \mathbb{R}.\end{aligned}$$

5. Find the equations of the tangent lines to the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 115]).

Solution. The intersection of the hyperbola $(\mathcal{H})$ with the line $(d) y = mx + n$ is given by the solution of the system

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

By substituting y in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \tag{9.8}$$

- If $a^2m^2 - b^2 = 0$, (or $m = \pm\frac{b}{a}$), then the equation (9.8) becomes

$$\pm 2bnx + a(n^2 + b^2) = 0.$$

  - If $n = 0$, there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);
  - If $n \neq 0$, there exists a unique solution (geometrically, a line d, which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);
- If $a^2m^2 - b^2 \neq 0$, then the discriminant of the equation (9.8) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

  The line $d : y = mx + n$ is tangent to the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ if the discriminant Δ of the equation (9.8) is zero, i.e. $a^2m^2 - n^2 - b^2 = 0$.

  - If $a^2m^2 - b^2 \geq 0$, i.e. $m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$, then $n = \pm\sqrt{a^2m^2 - b^2}$. The equations of the tangent lines to $\mathcal{H}$, having the angular coefficient m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{9.9}$$

  - If $a^2m^2 - b^2 < 0$, there are no tangent lines to $\mathcal{H}$, of angular coefficient m.

6. **(2p)** Find the equations of the tangent lines to the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$ which are orthogonal to the line $d : 4x + 3y - 7 = 0$.

Solution.

7. Find the equations of the tangent lines to the parabola $\mathcal{P}: \ y^2 = 2px$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 119]).

   Solution. The intersection beteen the porabla (P) and the line $(d)y = mx + n$ is given by the solution of the system
$$\begin{cases} y^2 = 2px \\ y = mx + n. \end{cases}$$
   This leads to a second degree equation
$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$
   having the discriminant
$$\Delta = 4p(2mn - p) \tag{9.10}$$
   The line $d : y = mx + n$ (with $m \neq 0$) is tangent to the parabola $\mathcal{P} : y^2 = 2px$ if the discriminant Δ which appears in (9.10) is zero, i.e. $2mn = p$. Then, the equation of the tangent line to $\mathcal{P}$, having the angular coefficient m, is
$$y = mx + \frac{p}{2m}. \tag{9.11}$$

8. **(2p)** Find the equation of the tangent line to the parabola $\mathcal{P} : \ y^2 - 8x = 0$, parallel to $d : 2x + 2y - 3 = 0$.

   Solution.

9. **(2p)** Find the equation of the tangent line to the parabola $\mathcal{P} : \ y^2 - 36x = 0$, passing through $P(2, 9)$.

   Solution.

10. **(3p)** Find the locus of the orthogonal projections of the center $O(0, 0)$ of the ellipse
$$E : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
    on its tangents.

Solution.

11. **(3p)** Find the locus of the orthogonal projections of the center $O(0,0)$ of the hyperbola

$$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

on its tangents.

Solution.

12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Recall that the gradient $\text{grad}(f)(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathcal{E}$ to its point

$M_0(x_0, y_0)$, where

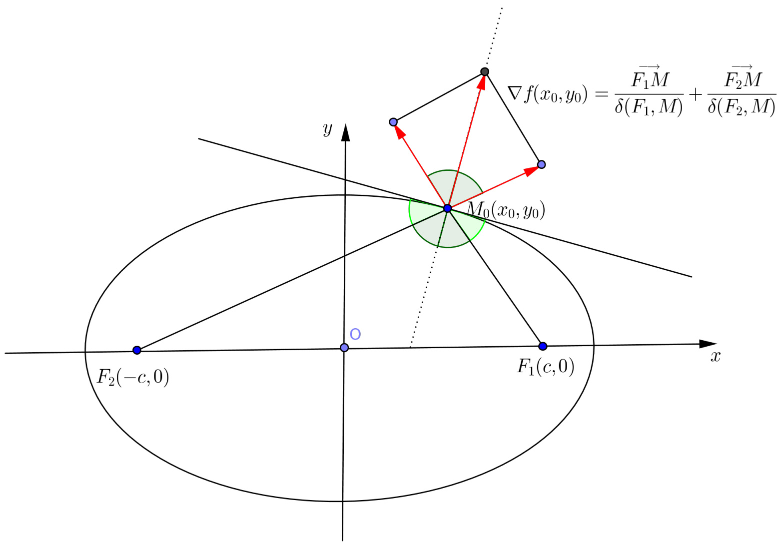
$$f(x,y) = \delta(F_1, M) + \delta(F_2, M) = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2}$$

and $M(x,y)$. Note that

$$f_x(x_0, y_0) = \frac{x_0 - c}{\delta(F_1, M_0)} + \frac{x_0 + c}{\delta(F_2, M_0)} \text{ and } f_y(x_0, y_0) = \frac{y}{\delta(F_1, M_0)} + \frac{y}{\delta(F_2, M_0)},$$

and shows that

$$\begin{aligned} \mathrm{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0 - c}{\delta(F_1, M_0)} + \frac{x_0 + c}{\delta(F_2, M_0)}, \frac{y_0}{\delta(F_1, M_0)} + \frac{y_0}{\delta(F_2, M_0)}\right) \\ &= \frac{(x_0 - c, y)}{\delta(F_1, M_0)} + \frac{(x_0 + c, y)}{\delta(F_2, M_0)} = \frac{\overrightarrow{F_1M_0}}{\delta(F_1, M_0)} + \frac{\overrightarrow{F_2M_0}}{\delta(F_2, M_0)}. \end{aligned}$$



The versors $\frac{\overrightarrow{F_1M_0}}{\delta(F_1, M_0)}$ and $\frac{\overrightarrow{F_2M_0}}{\delta(F_2, M_0)}$ point towards the exterior of the ellipse $\mathcal{E}$ and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{E})$ of the ellipse at $M_0(x_0, y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{E})$.

13. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).

Solution. Let $F_1(-c, 0)$, $F_2(c, 0)$ be the foci of the hyperbola $\mathcal{E} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Recall that the gradient $\mathrm{grad}(f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$ is a normal vector of the hyperbola $\mathcal{H}$ to its point $M_0(x_0, y_0)$, where

$$f(x,y) = \delta(F_2, M) - \delta(F_1, M) = \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} \tag{9.12}$$

on the left hand side branch of $\mathcal{H}$ and

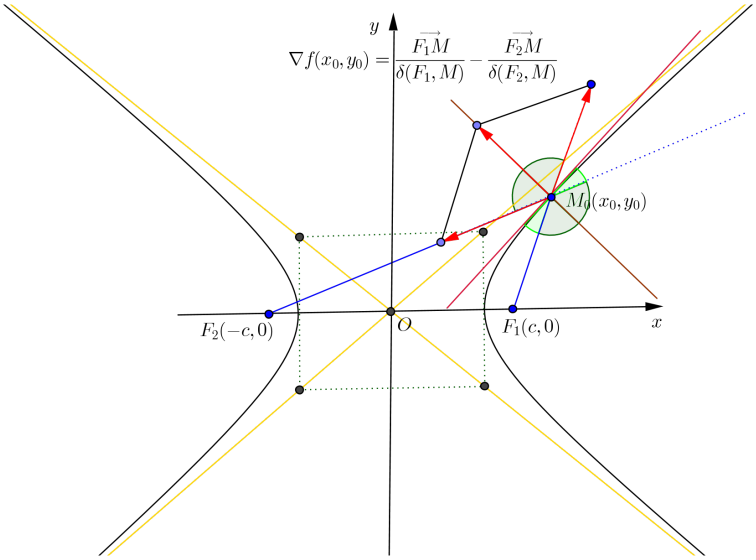
$$f(x,y) = \delta(F_1, M) - \delta(F_2, M) = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} \tag{9.13}$$

on the right hand side branch of $\mathcal{H}$ and $M(x,y)$. We shall only use the version (9.12) of f, as judgement for the version (9.13) works in a similar way. Note that

$$f_x(x_0, y_0) = \frac{x_0 - c}{\delta(F_1, M_0)} - \frac{x_0 + c}{\delta(F_2, M_0)} \text{ and } f_y(x_0, y_0) = \frac{y}{\delta(F_1, M_0)} - \frac{y}{\delta(F_2, M_0)},$$

and shows that

$$\begin{aligned}\operatorname{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0-c}{\delta(F_1,M_0)} - \frac{x_0+c}{\delta(F_2,M_0)}, \frac{y_0}{\delta(F_1,M_0)} - \frac{y_0}{\delta(F_2,M_0)}\right)\\ &= \frac{(x_0-c,y)}{\delta(F_1,M_0)} - \frac{(x_0+c,y)}{\delta(F_2,M_0)} = \frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)} - \frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$



The versors $\dfrac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $-\dfrac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the 'exterior' of the hyperbola $\mathcal{H}$[3] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{H})$ of the hyperbola at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{H})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{H})$.

14. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).

Solution. Let $F(\frac{p}{2},0)$ be the focus of the parabola $\mathcal{P}: \ y^2 = 2px$ and $d: x = -\frac{p}{2}$ be its director line. Recall that the gradient $\operatorname{grad}(f)(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$ is a normal vector of parabola $\mathcal{P}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y) = \delta(F,M) - \delta(M,d) = \sqrt{\left(x-\frac{p}{2}\right)^2 + y^2} - \left(x+\frac{p}{2}\right)$$

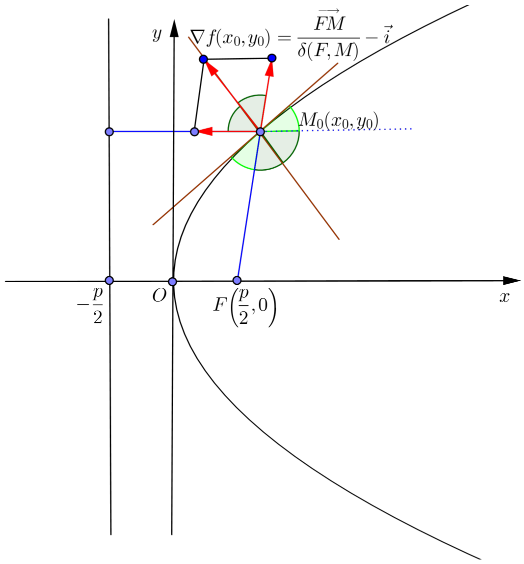
and $M(x,y)$. Note that

$$f_x(x_0,y_0) = \frac{x_0-\frac{p}{2}}{\delta(F,M_0)} - 1 \text{ and } f_y(x_0,y_0) = \frac{y_0}{\delta(F,M_0)},$$

and shows that

$$\begin{aligned}\operatorname{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0-\frac{p}{2}}{\delta(F,M_0)} - 1, \frac{y_0}{\delta(F,M_0)}\right)\\ &= \left(\frac{x_0-\frac{p}{2}}{\delta(F,M_0)}, \frac{y_0}{\delta(F,M_0)}\right) - (1,0) = \frac{\overrightarrow{FM_0}}{\delta(F,M_0)} - \mathbf{i}.\end{aligned}$$

[3]The exterior of a hyperbola is the nonconvex component of its complement



The versors $\dfrac{\overrightarrow{FM_0}}{\delta(F,M_0)}$ and $-\mathbf{i}$ point towards the 'exterior' of the parabola $\mathcal{P}$[4] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{FM_0}$ and $\mathbf{i}$ and (the sum) is also orthogonal to the tangent line $T_{M_0}(\mathcal{P})$ of the parabola at $M_0(x_0,y_0)$. This shows that the angle between the ray FM and the tangent line $T_{M_0}(\mathcal{P})$ equals the angle between Ox and the tangent $T_{M_0}(\mathcal{E})$.

[4]The exterior of a parabola is the nonconvex component of its complement

10 Week 10: Quadrics

10.1 The ellipsoid

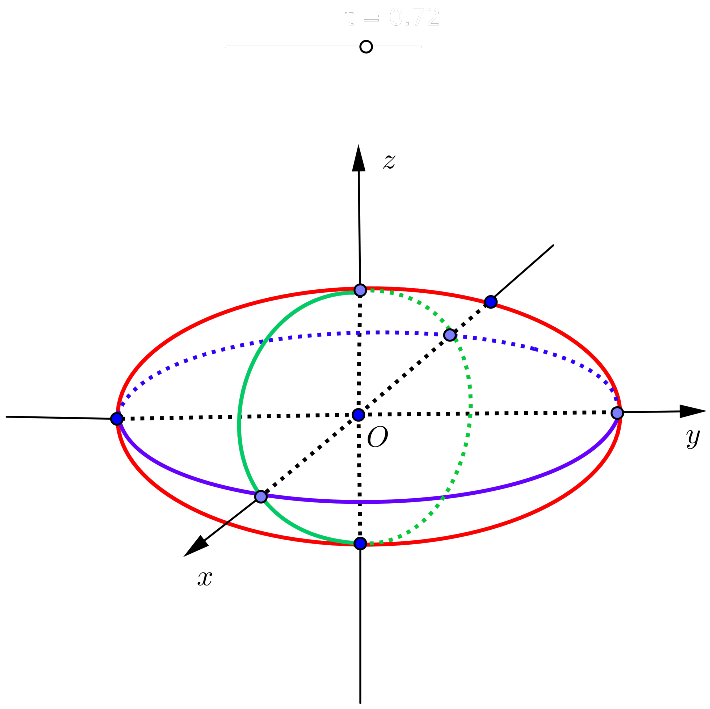
The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{10.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (10.1);

- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}$$



- The sections with planes parallel to xOy are given by setting $z = \lambda$ in (10.1). Then, a section is of equations $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$.

- If $|\lambda| < c$, the section is an ellipse

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$

- If $|\lambda| = c$, the intersection is reduced to one (tangency) point $(0, 0, \lambda)$;
- If $|\lambda| > c$, the plane $z = \lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to xOz or yOz are obtained in a similar way.

10.2 The hyperboloid of one sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{10.2}$$

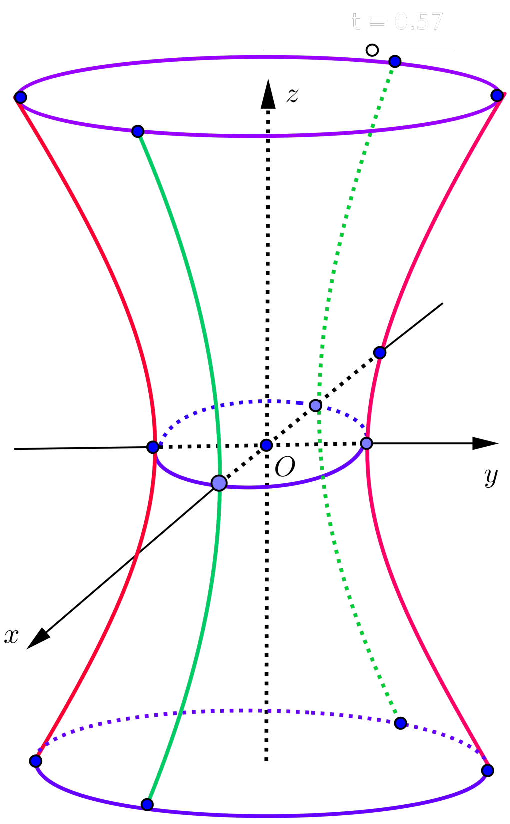
is called *hyperboloid of one sheet*.

