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Constructions of Multialgebras

PhD Thesis Summary

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2003

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Introduction

The first steps in the field of multialgebras were made in the 1930s in a paper presented by the French mathematician F. Marty at the 8th Congress of the Scandinavian Mathematicians (1934). In this paper, the author presented a generalization for groups, called hypergroup, as well as some of its properties. From the very next articles of Marty, one could foresee that these objects can be used as tools in other mathematical theories. Some of the connections between hyperstructures and other areas of mathematics (connections that I noticed while preparing this thesis) concern rational fractions (Marty: [49]), ordered sets (Benado: [1], [2], Călugareanu and Leoreanu: [8]), character tables of finite groups (Roth: [72], [74], McMullen and Price: [51]), binary relations (Rosenberg: [71], Corsini: [11], Corsini and Leoreanu: [14]), fuzzy sets (Corsini and Leoreanu: [12], [15], Leoreanu: [44]), abstract data types (Walicki and Meldal: [89]). I mention that the weak Cayley table, used by Johnson, Mattarei and Sehgal in [37] for determining 1 and 2-characters of a finite group, is an example of multiloop (although the authors do not specify it). Our perspective on the connections between multialgebras and other fields of research is completed by P. Corsini and V. Leoreanu in [16].

The first multialgebras that were studied are hypergroupoids, and semihypergroups and hypergroups. Later, there also appeared results concerning other hyperstructures such as hyperrings, hypermodules, hyperlattices. The Romanian mathematician Mihail Benado had a major contribution to the study of hyperlattices. Gratzer's and Pickett's papers ([27]) and ([67] respectively) are extremely important for the theory of multialgebras. In these works, multialgebras are seen as particularizations of the relational systems which generalize the notion of universal algebra and the results obtained here place multialgebras right next to the universal algebras. This is also proved by some of the properties established here, which extend some of the results that are already known for universal algebras. Within the framework defined by Grätzer and Pickett also lie works such as [36] (Höft and Howard), [33], [34] (Hansoul), [75] (Schweigert), [90] (Walicki and Białasik) and the more recent [4] (Breaz and Pelea), [59], [60], [61] (Pelea). We should notice that in the above mentioned papers of Hansoul, Walicki and Białasik the notion of multioperation does not include the fact that its images are nonempty sets. Multialgebras which have such multioperations are closer to the relational systems than to the universal algebras and in their case Grätzer's representation theorem ([27]) does not apply anymore.

Like in other theories, in the theory of multialgebras it is very important to obtain new objects starting from given objects and this is what the constructions of multialgebras do. The simplest constructions are the formation of submultialgebras and of factor multialgebras. The latter have been studied from the outset of this theory and this is not surprising because the first hypergroups emerged as a result of the factorization of a group modulo an equivalence relation determined by a subgroup and, later, G. Grätzer proved that any multialgebra can be obtained by an appropriate factorization of a universal algebra modulo an equivalence relation. Our research concerns a field where there already existed results on direct and subdirect products (of multialgebras), as well as some properties on direct limits of direct systems and inverse limits of inverse systems of particular hyperstructures. Among the constructions studied in this thesis, we mention submultialgebras and especially the submultialgebra generated by a subset, the factor multialgebras, the direct products, the direct limits of direct systems and the inverse limits of inverse systems of multialgebras.

The first chapter begins with an introductory paragraph where we present the notion of multialgebra and a few particular multialgebras which will be used along the thesis. Starting from a $\mathfrak{A}$ multialgebra, Pickett introduces in [67] a structure of universal algebra $\mathfrak{P}^*(\mathfrak{A})$ on the set $P^*(A)$ of the nonempty subsets of A. In the second paragraph of Chapter 1, we remind the way that are defined polynomial functions and the term functions of a universal algebra, as well as the way that the latter can be obtained from terms. This discussion occurs in the case of $\mathfrak{P}^*(\mathfrak{A})$, which allows the introduction of particular algebraic functions that prove to be useful in paragraphs 1.6 and 1.8.

The third paragraph concerns the submultialgebras of a multialgebra. In this paragraph, the original contribution is the characterization of the submultialgebra generated by a subset in a multialgebra (Theorem 1.3.13). Pickett noticed that, given a subset B of the support set A of a multialgebra $\mathfrak{A}$, B is a submultialgebra of the multialgebra $\mathfrak{A}$ if and only if $P^*(B)$ is a submultialgebra for the algebra $\mathfrak{P}^*(\mathfrak{A})$. Starting from this, we can write the submultialgebra generated by a subset X of a multialgebra $\mathfrak{A}$ as a union of all the images of one element subsets of X through term functions of $\mathfrak{P}^*(\mathfrak{A})$ (Theorem 1.3.13).

There exist several generalizations of the notion of homomorphism for multialgebras. We only use two of them: one that is called homomorphism (which results from considering the multialgebras as relational systems) and another one called ideal homomorphism (whose definition is analogous to the definition of universal algebra homomorphism). We used the fact that the ideal homomorphisms of multialgebras determine and are determined by a special class of equivalence relations defined on multialgebras, namely ideal equivalences (Pickett ([67])). In the same paper, Pickett mentions a connection between ideal homomorphisms of multialgebras and certain homomorphisms of the corresponding nonempty algebras of subsets. Whether by using this connection or directly, in paragraph 1.4 we established connections of the homomorphism between two multialgebras $\mathfrak{A}$, $\mathfrak{B}$ with the term functions of the universal algebras $\mathfrak{P}^*(\mathfrak{A})$, $\mathfrak{P}^*(\mathfrak{B})$. Thus, in Proposition 1.4.9 and Corollary 1.4.14 we proved that for these term functions and for the homomorphisms used here certain properties similar to the those established for universal algebras hold. We also showed that the ideal equivalences of a multialgebra $\mathfrak{A}$ are in close connection with certain congruences of the algebra $\mathfrak{P}^*(\mathfrak{A})$ (Theorem 1.4.5), which allows a characterization of the ideal equivalences of a multialgebra (Proposition 1.4.7). The original results in paragraph 1.3 and 1.4 were obtained together with Simion Breaz and published in [4].