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{cases} ; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases} ; \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{cases} ;$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases} ; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases} ; \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases} ;$$



Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

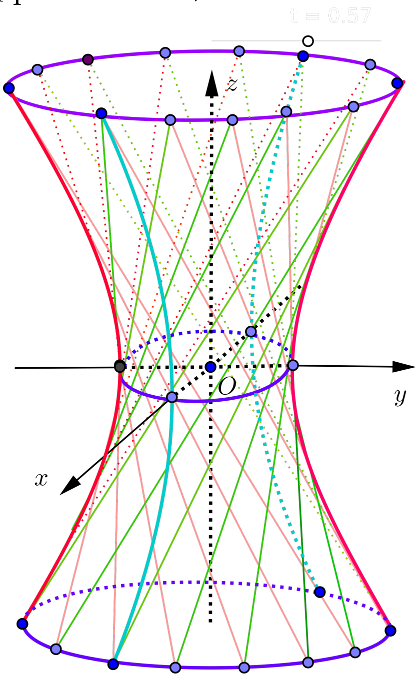
$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}\Leftrightarrow\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda:\begin{cases}\lambda\left(\frac{x}{a}+\frac{z}{c}\right)=1+\frac{y}{b}\\ \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right)\end{cases},\ \lambda\in\mathbb{R},$$

$$d'_\mu:\begin{cases}\mu\left(\frac{x}{a}+\frac{z}{c}\right)=1-\frac{y}{b}\\ \frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)\end{cases},\ \mu\in\mathbb{R}.$$

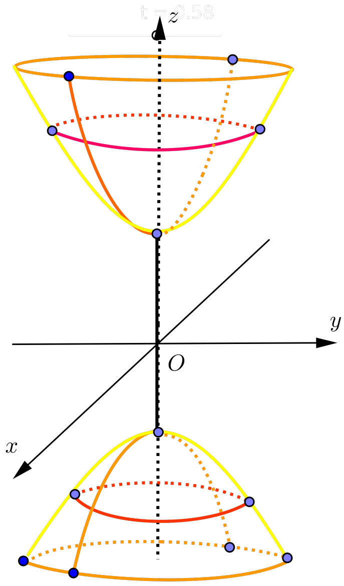
Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.



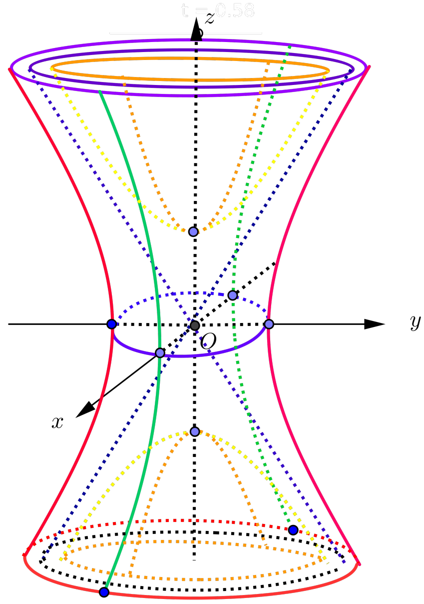
10.3 Th hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2:\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0,\qquad a,b,c\in\mathbb{R}_+^*. \tag{10.3}$$



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

$$\left\{\begin{array}{c}\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0\\ x=0\\ \text{a hyperbola;}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}-\frac{z^2}{c^2}+1=0\\ y=0\\ \text{a hyperbola}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0\\ z=0\\ \text{the empty set}\end{array}\right.;$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c}\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1-\frac{\lambda^2}{a^2}\\ x=\lambda\\ \text{hyperbolas}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1-\frac{\lambda^2}{b^2}\\ y=\lambda\\ \text{hyperbolas}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1+\frac{\lambda^2}{c^2}\\ z=\lambda.\end{array}\right.$$

- If $|\lambda| > c$, the section is an ellipse;
- If $|\lambda| = c$, the intersection reduces to the point of coordinates $(0, 0, \lambda)$;
- If $|\lambda| < c$, one obtains the empty set.

10.4 Elliptic Cones

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{10.4}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;
- The intersections with the coordinates planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0 \\ x = 0 \\ \text{two lines} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{two lines} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \\ \text{the point } O(0,0,0). \end{array}\right.$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{array}\right. ; \left\{\begin{array}{c} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas.} \end{array}\right. , \left\{\begin{array}{c} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{\lambda^2}{c^2} \\ z = \lambda. \\ \text{ellipses} \end{array}\right.$$

10.5 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*, \tag{10.5}$$

is called *elliptic paraboloid*.

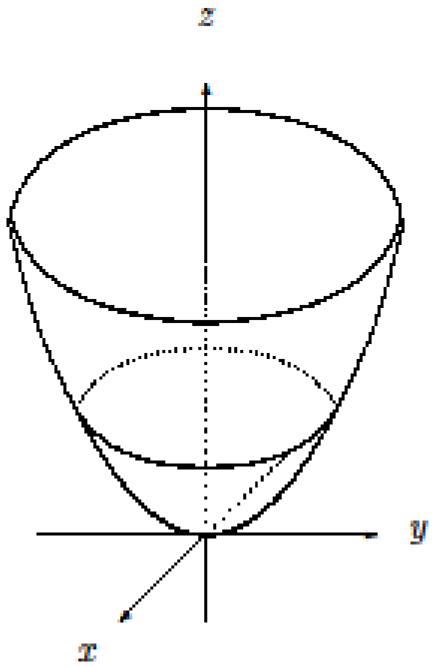
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{the point } O(0,0,0). \end{array}\right.$$

- The intersection with the planes parallel to the coordinate planes are $\left\{\begin{array}{c} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \end{array}\right.$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

and

$$\begin{cases} \dfrac{y^2}{q} = 2z - \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \qquad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases};$$



10.6 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q > 0. \tag{10.6}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\begin{cases} -\dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{cases};$$

- The intersection with the planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = -2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z + \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{cases}$$

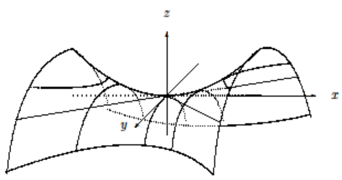
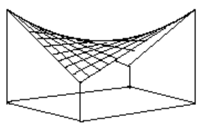
Remark: The hyperbolic paraboloid contains two families of lines. Since

$$\left(\frac{x}{\sqrt{p}}+\frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}}-\frac{y}{\sqrt{q}}\right)=2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}=\lambda \\ \lambda\left(\dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}\right)=2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}=\mu \\ \mu\left(\dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}\right)=2z \end{cases}, \mu \in \mathbb{R}.$$

10.7 Singular Quadrics

Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

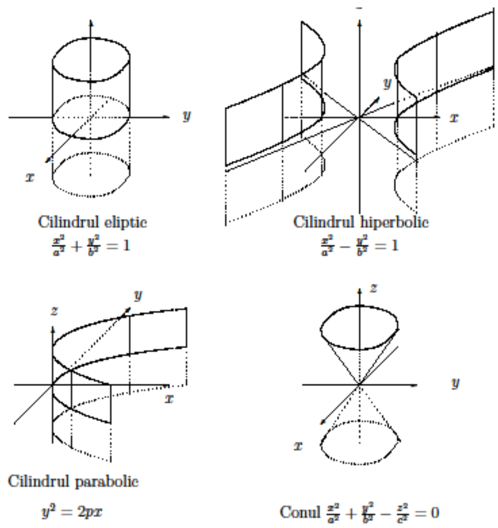
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0, \quad a,b>0 \text{ or } \frac{x^2}{a^2}+\frac{z^2}{c^2}-1=0, \frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0. \tag{10.7}$$

- The *hyperbolic cylinder* is the surface of equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0, \quad a,b>0 \text{ or } \frac{x^2}{a^2}-\frac{z^2}{c^2}-1=0, \frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0. \tag{10.8}$$

- The *parabolic cylinder* is the surface of equation

$$y^2=2px, \quad p>0, \quad (\text{or an alternative equation}). \tag{10.9}$$



10.8 Problems

1. Find the intersection points of the ellipsoid

$$\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$

with the line

$$\frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}$$

and write the equations of the tangent planes as well as the equations of the normal lines to the ellipsoid at the intersection points.

Solution.

2. Find the rectilinear generatrices of the quadric $4x^2 - 9y^2 = 36z$ which passes through the point $P(3\sqrt{2}, 2, 1)$.

   Solution.

3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1)\ \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

   which are parallel to the plane $(\pi)\ x + y + z = 0$.

Solution.

4. Find the locus of points on the hyperbolic paraboloid $(\mathcal{P}_h)\ y^2 - z^2 = 2x$ through which the rectilinear generatrices are perpendicular.

   Solution.

5. Compute the distance from $O(0,0,0)$ to the tangent plane $T_M(\mathcal{E})$ of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at some of its point $M(x,y,z) \in \mathcal{E}$.

Solution.

6. Show that the intersection between a straight line d and the sphere $S(O, r)$ is a singleton if and only if $\mathrm{dist}(O, d) = r$.

   Solution.