One of the most important results concerning multialgebras is G. Grätzer's characterization theorem ([27]), which proves that the study of multialgebras is a natural extension of the theory of universal algebras. The last paragraphs of Chapter 1 come within this context. One of the problems suggested by Grätzer in [27] is the following: *What are the factor multialgebras of a group, abelian group, lattice, ring and so on? Characterize these with a suitable axiom system.*. The fact that these particular universal algebras are defined by identities made us wonder what happens with the identities of an algebra after the factorization modulo an equivalence relation. If we study the definitions of the hyperstructures presented in P. Corsini's ([10]) and T. Vougiouklis's ([85], [86], [87], [88]) works — some of them also presented in the first paragraph of our thesis — we can see that it is necessary to adapt the notion of identity from universal algebras to multistructures. Thus, we introduce two types of identities for multialgebras: (strong) identities and weak identities. One answer to the first part of Grätzer's problem results immediately by establishing that the identities of a universal algebra usually become weak identities on the factor algebra. The identity of the algebra, as well as the equivalence relation we are dealing with can make the factor multialgebra satisfy this identity in a strong manner. In paragraph 1.5, we present a series of remarks which confirm these statements.

It is known that by the factorization of a universal algebra modulo a congruence that includes a relation we obtain a universal algebra in which any two elements in the given relation determine the same class. Then, our study tries to prove that the factorization of a universal algebra modulo an equivalence relation — which gave rise to multialgebras — can be seen as an "intermediate step" of such a factorization. This leads to the study of certain (ideal) equivalences which have the property that the factor multialgebras they determine are universal algebras. Such equivalences appear in the literature from the very first papers on hypergroups (Dresher, Ore ([19]) and Ore, Eaton ([22])). A series of important works about these equivalences of the hypergroupoids, semihypergroups, hypergroups, hyperrings and other particular multistructures have been published after 1990 and converge toward the study of the smallest equivalences of this kind. In Proposition 1.6.1. we gave a characterization of the equivalences of a $\mathfrak{A}$ multialgebra for which the factor multialgebra is a universal algebra and in Theorem 1.6.13 we determined the smallest equivalence $\alpha^*_{\mathfrak{A}}$ on A that has this property. If we apply Theorem 1.6.13 to semihypergroups, hypergroups and hyperrings, we obtain the fundamental relation of these hyperstructures (studied for example by (Corsini ([10]), Freni ([25]), Guţan ([31]), Vougiouklis ([83])). This is why we called the relation $\alpha^*_{\mathfrak{A}}$ a fundamental relation for the $\mathfrak{A}$ multialgebra, too. We called the factor universal algebra it determines a fundamental algebra. Using Theorem 1.6.8, where we proved that any homomorphism between two multialgebras induces a homomorphism between the corresponding fundamental algebras, we defined a covariant functor from the category of multialgebras of a given type into the category of the universal algebras of the same type (Observation 1.6.21).

Naturally, we get to the following question: what happens with the (strong or weak) identities of a multialgebra after their factorization modulo the fundamental relation? In Proposition 1.7.1, we conclude that they become identities of the universal algebra obtained by factorization. It is easy to notice that an identity of the fundamental algebra does not have to originate in an identity, be it weak, of the given multialgebra. However, we established a class of multialgebras — complete multialgebras — which have the property that any identity of the fundamental algebra is verified, at least in a weak manner, on the original multialgebra — see the proof of Proposition 1.7.6 and Proposition 1.7.11.

In the last paragraph of Chapter 1, we determined the smallest equivalence for which the factor multialgebra is a universal algebra for which a given identity is verified (Theorem 1.8.3). By applying this theorem to the case of (semi)hypergroups and to the identity which expresses the commutativity of the hyperproduct, one finds the relation introduced by Freni in [26] in order to obtain a characterization of the derived hypergroup of a hypergroup. In this thesis, we proved that the multialgebra which results from a universal algebra $\mathfrak{B}$ by factorization modulo an equivalence relation ρ becomes, after the factorization by the relation introduced by us isomorphic to the same algebra as the one obtained as a factor algebra of the universal algebra $\mathfrak{B}$ modulo the smallest congruence that contains the equivalence ρ, as well as the pairs of elements of $\mathfrak{B}$ which become equal thanks to the identity we are using (Theorem 1.8.9). Using Theorem 1.8.9, we established a connection between the derived subgroup of a group and the derived subhypergroup of its factor hypergroup modulo an equivalence relation determined by a subgroup (Example 1.8.10).

The results presented in the paragraphs 1.5, 1.6, 1.7, 1.8 are original and have been published, accepted for publication or submitted for publication. Thus, the main result in paragraph 1.6 (Theorem 1.6.13) was published in [59] and the final part of paragraph 1.6 and paragraphs 1.7 and 1.8 constitute a paper written with Professor Ioan Purdea and which is still a preprint ([66]).

In Chapter 2, we study certain properties concerning direct products, direct limits of direct systems and inverse limits of multialgebras. All these constructions are generalizations of the properties established for universal algebras and they are natural since one obtains objects of the category of multialgebras corresponding to the similar constructions in the category theory. We mention that in [90] Walicki and Białasik obtained results concerning categorical constructions of multialgebras. In this paper, the authors proved that multialgebras together with homomorphisms form a category with finite products, equalizers, finite coproducts , coequalizers, and, consequently, with finite limits and colimits. But the construction of the equalizers and of the coproducts uses very much the fact that the image of a multioperation can be empty, so the possibility of transferring these properties to categories whose objects are the multialgebras characterized by Grätzer's theorem becomes uncertain. As an example supporting this statement, we underline the fact that the multialgebras of type $\tau = (n_\gamma)_{\gamma<o(\tau)}$ form a subcategory in the category of the relational systems of type $(n_\gamma + 1)_{\gamma<o(\tau)}$ which is not closed under inverse limits.

Chapter 2 is a collection of results belonging to the author of the thesis. In the first paragraph, we present a series of properties of the direct product of multialgebras, most of them published in [61]. In [90], Walicki and Białasik proved that for two multialgebras, the direct product is the product in the category of multialgebras. Without significantly modifying the proof, this property can be proved for arbitrary families of multialgebras (Proposition 2.1.1). We showed that the direct product satisfies the identities verified by the given multialgebras (Propositions 2.1.3 and 2.1.4) and that the direct product of complete multialgebras is a complete multialgebra (Proposition 2.1.7). Another problem we studied for each construction is whether and when the functor obtained by factorization modulo the fundamental relation preserves them. Paragraph 2.2 presents the results obtained in [63], a paper accepted for publication in Italian Journal of Pure and Applied Mathematics. In Example 2.2.1, we show that the functor introduced in paragraph 1.6 usually does not commute with the multialgebra products. However, for a more restricted case, we found a necessary and sufficient condition for the fundamental algebra of the direct product of a family of multialgebras to be isomorphic with the direct product of the corresponding fundamental algebras (Proposition 2.2.2). The condition in Proposition 2.2.2 is quite complicated but it leads to a sufficient condition (Corollary 2.2.3) which helps us establish when the above property holds for hypergroups and for complete multialgebras (Theorems 2.2.9 and 2.2.12).

In paragraph 2.3, we constructed the direct limit of a direct system of multialgebras $\mathcal{A}$. We considered the direct system $\mathcal{A}$ to have a (directed) ordered carrier $(I, \leq)$ and we showed that certain properties of the ordered set $(I, \leq)$ facilitate this construction (Proposition 2.3.8 and Theorem 2.3.10). We proved that a class of multialgebras closed under the formation of the isomorphic images is closed under the formation of direct limits of arbitrary direct systems if and only if it is closed under the formation of direct limits of well ordered direct systems (Theorem 2.3.10). If we consider that in a direct system of multialgebras all the homomorphisms are ideal, it results immediately that if our multialgebras are algebras, then we obtain the construction with the same name from universal algebras. In this case, from the above mentioned results we obtain Lemma 7, Theorem 2 and Theorem 4 from [29, §21]. Some of the properties established by Romeo in [70] and Leoreanu in [40] and [46] can be obtained by using Proposition 2.3.8 and Propositions 2.3.14, 2.3.16, which state that the direct limit of a direct system of multialgebras which satisfy a given identity, weak or strong, verifies this identity. From Theorem 2.4.1, in which we showed that the functor F, determined by the factorization modulo the fundamental relation, is a left adjoint for the inclusion functor, we deduce that F commutes with the direct limits of direct systems of multialgebras (Corollary 2.4.2).

The last construction we present in this thesis is the construction of inverse limits of inverse systems of multialgebras. At the beginning of the paragraph 2.5, we prove that the inverse limit of an inverse system of multialgebras of type $\tau = (n_\gamma)_{\gamma < o(\tau)}$ in the category of the relational systems of type $(n_\gamma + 1)_{\gamma < o(\tau)}$ is not always a multialgebra (Example 2.5.6). We also prove a series of properties analogous to those which hold for direct limits (Propositions 2.5.14 and Theorems 2.5.14 and 2.5.17). An important part of these results consists in establishing some conditions in which the inverse limit of an inverse system of multialgebras is a multialgebra. Let K be a class of multialgebras of type τ closed under isomorphic images. In Theorem 2.5.17, we established a necessary and sufficient condition for the inverse limit of an inverse system of multialgebras from K to be a multialgebra in K. In the paragraph 2.6, we study the commutativity of the functor F determined by the fundamental relation with the inverse limits of inverse systems. In general, this functor does not commute with the inverse limits (Example 2.6.2). Starting from Proposition 2.5.14 and Theorem 2.5.14, in the last paragraph we proved Propositions 2.6.3 and 2.6.4 which present the necessary and sufficient conditions (in certain cases) for the fundamental algebra of the inverse limit of an inverse system of multialgebras to be the inverse limit of the inverse system formed with the corresponding fundamental algebras. We considered the functor F to be defined on subcategories of multialgebras such as those in Theorem 2.5.17 and, using Propositions 2.6.3 and 2.6.4, we established a necessary and sufficient condition for this functor to commute with the inverse limits of inverse families (Theorem 2.6.5).

I am very grateful to my family, who proved extremely patient and understanding all along these six years. I want to thank Professor Ioan Purdea for the discussions and the remarks which made this thesis possible, Professor Nicolae Both and Professor George Georgescu for their suggestions and my friend Simion Breaz for the mathematical dialogues which had an influence on the present work. I also thank my teachers Professor Grigore Călugăreanu, Professor Rodica Covaci, Professor Andrei Mărcus and my colleagues Septimiu Crivei, Christian Săcărea, Ciprian Modoi, Csaba Szanto, Iluska Bonta, Camelia Dicu, for the friendly environment in the Chair of Algebra and for the advice they offered me so generously.