11 Week 11: Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x,y,z) : F(x,y,z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x,y,z) = 0$. For example the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1 = \{M(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v)\},$$

where $x,y,z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}, \quad (u,v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x,y,z) : F(x,y,z) = 0, G(x,y,z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x,y,z) : x = x(t), y = y(t), z = z(t)\},$$

where $x,y,z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x,y,z;\lambda) = 0 \\ F_2(x,y,z;\lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x,y,z;\lambda,\mu) = 0 \\ F_2(x,y,z;\lambda,\mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda,\mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

11.1 Cylindrical Surfaces

Definition 11.1. The surface generated by a variable line, called *generatrix*, which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called *cylindrical surface*. The curve $\mathcal{C}$ is called the *director curve* of the cylindrical surface.

Theorem 11.1. *The cylindrical surface, with the generatrix parallel to the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{11.1}$$

Proof. The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

is compatible. By eliminating λ and μ between four equations of the system, one obtains a *necessary condition* $\varphi(\lambda, \mu) = 0$ for the parameters λ and μ in order to nonempty intersection between the line $d_{\lambda,\mu}$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. □

Remark 11.1. Any equation of the form (11.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example 11.1. Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x+y=\lambda \\ z=\mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2-2y^2-z=0 \\ x-1=0 \\ x+y=\lambda \\ z=\mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

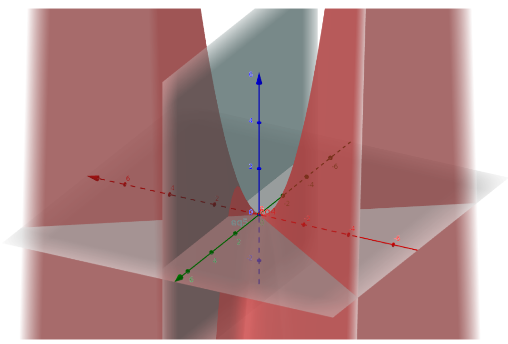
$$\begin{cases} x=1 \\ y=\lambda-1 \\ z=\mu \end{cases}$$

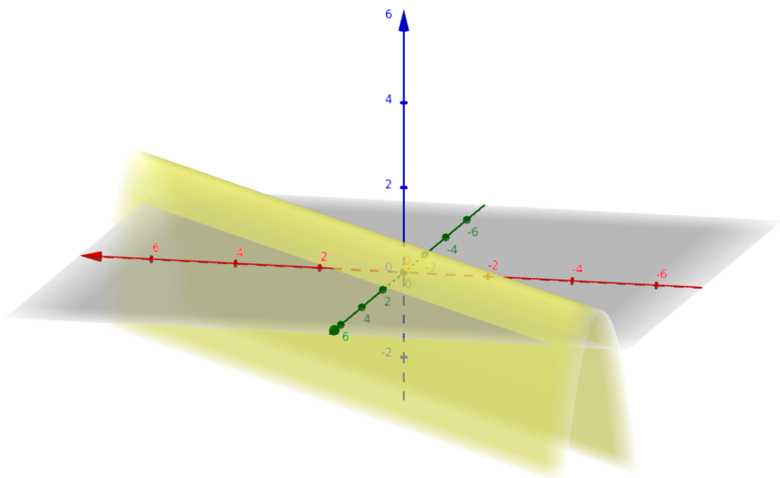
and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda-1)^2+\mu-1=0.$$

Thus, the equation of the required cylindrical surface is

$$2(x+y-1)^2+z-1=0.$$





11.2 Conical Surfaces

Definition 11.2. The surface generated by a variable line, called *generatrix*, which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called *conical surface*. The point V is called the *vertex* of the surface and the curve $\mathcal{C}$ *director curve*.

Theorem 11.2. *The conical surface, of vertex $V(x_0,y_0,z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \end{cases},$$

(V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right)=0. \tag{11.2}$$

Proof. The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases} .$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating x, y and z in the the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases} ,$$

i.e.

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

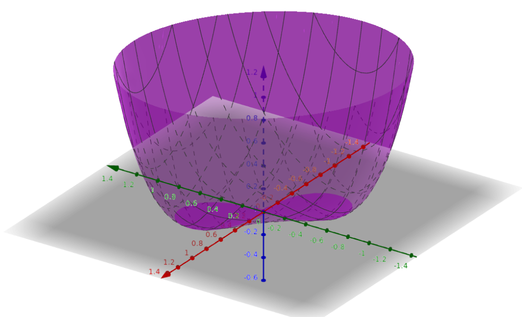
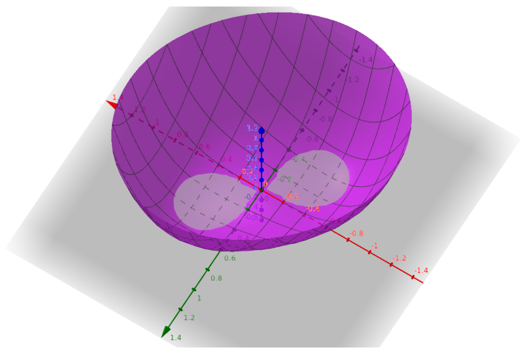
Remark 11.2. If φ is a polynomial function, then the equation (11.2) can be written in the form

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$. If φ is polynomial and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example 11.2. Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases} .$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases} .$$

The system of equations
$$\begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \\ x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \end{cases}$$
must be compatible. A solution is
$$\begin{cases} x = 1-\lambda \\ y = 1-\mu \\ z = 0 \end{cases},$$
and, replaced in the first equation of the system, gives the compatibility condition
$$[(1-\lambda)^2 + (1-\mu)^2]^2 - (1-\lambda)(1-\mu) = 0.$$
The equation of the conical surface is obtained by eliminating the parameters λ and μ in
$$\begin{cases} x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \\ ((1-\lambda)^2 + (1-\mu)^2)^2 - (1-\lambda)(1-\mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x-1}{z-1}$ and $\mu = \dfrac{y-1}{z-1}$ and replacing in the compatibility condition, one obtains
$$\left[\left(\frac{z-x}{z-1}\right)^2 + \left(\frac{z-y}{z-1}\right)^2\right]^2 - \left(\frac{z-x}{z-1}\right)\left(\frac{z-y}{z-1}\right) = 0,$$
or
$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

11.3 Conoidal Surfaces

Definition 11.3. The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called *conoidal surface*. The curve $\mathcal{C}$ is the *director curve* and the plane π is the *director plane* of the conoidal surface.

Theorem 11.3. *The conoidal surface whose generatrix intersects the line*
$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$
and the curve
$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$
and has the director plane $\pi = 0$, (π is not parallel to d and that $\mathcal{C}$ is not contained into π), is characterized by an equation of the form
$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{11.3}$$

Proof. An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are
$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu \pi_2 \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu \pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

By expressing λ and μ, one obtains (11.3). □

Example 11.3. Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

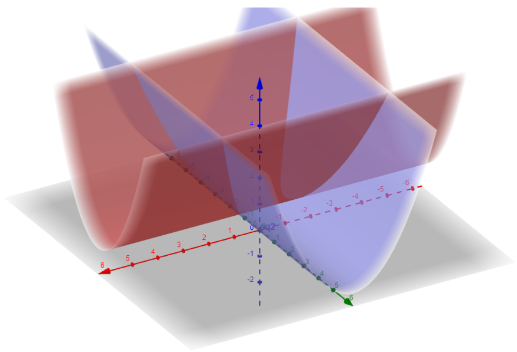
$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

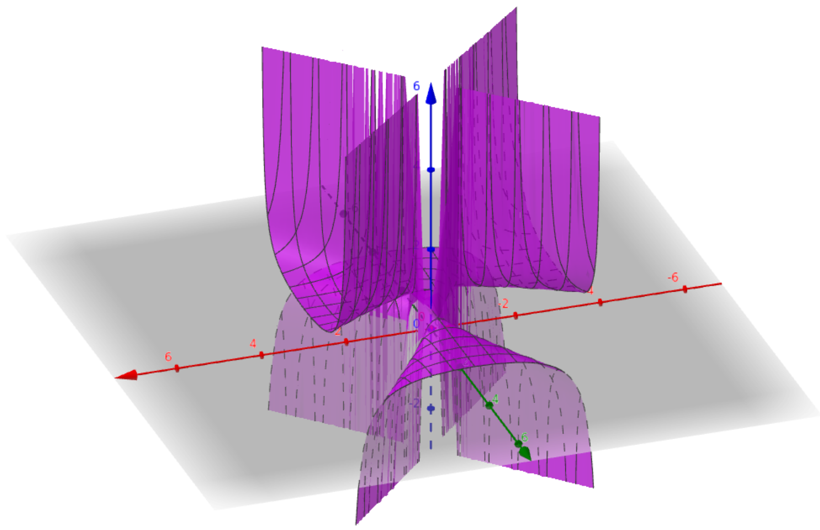
one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{x}{y}$ and $\mu = z$, the equation of the conoidal surface is