Cosmin Pelea

Cluj–Napoca, September 2003.

Chapter 1

Multialgebras. Submultialgebras. Factor multialgebras

1.1 Multialgebras. Definitions. Particular cases

Let A be a set and let $P^*(A)$ be the set of the nonempty subsets of A.

Definition 1.1.1. Let $n \in \mathbb{N}$ be a nonnegative integer. A mapping $A^n \to P^*(A)$ is called *n-ary multioperation* on A.

Remark 1.1.1. There exist nullary multioperations on A if and only if A is not empty.

Let us consider a sequence of nonnegative integers $\tau = (n_\gamma)_{\gamma < o(\tau)} = (n_0, n_1, \ldots, n_\gamma, \ldots)$ indexed with a set of ordinal numbers $\{\gamma \mid \gamma < o(\tau)\}$ and for any $\gamma < o(\tau)$ let us consider a symbol $\mathbf{f}_\gamma$ of an n_γ-ary multioperation.

Definition 1.1.2. A *multialgebra of type* τ, $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$, consists of a set A and a family of multioperations $(f_\gamma)_{\gamma < o(\tau)} = (f_0, f_1, \ldots, f_\gamma, \ldots)$ such that each f_γ is an n_γ-ary multioperation having the symbol $\mathbf{f}_\gamma$. The set A is called *the support* (*set*) of the multialgebra $\mathfrak{A}$.

Example 1.1.3. [47] Let $(G, \cdot)$ be a group, H a subgroup of G and let $G/H = \{xH \mid x \in G\}$. The equality $xH \cdot yH = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}$ defines an operation on G/H if and only if the subgroup H is normal. In general, the above equality defines a binary multioperation on G/H.

Let us remind a few particular multialgebras which will appear in our thesis:

Hypergroupoid. A multialgebra $(H, \circ)$ with one binary multioperation is called *hypergroupoid.* If $a, b \in H$, the image $a \circ b$ of the pair (a, b) will be called *hyperproduct.* Sometimes we will use the term *hyperproduct* for the binary multioperation of a hypergroupoid. If $A, B \subseteq H$ then $A \circ B = \bigcup\{a \circ b \mid a \in A,\ b \in B\}$.

Semihypergroup. A hypergroupoid $(H, \circ)$ having the multioperation $\circ$ associative, i.e. $a \circ (b \circ c) = (a \circ b) \circ c$ for any $a, b, c \in H$, is called *semihypergroup.* If $a_1, \ldots, a_n \in H$ the nonempty subset $a_1 \circ \cdots \circ a_n$ of H will be called *hyperproduct with n factors.* By replacing the associativity of $\circ$ with the condition

$a \circ (b \circ c) \cap (a \circ b) \circ c \neq \emptyset$ for any $a, b, c \in H$, called *weak associativity*, one obtains the definition for *H_v-semigroup*.

Hypergroup. Let H be a nonempty set. A semihypergroup $(H, \circ)$ for which

$$a \circ H = H \circ a = H \text{ for all } a \in H, \tag{1.1.1}$$

is called *hypergroup*. An H_v-semigroup satisfying (1.1.1) is called *H_v-group*. The multialgebra from Example 1.1.3 is a hypergroup. From (1.1.1) it follows that the equalities

$$a/b = \{x \in H \mid a \in x \circ b\},\ b\backslash a = \{x \in H \mid a \in b \circ x\} \tag{1.1.2}$$

define two binary multioperations on H. So, the hypergroups and the H_v-groups can be seen as multialgebras $(H, \circ, /, \backslash)$ of type $\tau = (2, 2, 2)$. Let us also notice that if $H \neq \emptyset$ and $(H, \circ, /, \backslash)$ is a multialgebra of type $\tau = (2, 2, 2)$ such that the multioperation $\circ$ is associative (weak associative) and the multioperations $/$ and $\backslash$ can be obtained from $\circ$ using (1.1.2) then $(H, \circ)$ is a hypergroup (H_v-group).

Canonical hypergroup. A nonempty set H together with a binary multioperation $+$ is a *canonical hypergroup* if: (i) $+$ is associative; (ii) $+$ is commutative ($a + b = b + a$, for all $a, b \in H$); (iii) there exists a $0 \in H$ such that $0 + a = a$, for any $a \in H$; (iv) for any $a \in H$, there exists $-a \in H$ which verifies the following property: if $b, c \in H$ such that $c \in a + b$ then $b \in (-a) + c$.

This multialgebra is a hypergroup, 0 is unique with the given property, $0 \in a + (-a)$, and $-a$ is unique with this property, so the canonical hypergroups can be seen as multialgebras $(H, +, /, \backslash, 0, -)$ with $+, /, \backslash$ binary multioperations, 0 nullary operation and $-$ unary operation.

Hyperring (in the general sense). A *hyperring (in the general sense)* is a multialgebra $(R, +, \cdot)$ for which $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup and for all $a, b, c \in R$ the inclusions $a \cdot (b + c) \subseteq a \cdot b + a \cdot c$, $(b + c) \cdot a \subseteq b \cdot a + c \cdot a$ hold. If $/$ and $\backslash$ are the multioperations defined in the hypergroup $(R, +)$ by (1.1.2) then the hyperring R can be seen as a multialgebra $(R, +, /, \backslash, \cdot)$ of type $(2, 2, 2, 2)$, with $+, \cdot$ associative multioperations which verify the above inclusions.

Krasner hyperring. A *Krasner hyperring* $(A, +, \cdot, 0)$ consists of a set A, and two binary multioperations (on A) $+, \cdot$ having the following properties: i) $(A, +, 0)$ is a canonical hypergroup with; ii) $(A, \cdot)$ is a semigroup; iii) $0 \cdot a = a \cdot 0 = 0$, for any $a \in A$; iv) the operation $\cdot$ is distributive with respect to the multioperation $+$, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$. A Krasner hyperring can be seen as a multialgebra $(A, +, /, \backslash, 0, -, \cdot)$ with $+, /, \backslash$ binary multioperations and $0, -, \cdot$ operations with the arities $0, 1, 2$, respectively.

The universal algebras are particular cases of multialgebras. So, the semigroups are particular cases of semihypergroups, the groups are particular cases of hypergroups, the Abelian groups are particular cases of canonical hypergroups, and the rings are particular cases of rings.

A multioperation f_γ of a multialgebra of type τ can be seen as a $n_\gamma + 1$-ary relation r_γ as follows:

$$(a_0, \dots, a_{n_\gamma - 1}, a_{n_\gamma}) \in r_\gamma \Leftrightarrow a_{n_\gamma} \in f_\gamma(a_0, \dots, a_{n_\gamma - 1}). \tag{1.1.3}$$

So, the multialgebras are particular relational systems, more general than universal algebras.

1.2 The algebra of the nonempty subsets of a multialgebra

A multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ of type τ determines a structure of universal algebra of type τ on $P^*(A)$ with the operations defined as follows:

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid a_i \in A_i,\ i \in \{0, \dots, n_\gamma - 1\}\}, \tag{1.2.1}$$

for any $\gamma < o(\tau)$ and $A_0, \dots, A_{n_\gamma-1} \in P^*(A)$. We will denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$ and we will call it the (universal) algebra of the nonempty subsets of the multialgebra $\mathfrak{A}$.

Remark 1.2.1. If A_i, B_i $(i \in \{0, \dots, n_\gamma - 1\})$ are nonempty subsets of A such that $A_i \subseteq B_i$ then $f_\gamma(A_0, \dots, A_{n_\gamma-1}) \subseteq f_\gamma(B_0, \dots, B_{n_\gamma-1})$.

Let us consider a nonnegative integer n and the universal algebra $\mathfrak{P}^*(\mathfrak{A}) = (P^*(A), (f_\gamma)_{\gamma<o(\tau)})$. We will denote by $P^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ the set of the n-ary polynomial functions of the algebra $\mathfrak{P}^*(\mathfrak{A})$ and by $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ the algebra $(P^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A})), (f_\gamma)_{\gamma<o(\tau)})$.

Remark 1.2.2. If $n \in \mathbb{N}$, $p \in P^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ and the nonempty subsets $A_0, \dots, A_{n-1}$, $B_0, \dots, B_{n-1}$ of A are such that $A_0 \subseteq B_0, \dots, A_{n-1} \subseteq B_{n-1}$ then $p(A_0, \dots, A_{n-1}) \subseteq p(B_0, \dots, B_{n-1})$.

In the algebra $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ we consider the subalgebra $P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ generated by the functions e^n_i, $i \in \{0, \dots, n-1\}$. The algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A})) = (P^{(n)}(\mathfrak{P}^*(\mathfrak{A})), (f_\gamma)_{\gamma<o(\tau)})$ is the algebra of the n-ary term functions of $\mathfrak{P}^*(\mathfrak{A})$. We mention that the algebra $\mathfrak{P}^{(0)}(\mathfrak{P}^*(\mathfrak{A}))$ exists if and only if the multialgebra $\mathfrak{A}$ has no nullary multioperations.

For each $a \in A$, we denote the polynomial function $c^n_{\{a\}}$ by c^n_a and by $P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ the subalgebra of $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ generated by the subset $\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \dots, n-1\}\}$.

Let $n \in \mathbb{N}$. Starting from the symbols $(\mathbf{f}_\gamma)_{\gamma<o(\tau)}$ and $\mathbf{x}_0, \dots, \mathbf{x}_{n-1}$ one can construct the algebra of the n-ary terms. We denote by $\mathbf{P}^{(n)}(\tau)$ the set of the n-ary terms. For any $\gamma < o(\tau)$ the equality $f_\gamma(\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1}) = \mathbf{f}_\gamma(\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1})$ defines an n_γ-ary operation on $\mathbf{P}^{(n)}(\tau)$. So, one obtains the algebra $\mathfrak{P}^{(n)}(\tau)$ of the n-ary terms. The algebra $\mathfrak{P}^{(0)}(\tau)$ exists if and only if there exists $\gamma < o(\tau)$ such that $\mathbf{f}_\gamma$ is the symbol of a nullary multioperation.

Remark 1.2.6. [29, Corollary 8.1] Any n-ary term function p of the algebra $\mathfrak{P}^*(\mathfrak{A})$ is induced by an n-ary term $\mathbf{p}$.

Notation. The term function induced by $\mathbf{p}$ on the algebra $\mathfrak{P}^*(\mathfrak{A})$ will be denoted by p or by $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}$.

1.3 The lattice of the submultialgebras. The generated submultialgebra

Definition 1.3.1. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and $B \subseteq A$. We will say that B is a *submultialgebra* of $\mathfrak{A}$ if for any $\gamma < o(\tau)$ and for all $b_0, \dots, b_{n_\gamma-1} \in B$, $f_\gamma(b_0, \dots, b_{n_\gamma-1}) \subseteq B$.