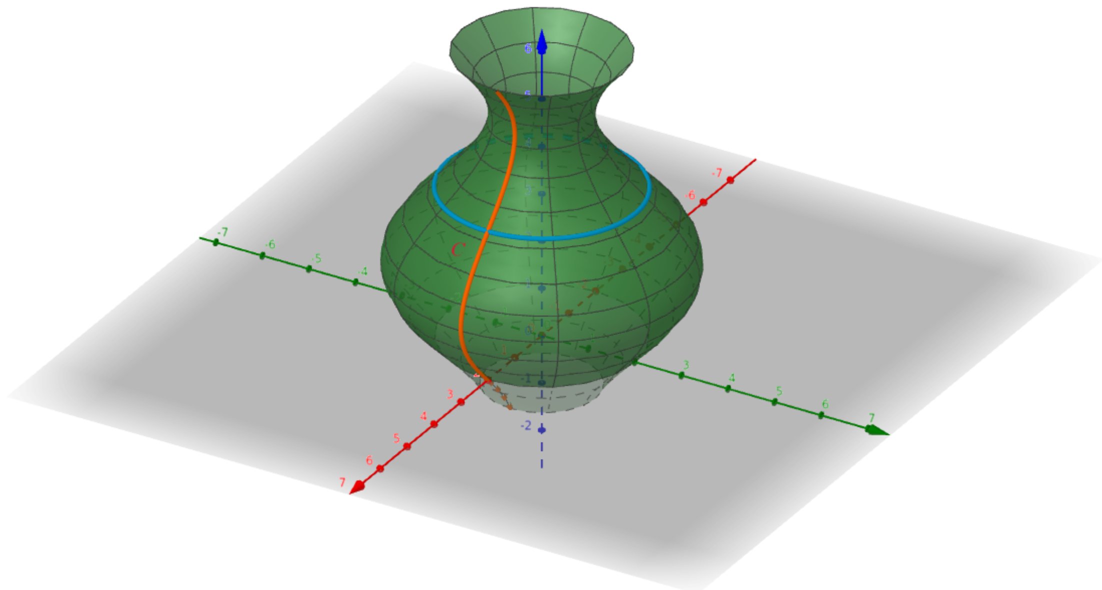
$$2x^2z - 2y^2z - 2x^2 + y^2 = 0. \tag{11.4}$$





11.4 Revolution Surfaces

Definition 11.4. The surface generated by rotating of a given curve $\mathcal{C}$ around a given line d is said to be a *revolution surface*.



Theorem 11.4. *The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \tag{11.5}$$

Proof. An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda. \\ px + qy + rz = \mu \end{cases}$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda,\mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (11.5). □

11.5 Problems

1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C) \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.

Solution.

2. A disk of radius 1 is centered at the point $A(1,0,2)$ and is parallel to the plane yOz. A source of light is placed at the point $P(0,0,3)$. Characterize analitically the shadow of the disk rushed over the plane xOy.

   Solution. Consider the conical surface of vertex P whose director curve is the circle of radius 1 which is centered at the point $A(1,0,2)$ and is parallel to the plane yOz. The shadow of the disk rushed over the plane xOy is the convex component of the complement, in the plane xOy, of the intersection curve between the plane xOy and the described conical surface.

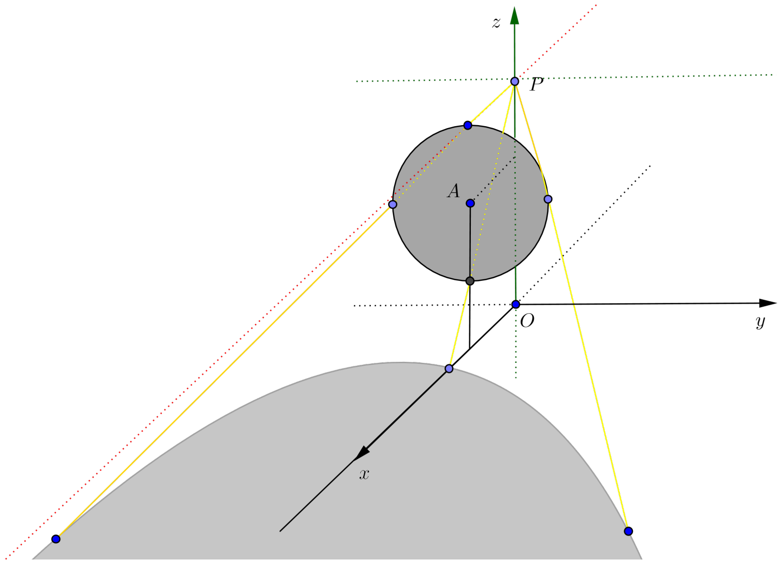
   In order to find the equation of the conical surface we consider the lines

$$(Oz)\left\{\begin{array}{l} x=0 \\ y=0 \end{array}\right. \text{ and } (d)\left\{\begin{array}{l} x=0 \\ z=3 \end{array}\right.$$

   as well as the family of lines

$$(\Delta_{\lambda\mu})\left\{\begin{array}{l} y-\lambda x=0 \\ z-3-\mu x=0 \end{array}\right.$$

depending on the parameters λ and μ of the reduced pencils of lines $x - \lambda y = 0$ through Oz and $z - 3 - \mu z = 0$ through d.



The circle C which borders the disk is given by the equations

$$(C) \begin{cases} (x-1)^2 + y^2 + (z-2)^2 = 1 \\ x = 1. \end{cases}$$

The intersection point of the line $\Delta_{\lambda\mu}$ with the plane of the circle is described by the system

$$\begin{cases} x = 1 \\ y - \lambda x = 0 \\ z - 3 - \mu x = 0 \end{cases}$$

which has the solution

$$(\Delta_{\lambda\mu} \cap (x=1)) \begin{cases} x = 1 \\ y = \lambda \\ z = 3 + \mu. \end{cases} \tag{11.6}$$

By imposing the condition on the intersection point (11.6) to belong the other surface which defines C, namely the sphere $(x-1)^2 + y^2 + (z-2)^2 = 1$, we obtain the relation $\lambda^2 + (\mu+1)^2 = 1$, between λ and μ, in order to have concurrence between $\Delta_{\lambda\mu}$ and C. The equation of the conical surface is

$$\left(\frac{y}{x}\right)^2 + \left(\frac{z-3}{x} + 1\right)^2 = 1, \text{ or } y^2 + (x+z-3)^2 = x^2.$$

The latter equation is equivalent with

$$y^2 + z^2 + 2xz - 6x - 6z + 9 = 0.$$

Its intersection curve with the plane xOy is the parabola

$$(\mathcal{P} \begin{cases} z = 0 \\ y^2 - 6x + 9 = 0. \end{cases}$$

The convex compoennt of the compelment $xOy \setminus \mathcal{P}$ coincides with the required shadow and is characterized by the following system

$$\begin{cases} y^2 - 6x + 9 & \leq 0 \\ z = 0. \end{cases}$$

3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line (d) and the circle (C) and remains orthogonal to (d). (The Willis conoid)

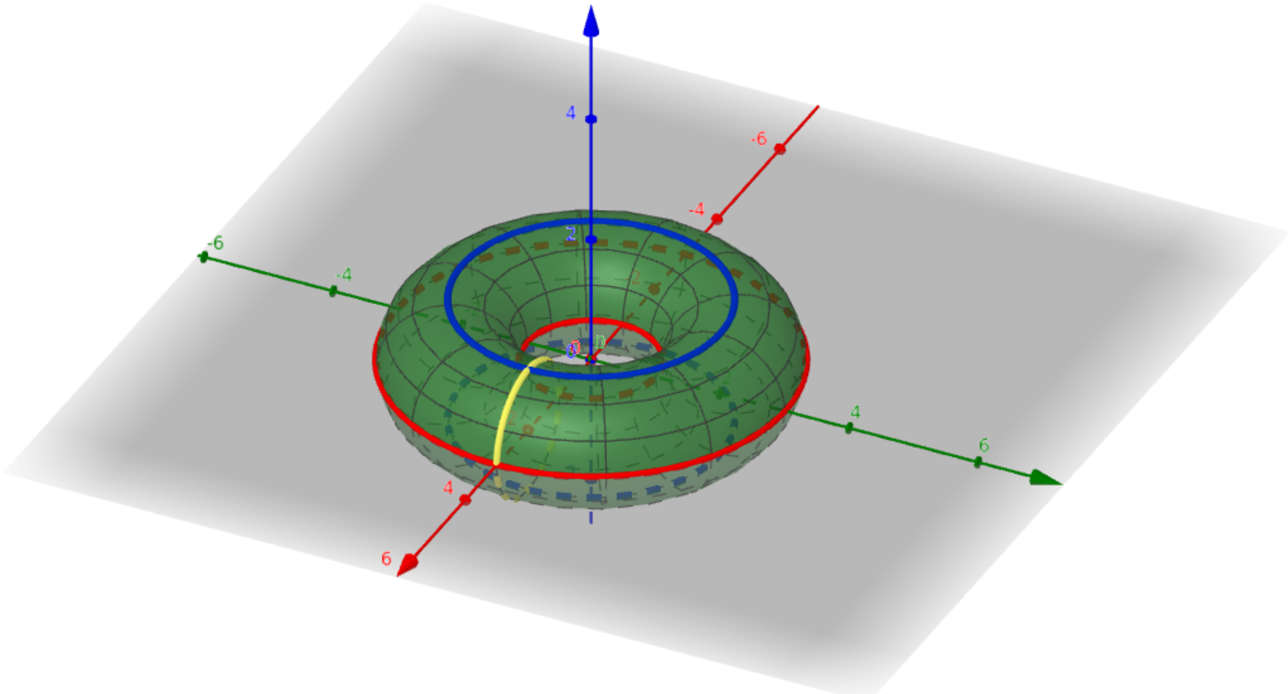
   Solution.

4. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

   Solution.

5. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus[5]

[5]The torus is a regular surface.



Solution.

12 Week 12. Transformations

12.1 Transformations of the plane

Definition 12.1. An *affine transformation* of the plane is a perturbation by a translation of a linear transformation, i.e.

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax + by + c, dx + ey + f), \tag{12.1}$$

for some constant real numbers a, b, c, d, e, f.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y) = [x\ y] \begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \\ &= [L]\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix},\ \text{where } [L] = \begin{bmatrix} a & b \\ d & e \end{bmatrix}. \end{aligned}$$

Lemma 12.1. *If $(aB - bA)^2 + (dB - eA)^2 > 0$, then the affine transformation (14.1) maps the line*

$$(d)\ Ax + By + C = 0$$

to the line

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

If $aB - bA = dB - eA = 0$, then $ae - bd = 0$ and $L\big|_d$ is the constant map $\left(\frac{cB-bC}{B}, \frac{fB-eC}{B}\right)$.

Definition 12.2. An affine transformation (14.1) is said to be *singular* if

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

12.1.1 Translations

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 12.3. The *translation* of vector $(h,k) \in \mathbb{R}^2$ is the affine transformation

$$T(h,k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ [T(h,k)]\,(x,y) = (x + h, y + k).$$

Thus

$$[T(h,k)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + h \\ y + k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h,k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the translation $T(h,k)$ is non-singular (invertible) and $(T(h,k))^{-1} = T(-h,-k)$.

12.1.2 Scaling about the origin

Definition 12.4. The *scaling about the origin* by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is the affine transformation

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ [S(s_x, s_y)]\,(x, y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$\left[S(s_x, s_y)^c\right]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[S(s_x, s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

12.1.3 Reflections

Definition 12.5. The *reflections about the x-axis* and the y-axis respectively are the affine transformation

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ r_x(x, y) = (x, -y), \ r_y = (-x, y).$$

Thus

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ Similarly } [r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1, 1)$ and $r_y = S(1, -1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 12.6. The *reflection* $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the line* l maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is

$$r_l(x, y) = \left(\frac{b^2 - a^2}{a^2 + b^2}x - \frac{2ab}{a^2 + b^2}y - \frac{2ac}{a^2 + b^2}, -\frac{2ab}{a^2 + b^2}x + \frac{a^2 - b^2}{a^2 + b^2}y - \frac{2bc}{a^2 + b^2}\right).$$

Thus

$$\begin{aligned} [r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y - \dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2}x + \dfrac{a^2 - b^2}{a^2 + b^2}y - \dfrac{2bc}{a^2 + b^2} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \dfrac{2ac}{a^2 + b^2} \\ \dfrac{2bc}{a^2 + b^2} \end{bmatrix}, \end{aligned}$$

i.e.

$$[r_l] = \frac{1}{a^2 + b^2}\begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

12.1.4 Rotations

Definition 12.7. The *rotation* $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin through an angle θ maps a point $M(x,y)$ into a point $M'(x',y')$ with the properties that the segments $[OM]$ and $[OM']$ are congruent and the $m(\widehat{MOM'}) = \theta$. If $\theta > 0$ the rotation is supposed to be *anticlockwise* and for $\theta < 0$ the rotation is *clockwise*. If $(x,y) = (r\cos\varphi, r\sin\varphi)$, then the coordinates of the rotated point are$\big(r\cos(\theta+\varphi), r\sin(\theta+\varphi)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, i.e.

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Thus

$$\begin{aligned}[R^c_\theta]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e.

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that the rotation R_θ is non-singular (invertible) and $R_\theta^{-1} = R_{-\theta}$.

12.1.5 Shears

Definition 12.8. Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines d with direction v and the oriented distance δ from the origin. The *shear* about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x,y)$ on d to the point $M' = M + r\delta v$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distacnce δ from the origin to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v,r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is

$$\begin{aligned}Sh(v,r)(x,y) &= (x,y) + r\delta(v_1, v_2) \\ &= (x,y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x,y) + \big(-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y\big) \\ &= \big((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y\big)\end{aligned}$$

Thus

$$\begin{aligned}[Sh(v,r)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (1 - rv_1v_2)x + rv_1^2y \\ -rv_2^2x + (1 + rv_1v_2)y \end{bmatrix} \\ &= \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e. $[Sh(v,r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}$.

12.2 Problems

1. Find the image of the triangle ABC through the reflection in the line (d) $x - y = 2$, where $A(-1, 2)$, $B(-2, -1)$ and $C(3, 3)$.

   Solution.

2. Find the image of the triangle ABC through the clockwise rotation of angle 30°, where $A(6,4)$, $B(6,2)$ and $C(10,6)$.

   Solution.

3. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5},1/\sqrt{5}\right),1.5\right)$.

   Solution.

4. Let $M(x, y)$ be a mobile point on the ellipse $(\mathcal{E})$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the locus of centroids of the triangles MFF', where F and F' are the foci of the ellipse, is the image through a scalling of equal factors (a homothety) of the given ellipse $\mathcal{E}$. Find the equation of the locus..

   Solution.

5. Consider the line $(d)\ ax + by + c = 0$ and the points $A, B \notin d$. Find the coordinates of the point $M \in d$ such that $\operatorname{dist}(A, M) + \operatorname{dist}(M, B)$ is minimal.

   Solution.

13 Week 13

13.1 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax+by+c, dx+ey+f)$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the line matrices $[x\ y] \in \mathbb{R}^{1\times 2}$ and implicitely $\mathbb{R}^2$ with $\mathbb{R}^{1\times 2}$:

$$L[x\ y] = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

   (b) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and implicitely $\mathbb{R}^2$ cu $\mathbb{R}^{2\times 1}$:

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

2. $\begin{cases} x' = ax+by+c \\ y' = dx+ey+f. \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$

   Observe that the representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

In this lesson we identify the points $(x,y) \in \mathbb{R}^2$ with the points $(x,y,1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx, ry, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x,y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ r \end{bmatrix},\ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x,y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x,y,w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x,y,w) \in \mathbb{RP}^2$, $w \neq 0$ şi $\left(\frac{x}{w}, \frac{y}{w}, 1\right)$represent the same element of $\mathbb{RP}^2$.