Remark 1.3.1. If B is a submultialgebra of $\mathfrak{A}$ then the set B and the restrictions $f_\gamma|_{B^{n_\gamma}} : B^{n_\gamma} \to P^*(B)$ of the multioperations f_γ form a multialgebra $\mathfrak{B}$ of type τ.

Notations. For a multialgebra $\mathfrak{A}$ we will denote by $S(\mathfrak{A})$ the set of its submultialgebras, and for a submultialgebra B, the multioperation $f_\gamma|_{B^{n_\gamma}}$ of the multialgebra $\mathfrak{B}$ will be denoted by f_γ.

Example 1.3.2. Let $(H, \circ)$ be hypergroupoid. $S \subseteq H$ is a *subhypergroupoid* of H if S is a submultialgebra for $(H, \circ)$. A subhypergroupoid of a semihypergroup is called *subsemihypergroup*.

Remark 1.3.3. Let H be a hypergroup and $S \subseteq H$ a nonempty set. If we see the hypergroup H as a multialgebra $(H, \circ, /, \backslash)$ then the submultialgebras which have nonempty support sets are the closed subhypergroups (see [10]) and $S(H, \circ, /, \backslash)$ is the union of the set of the closed subhypergroups of H with $\{\emptyset\}$.

Theorem 1.3.4. *[67, Theorem 3] Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ and $B \subseteq A$. $P^*(B)$ is a subalgebra of $\mathfrak{P}^*(\mathfrak{A})$ if and only if B is a submultialgebra of $\mathfrak{A}$.*

Corollary 1.3.6. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ, B a submultialgebra of $\mathfrak{A}$, $n \in \mathbb{N}$ and $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$. If $b_0, \ldots, b_{n-1} \in B$ then $p(b_0, \ldots, b_{n-1}) \subseteq B$.*

In [33, Lemma 1] is presented the following result (even in a more general case which concerns a multialgebra whose multioperations are not necessarily finitary):

Lemma 1.3.7. *The set $S(\mathfrak{A})$ is an algebraic closure system on A.*

Corollary 1.3.9. *If $X \subseteq A$ then $\langle X \rangle = \bigcap \{B \in S(\mathfrak{A}) \mid X \subseteq B\}$ is a submultialgebra of $\mathfrak{A}$.*

Definition 1.3.10. $\langle X \rangle$ will be called the *submultialgebra* of $\mathfrak{A}$ *generated by the subset* X.

Theorem 1.3.13. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ, $X \subseteq A$. Then $a \in \langle X \rangle$ if and only if there exist $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $x_0, \ldots, x_{n-1} \in X$ such that $a \in p(x_0, \ldots, x_{n-1})$.*

1.4 Multialgebra homomorphisms

Maybe the most natural way to define an homomorphism between two multialgebras is the one provided by considering the multialgebras as relational systems.

Definition 1.4.1. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ and $\mathfrak{B} = (B, (f_\gamma)_{\gamma < o(\tau)})$ be two multialgebras of the same type τ. A map $h : A \to B$ is a *homomorphism* between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ if for any $\gamma < o(\tau)$ and $a_0, \ldots, a_{n_\gamma - 1} \in A$ we have

$$h(f_\gamma(a_0, \ldots, a_{n_\gamma - 1})) \subseteq f_\gamma(h(a_0), \ldots, h(a_{n_\gamma - 1})). \tag{1.4.1}$$

Often, the homomorphisms we are dealing with have the property that the inclusion (1.4.1) is an equality. As in [67], we will call such a homomorphism, *ideal homomorphism*. Let us mention that the ideal homomorphisms are called *relational homomorphisms* in [75] – but we will not use this term in order to avoid the confusion with the notion of homomorphism between relational systems – and *tight homomorphisms* in [90].

Remark 1.4.3. If H and H' are hypergroups and we see them as multialgebras with three binary multioperations as in the paragraph 1.1 then a homomorphism between the hypergroupoids $(H, \circ)$ and $(H', \circ)$ is a homomorphism between the multialgebras $(H, \circ, /, \backslash)$ and $(H', \circ, /, \backslash)$.

Definition 1.4.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two multialgebras of the same type. A bijective map $h : A \to B$ for which h and h^{-1} are multialgebra homomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ is called *isomorphism*.

Remark 1.4.4. A bijective multialgebra homomorphism is an isomorphism if and only if it is an ideal homomorphism.

Definition 1.4.3. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and let ρ be an equivalence relation on A. The relation ρ is called *ideal equivalence* on $\mathfrak{A}$ if for any $\gamma < o(\tau)$ and for any $x_i, y_i \in A$ having the property that $x_i \rho y_i$ for all $i \in \{0, \ldots, n_\gamma - 1\}$ we have

$$a \in f_\gamma(x_0, \ldots, x_{n_\gamma-1}) \Rightarrow \exists b \in f_\gamma(y_0, \ldots, y_{n_\gamma-1}) \text{ such that } a\rho b.$$

Let A be a set and let $P^*(A)$ be the set of its nonempty subsets. Let ρ be an equivalence relation on A and let $\overline{\rho}$ be the relation defined on $P^*(A)$ as follows:

$$A\overline{\rho}B \Leftrightarrow \forall a \in A,\ \exists b \in B \text{ such that } \ a\rho b \text{ şi } \forall b \in B,\ \exists a \in A \text{ such that } \ a\rho b.$$

One immediately notices that $\overline{\rho}$ is an equivalence relation on $P^*(A)$.

Theorem 1.4.5. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and let ρ be an equivalence relation on A. The relation ρ is an ideal equivalence on $\mathfrak{A}$ if and only if the relation $\overline{\rho}$ is a congruence relation on $\mathfrak{P}^*(\mathfrak{A})$.*

Proposition 1.4.7. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and let ρ be an equivalence relation on A. The following statements are equivalent:*
(a) ρ is an ideal equivalence on $\mathfrak{A}$;
(b) for any $\gamma < o(\tau)$ and any elements $x_i, y_i \in A$ such that $x_i\rho y_i$ for all $i \in \{0, \ldots, n_\gamma - 1\}$ we have $f_\gamma(x_0, \ldots, x_{n_\gamma-1})\overline{\rho} f_\gamma(y_0, \ldots, y_{n_\gamma-1})$;
(c) for any $\gamma < o(\tau)$, any $a, b, x_i \in A$ $(i \in \{0, \ldots, n_\gamma - 1\})$ such that $a\rho b$ and any $i \in \{0, \ldots, n_\gamma - 1\}$ we have $f_\gamma(x_0, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_\gamma-1})\overline{\rho} f_\gamma(x_0, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_\gamma-1})$;
(d) for any $n \in \mathbb{N}$, any $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and any elements $x_i, y_i \in A$ with $x_i\rho y_i$ $(i \in \{0, \ldots, n-1\})$ we have $p(x_0, \ldots, x_{n-1})\overline{\rho} p(y_0, \ldots, y_{n-1})$.

Proposition 1.4.9. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ and $\mathfrak{B} = (B, (f_\gamma)_{\gamma<o(\tau)})$ be multialgebras of the same type τ, let $h : A \to B$ be a homomorphism, $n \in \mathbb{N}$ and $p \in \mathbf{P}^{(n)}(\tau)$. For any $a_0, \ldots, a_{n-1} \in A$ we have $h(p(a_0, \ldots, a_{n-1})) \subseteq p(h(a_0), \ldots, h(a_{n-1}))$.*

Given a multialgebra $\mathfrak{A}$ and an equivalence relation ρ on A the equalities:

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1}),\ a_i\rho b_i,\ i \in \{0, \ldots, n_\gamma - 1\}\}$$

define multioperations on A/ρ, so $\mathfrak{A}/\rho = (A/\rho, (f_\gamma)_{\gamma<o(\tau)})$ is a multialgebra. We will call it the *factor multialgebra* determined by (or, modulo) ρ. The canonical projection $\pi_\rho : A \to A/\rho$, $\pi_\rho(a) = \rho\langle a\rangle$ is a multialgebra homomorphism.

Applying Proposition 1.4.9 to $\pi_\rho : A \to A/\rho$ we have:

$$\{\rho\langle a\rangle \mid a \in (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}(a_0, \ldots, a_{n-1})\} \subseteq (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle) \tag{1.4.2}$$

for any $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$.

Remark 1.4.10. The inclusion (1.4.2) holds if we replace the term functions $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}$ and $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$ by the polynomial $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and, respectively, $p' \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho))$, where the polynomial function p' corresponding to p is obtained as follows:
(i) if $p = c_a^n$ then $p' = c_{\rho\langle a\rangle}^n$;
(ii) if $p = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A})}$ then $p' = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$;
(iii) if $p = f_\gamma(p_0, \dots, p_{n_\gamma-1})$ and the functions that correspond to $p_0, \dots, p_{n_\gamma-1} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ are $p'_0, \dots, p'_{n_\gamma-1} \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho))$, respectively, then $p' = f_\gamma(p'_0, \dots, p'_{n_\gamma-1})$.

Since the polynomial function p' is obtained in the same manner as p, with major changes only in step (i), we will write p instead of p' and, consequently, we have:

$$\{\rho\langle a\rangle \mid a \in p(a_0, \dots, a_{n-1})\} \subseteq p(\rho\langle a_0\rangle, \dots, \rho\langle a_{n-1}\rangle). \tag{1.4.3}$$

Theorem 1.4.11. *[67, Theorem 1] Let $\mathfrak{A}$ be a multialgebra of type τ. If ρ is an ideal equivalence on $\mathfrak{A}$ then $\pi_\rho : A \to A/\rho$, $\pi_\rho(a) = \rho\langle a\rangle$, is an ideal homomorphism. Conversely, if $h : A \to B$ is an ideal homomorphism between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type then the relation $\rho_h = \{(x, y) \in A \times A \mid h(x) = h(y)\}$ is an ideal equivalence on $\mathfrak{A}$. Moreover, the correspondence $h(a) \mapsto \pi_{\rho_h}(a)$ is an isomorphism between the multialgebras $h(\mathfrak{A})$ and $\mathfrak{A}/\rho_h$.*

Let h be an ideal homomorphism of multialgebras between $\mathfrak{A}$ and $\mathfrak{B}$. The homomorphism h induces a map $h_* : P^*(A) \to P^*(B)$ defined by $h_*(X) = h(X) = \{h(x) \mid x \in X\}$ for any $\emptyset \neq X \subseteq A$.

Theorem 1.4.12. *[67, Theorem 2] The map h_* is a homomorphism of universal algebras between $\mathfrak{P}^*(\mathfrak{A})$ and $\mathfrak{P}^*(\mathfrak{B})$ if and only if h is an ideal homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.*

Corollary 1.4.14. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ and $\mathfrak{B} = (B, (f_\gamma)_{\gamma<o(\tau)})$ be multialgebras of type τ, $h : A \to B$ an ideal homomorphism, $n \in \mathbb{N}$ and $p \in \mathbf{P}^{(n)}(\tau)$. For any $a_0, \dots, a_{n-1} \in A$ we have $h(p(a_0, \dots, a_{n-1})) = p(h(a_0), \dots, h(a_{n-1}))$.*

Remark 1.4.15. It is easy to observe that the multialgebras of type τ, with the multialgebra homomorphisms and the usual map composition form a category. We will denote it by $\mathbf{Malg}(\tau)$. The category of universal algebras of type τ, denoted here by $\mathbf{Alg}(\tau)$, is, obviously, a subcategory in $\mathbf{Malg}(\tau)$.