Remark 13.1. The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^3 \setminus \{0\}$:

$$(x,y,w) \sim (\alpha,\beta,\gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x,y,w) = r(\alpha,\beta,\gamma).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$. By the column matrix

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

we also denote the equivalence class of $(x,y,w) \in \mathbb{R}^3 \setminus \{0\}$. The meaning of this notation will be understood, each time, from the context.

Definition 13.1. A *projective transformation* of the projective plane $\mathbb{RP}^2$ is a transformation

$$L : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2,\ L\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax+by+cw \\ dx+ey+fw \\ gx+hy+kw \end{bmatrix}, \tag{13.1}$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. Note that the projective transformation L is defined by its *homogeneous transformation matrix*

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

Observe that a projective transformation (14.2) is well defined since

$$L\begin{bmatrix} rx \\ ry \\ rw \end{bmatrix} = \begin{bmatrix} arx+bry+crw \\ drx+ery+frw \\ grx+hry+krw \end{bmatrix} = \begin{bmatrix} r(ax+by+cw) \\ r(dx+ey+fw) \\ r(gx+hy+kw) \end{bmatrix}.$$

If $g = h = 0$ and $k \neq 0$, then the projective transformation (14.2) is said to be *affine*. The restriction of the affine transformation (14.2), which corresponds to the situation $g = h = 0$ and $k = 1$, to the subspace $w = 1$, has the form

$$L\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cw \\ dx+ey+fw \\ 1 \end{bmatrix}, \tag{13.2}$$

i.e.

$$\begin{cases} x' = ax+by+c \\ y' = dx+ey+f. \end{cases} \tag{13.3}$$

Remark 13.2. If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2)\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Remark 13.3. If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.

13.2 Transformations of the plane in homogeneous coordinates

In this section we shall identify an affine transformation of $\mathbb{RP}^2$ with its homogeneous transformation matrix

13.3 Translations and scalings

• The homogeneous transformation matrix of the translation $T(h,k)$ is

$$T(h,k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13.4 Reflections

• The homogeneous transformation matrix of reflection r_x about the x-axis is

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_y about the y-axis is

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_l about the line $l : ax + by + c = 0$ is

$$r_l = \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} & -\dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} & -\dfrac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Example 13.1. Consider a line (d) $ax + by + c$ whose slope is $\text{tg}\theta = -\dfrac{a}{b}$. By using the observation that the reflection r_d in the line d is the following concatenation (product)

$$T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b),$$

one can show that the homogeneous transformation matrix of r_d is

$$\begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} & -\dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} & -\dfrac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. The homogeneous matrix of the concatenation

$$T(0,-c/b)\circ R_\theta \circ r_x \circ R_{-\theta}\circ T(0,c/b)$$

is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\sin\theta\cos\theta & 2\frac{c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta & \frac{c}{b}\left(\sin^2\theta-\cos^2\theta-1\right) \\ 0 & 0 & 1 \end{bmatrix}. \tag{13.4}$$

Since $\mathrm{tg}\theta=-\frac{a}{b}$, it follows that $\frac{a^2}{b^2}=\frac{\sin^2\theta}{\cos^2\theta}=\frac{\sin^2\theta}{1-\sin^2\theta}=\frac{1-\cos^2\theta}{\cos^2\theta}$, namely

$$\sin^2\theta=\frac{a^2}{a^2+b^2} \text{ and } \cos^2\theta=\frac{b^2}{a^2+b^2}.$$

Thus

$$\sin\theta=\pm\frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos\theta=\mp\frac{b}{\sqrt{a^2+b^2}}, \text{ as } \frac{\sin\theta}{\cos\theta}=\mathrm{tg}\theta=-\frac{a}{b}.$$

Therefore $\sin\theta\cos\theta=-\frac{ab}{a^2+b^2}$ and the matrix (13.4) becomes

$$\begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{c}{b}\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{c}{b}\left(\frac{a^2-b^2}{a^2+b^2}-1\right) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

13.5 Rotations

The homogeneous transformation matrix of the rotation R_θ about the origin through an angle θ is

$$R_\theta=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 13.2. The homogeneous transformation matrix of the product (concatenation) $T(h,k)\circ R_\theta$ is the product

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to find the homogeneous transformation matrix of the inverse transformation

$$(T(h,k)\circ R_\theta)^{-1}=R_\theta^{-1}\circ T(h,k)^{-1}=R_{-\theta}\circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k)\circ R_\theta$ we can either multiply the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1}=R_\theta^{-1}$ and $T(h,k)^{-1}=$

$T(-h,-k)$ or use the next proposition. The product of the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ is

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 13.1. *A homogeneous transformation L is invertible if and only if its homogeneous transformation matrix, say T, is invertible and T^{-1} is the transformation matrix of L^{-1}.*

Proof. Suppose that L has an inverse L^{-1} with transformation matrix T_1. The product transformation $L \circ L^{-1} = id$ has the transformation matrix $TT_1 = I_3$. Similarly, $L^{-1} \circ L = I_3$ has the transformation matrix $T_1T = I_3$. Thus $T_1 = T^{-1}$. Conversely, assume that T has an inverse T^{-1}, and let L_1 be the homogeneous transformation defined by T^{-1}. Since $TT^{-1} = I_3$ and $T^{-1}T = I_3$, it follows that $L \circ L_1 = I$ and $L_1 \circ L = I$. Hence L_1 is the inverse transformation of L.

Example 13.3. The homogeneous transformation matrix of inverse

$$(T(h,k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h,k)^{-1} = R_{-\theta} \circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k) \circ R_\theta$ is the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

13.6 Shears

The homogeneous transformation matrix of the shear is

$$[Sh(v,r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1v_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13.7 Problems

1. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)

   Solution

2. Find the homogeneous matrix of the product (concatenation) $S(3,2) \circ R_{\frac{3\pi}{2}}$.

   Solution

3. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ.

   Solution The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned}
R_\theta(x_0,y_0) &= T(x_0,y_0)R_\theta T(-x_0,-y_0) \\
&= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

   Thus, the equations of the required rotation are:

$$\begin{cases} x' &= x\cos\theta - y\sin\theta - x_0\cos\theta + y_0\sin\theta + x_0 \\ y' &= x\sin\theta + y\cos\theta - x_0\sin\theta - y_0\cos\theta + y_0. \end{cases}.$$

4. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

   Solution

14 Week 14

14.1 Transformations of the space

Definition 14.1. An affine transformation of the spacee is a perturbation bya atranslation of a linear transformation, i.e.

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n), \tag{14.1}$$

for some constant real numbers a,b,c,d,e,f,g,h,k,l,m,n.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y,z) = [x\ y\ z]\begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

Definition 14.2. An affine transformation (14.1) is said to be *singular* if

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix} = 0.$$

and non-singular otherwise.

14.1.1 Translations

The *translation of* $\mathbb{R}^3$ *of vector* $(h,k,l) \in \mathbb{R}^3$ is the affine transformation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l).$$

Its associated transformation is

$$T(h,k,l)^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ T(h,k,l)^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} h \\ k \\ l \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[T(h,k,l)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 + h \\ w_2 = x_2 + k \\ w_3 = x_3 + l \end{cases}.$$

14.1.2 Scaling about the origin

The *scaling about the origin* by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is the affine transformation

$$S(s_x, s_y, s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x, s_y, s_z)\right](x, y, z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z).$$

Thus

$$\left[S(s_x, s_y, s_z)^c\right]\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

i.e.

$$[S(s_x, s_y, s_z)] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is non-singular (invertible) and $(S(s_x, s_y, s_z))^{-1} = S(s_x^{-1}, s_y^{-1}, s_z^{-1})$.

14.1.3 Reflections about planes

1. The *reflection of* $\mathbb{R}^3$ *through the xy-plane is* $r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Its associated transformation is

$$r_{xy}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xy}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = x_2 \\ w_3 = -x_3 \end{cases}.$$

2. The *reflection of* $\mathbb{R}^3$ *through the xz-plane is* $r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$. Its associated transformation is

$$r_{xz} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xz}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = -x_2 \\ w_3 = x_3 \end{cases}.$$

3. The *reflection of* $\mathbb{R}^3$ *through the yz-plane is* $r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Its associated transformation is

$$r_{yz}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{yz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{yz}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = -x_1 \\ w_2 = x_2 \\ w_3 = x_3 \end{cases}.$$

4. The *reflection of* $\mathbb{R}^3$ *through an arbitrary plane* $\pi : ax_1 + bx_2 + cx_3 + d = 0$ *is* $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$r_\pi(x,y,z) = \left(\frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2}, \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2}, \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \right).$$

Its associated transformation $r_\pi : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ is given by

$$\begin{aligned} r_\pi^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} \frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2}\left(\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2d\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right). \end{aligned}$$

which shows that its standard matrix and equations are:

$$[r_\pi] = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

and

$$\begin{cases} w_1 = \frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ w_2 = \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ w_3 = \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{cases}$$

14.1.4 Rotations

The *rotation operator of* $\mathbb{R}^3$ *through a fixed angle* θ *about an oriented axis*, rotates about the axis of rotation each point of $\mathbb{R}^3$ in such a way that its associated vector sweeps out some portion of the cone determine by the vector itself and by a vector which gives the direction and the orientation of the considered oriented axis. The angle of the rotation is measured at the base of the cone and it is measured clockwise or counterclockwise in relation with a viewpoint along the axis looking toward the origin. As in $\mathbb{R}^2$, the positives angles generates counterclockwise rotations and negative angles generates clockwise roattions. The counterclockwise sense of rotaion can be determined by the right-hand rule: If the thumb of the right hand points the direction of the direction of the oriented axis, then the cupped fingers points in a counterclockwise direction. The rotation operators in $\mathbb{R}^3$ are linear.

For example

1. The counterclockwise rotation about the positive x-axis through an angle θ has the equations

$$\begin{aligned} w_1 &= x \\ w_2 &= y\cos\theta - z\sin\theta \\ w_3 &= y\sin\theta + z\cos\theta \end{aligned},$$

its standard matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

2. The counterclockwise rotation about the positive y-axis through an angle θ has the equations

$$\begin{array}{lcl} w_1 & = & x\cos\theta + z\sin\theta \\ w_2 & = & y \\ w_3 & = & -x\sin\theta + z\cos\theta \end{array},$$

its standard matrix is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

3. The counterclockwise rotation about the positive z-axis through an angle θ has the equations

$$\begin{array}{lcl} w_1 & = & x\cos\theta - y\sin\theta \\ w_2 & = & x\sin\theta + y\cos\theta \\ w_3 & = & z \end{array},$$

its standard matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14.2 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n),$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the line matrices $[x\ y\ z] \in \mathbb{R}^{1\times 3}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{1\times 3}$. With this identification, the action of L is given by

$$L[x\ y\ z] = [x\ y\ z] \begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

   (b) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the column matrices $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^{3\times 1}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{3\times 1}$. We de note by $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ the associated map via this identification, and its action is given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

2. $$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this section we identify the points $(x, y, z) \in \mathbb{R}^3$ with the points $(x, y, z, 1) \in \mathbb{R}^4$ and even with the punctured lines of $\mathbb{R}^4$, (rx, ry, rz, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y, z) \in \mathbb{R}^3$ with the punctured lines of $\mathbb{R}^4$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ rz \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y, z) \in \mathbb{R}^3$. The set of homogeneous coordinates (x, y, z, w) will be denoted by $\mathbb{RP}^3$ and call it the real *projective space*. The homogeneous coordinates $(x, y, z, w) \in \mathbb{RP}^3$, $w \neq 0$ and $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$ represent the same element of $\mathbb{RP}^3$.

Remark 14.1. The projective space $\mathbb{RP}^3$ is actually the quotient set $(\mathbb{R}^4 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^4 \setminus \{0\}$:

$$(x, y, z, w) \sim (\alpha, \beta, \gamma, \delta) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, z, w) = r(\alpha, \beta, \gamma, \delta).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^3$. By the column matrix

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

we also denote the equivalence class of $(x, y, z, w) \in \mathbb{R}^3 \setminus \{0\}$. The meaning of this notation will be understood, each time, from the context.

Definition 14.3. A *projective transformation* of the projective space $\mathbb{RP}^3$ is a transformation

$$L : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3, \ L \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \\ kx + ly + mz + nw \\ px + qy + rz + sw \end{bmatrix}, \tag{14.2}$$

where $a, b, c, d, e, f, g, h, k, l, m, n, p, q, r, s \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}$$

is called the *homogeneous transformation matrix* of L.

Observe that a projective transformation (14.2) is well defined since

$$L\begin{bmatrix} tx \\ ty \\ tz \\ tw \end{bmatrix} = \begin{bmatrix} atx+bty+ctz+dtw \\ etx+fty+gtz+htw \\ ktx+lty+mtz+ntw \\ ptx+qty+rtz+tsw \end{bmatrix} = \begin{bmatrix} t(ax+by+cz+dw) \\ t(ex+fy+gz+hw) \\ t(kx+ly+mz+nw) \\ t(px+qy+rz+sw) \end{bmatrix}.$$

If $p=q=r=0$ and $s\neq 0$, then the projective transformation (14.2) is said to be *affine*. The restriction of the affine transformation (14.2), which corresponds to the situation $p=q=r=0$ and $s=1$, to the subspace $w=1$, has the form

$$L\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \\ 1 \end{bmatrix}, \tag{14.3}$$

i.e.

$$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n. \end{cases} \tag{14.4}$$

Remark 14.2. If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

then

$$(L_1\circ L_2)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Remark 14.3. If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.

14.3 Transformations of the space in homogeneous coordinates

14.3.1 Translations

The homogeneous transformation matrix of the translation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l)$$

is

$$\begin{bmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

14.3.2 Scaling about the origin

The homogeneous transformation matrix of the scaling

$$S(s_x, s_y, s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x, s_y, s_z)\right](x, y, z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z)$$

is

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

14.3.3 Reflections about planes

1. The homogeneous transformation matrix of the reflection

$$r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$$

of $\mathbb{R}^3$ through the xy-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The homogeneous transformation matrix of the reflection

$$r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$$

of $\mathbb{R}^3$ through the yz-plane is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The homogeneous transformation matrix of the reflection

$$r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

of $\mathbb{R}^3$ through the xz-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. The homogeneous transformation matrix of the reflection $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} r_\pi(x, y, z) = \Bigg(& \frac{(-a^2 + b^2 + c^2)x - 2aby - 2acz - 2ad}{a^2 + b^2 + c^2}, \\ & \frac{-2abx + (a^2 - b^2 + c^2)y - 2bcz - 2bd}{a^2 + b^2 + c^2}, \\ & \frac{-2acx - 2bcy + (a^2 + b^2 - c^2)z - 2cd}{a^2 + b^2 + c^2} \Bigg). \end{aligned}$$

through an arbitrary plane $\pi : ax_1 + bx_2 + cx_3 + d = 0$ is

$$\begin{bmatrix} \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} & \frac{-2ab}{a^2+b^2+c^2} & \frac{-2ac}{a^2+b^2+c^2} & \frac{-2ad}{a^2+b^2+c^2} \\ \frac{-2ab}{a^2+b^2+c^2} & \frac{a^2-b^2+c^2}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{-2bd}{a^2+b^2+c^2} \\ \frac{-2ac}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{a^2+b^2-c^2}{a^2+b^2+c^2} & \frac{-2cd}{a^2+b^2+c^2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2 + c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac & -2ad \\ -2ab & a^2-b^2+c^2 & -2bc & -2bd \\ -2ac & -2bc & a^2+b^2-c^2 & -2cd \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{bmatrix}.$$

14.3.4 Rotations

1. The homogeneous transformation matrix of the counterclockwise rotation about the positive x-axis through an angle θ is
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. The homogeneous transformation matrix of the counterclockwise rotation about the positive y-axis through an angle θ is
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The homogeneous transformation matrix of the counterclockwise rotation about the positive z-axis through an angle θ is
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

14.4 Problems

1. Find the homogeneous transformation matrix of the product (concatenation)
$$T(1,1,-2) \circ Rot_y(\pi/6),$$
where $Rot_y(\pi/6)$ stands for the rotation about the positive y-axis through an angle θ.

2. Find the homogeneous transformation matrix of the rotation through an angle θ, of the space, about an arbitrary line.

3. Find the homogeneous transformation matrix of the rotation through an angle θ about the line PQ, where $P(2,1,5)$ and $Q(4,7,2)$.

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