1.5 Factor multialgebras of universal algebras. A characterization theorem for multialgebras

G. Grätzer shows in [27] that any multialgebra can be obtained as in Example 1.1.3. Let $\mathfrak{B} = (B, (f_\gamma)_{\gamma<o(\tau)})$ be a universal algebra and let ρ be an equivalence relation on B. On the set B/ρ of the equivalence classes $\rho\langle b\rangle$ of the elements $b \in B$, Grätzer defines, for any $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle b_0\rangle, \dots, \rho\langle b_{n_\gamma-1}\rangle) = \{\rho\langle c\rangle \mid c = f_\gamma(c_0, \dots, c_{n_\gamma-1}),\ c_i \in \rho\langle b_i\rangle,\ i = 0, \dots, n_\gamma - 1\}.$$

It results a multialgebra $\mathfrak{B}/\rho$. Such a multialgebra is called *concrete multialgebra*.

Theorem 1.5.1. *[27, Theorem] Any multialgebra is concrete.*

Let $\mathfrak{A}$ be a multialgebra of type τ and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. As in case of universal algebras, we will say that the *n-ary identity* (or the *strong n-ary identity*) $\mathbf{q} = \mathbf{r}$ is satisfied on the multialgebra $\mathfrak{A}$ if $q(a_0, \dots, a_{n-1}) = r(a_0, \dots, a_{n-1})$ for any $a_0, \dots, a_{n-1} \in A$. We will say that the *weak identity* (the notation is intended to be as suggestive as possible) $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on the multialgebra $\mathfrak{A}$ if $q(a_0, \dots, a_{n-1}) \cap r(a_0, \dots, a_{n-1}) \neq \emptyset$ for any $a_0, \dots, a_{n-1} \in A$.

Many important particular multialgebras can be defined by using (strong and/or weak) identities.

Example 1.5.2. The semihypergroups are multialgebras of type (2) which satisfy the identity

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 = \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2). \tag{1.5.1}$$

The H_v-semigroups can be defined in a similar way, replacing (1.5.1) by

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 \cap \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \neq \emptyset. \tag{1.5.1'}$$

Remark 1.5.3. Associating to a hypergroup $(H, \circ)$ a multialgebra $(H, \circ, /, \backslash)$ with the multioperations $/, \backslash$ given by (1.1.2) we obtain an injective map (which is not bijective) from the class of hypergroups into the class of those multialgebras of type $\tau = (2, 2, 2)$ which satisfy the identities (1.5.1), (1.5.2) and (1.5.3). A similar remark can be obtained for H_v-groups if we replace (1.5.1) by (1.5.1′).

Example 1.5.5. Using the previous remarks, we can see a canonical hypergroup as a multialgebra $(H, +, /, \backslash, 0, -)$ of type $(2, 2, 2, 0, 1)$ having the property that $(H, +, /, \backslash)$ is a hypergroup and which verifies the identities $\mathbf{x}_0 + \mathbf{x}_1 = \mathbf{x}_1 + \mathbf{x}_0,\ \mathbf{x}_0 + 0 = \mathbf{x}_0,\ \mathbf{x}_0/\mathbf{x}_1 = -(\mathbf{x}_1/\mathbf{x}_0)$.

Example 1.5.6. A Krasner hyperring can be seen as a multialgebra $(H, +, /, \backslash, 0, -, \cdot)$ of type $\tau = (2, 2, 2, 0, 1, 2)$ with $(H, +, /, \backslash, 0)$ canonical hypergroup, $\cdot$ binary operation which, in addition to the identities of the semigroup $(H, \cdot)$, verifies the identities $\mathbf{x}_0 \cdot 0 = 0,\ 0 \cdot \mathbf{x}_0 = 0,\ \mathbf{x}_0 \cdot (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_0 \cdot \mathbf{x}_1 + \mathbf{x}_0 \cdot \mathbf{x}_2,\ (\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{x}_0 = \mathbf{x}_1 \cdot \mathbf{x}_0 + \mathbf{x}_2 \cdot \mathbf{x}_0$.

Remark 1.5.7. Let $\mathfrak{B}$ be a universal algebra, ρ an equivalence relation on B and $\mathfrak{B}/\rho$ be the corresponding factor multialgebra. If we take p as in Remark 1.4.10 and $b_0, \dots, b_{n-1} \in B$ we have

$$p(\rho\langle b_0\rangle, \dots, \rho\langle b_{n-1}\rangle) \supseteq \{\rho\langle c\rangle \mid c = p(c_0, \dots, c_{n-1}),\ b_i \rho c_i,\ i \in \{0, \dots, n-1\}\}. \tag{1.5.4}$$

Remark 1.5.8. It follows immediately that if $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ and the identity $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{B}$ then the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{B}/\rho$.

In general, the inclusion (1.5.4) is not an equality. Also, the weak identity established on $\mathfrak{B}/\rho$ is not, in general, a strong one.

Example 1.5.9. Let $(\mathbb{Z}_5, +)$ be the additive cyclic group of order 5 and let $\mathbb{Z}_5$ be the equivalence relation $\rho = (\{0, 1\} \times \{0, 1\}) \cup (\{2\} \times \{2\}) \cup (\{3, 4\} \times \{3, 4\})$. In the factor hypergroup we have $(\rho\langle 2\rangle + \rho\langle 2\rangle) + \rho\langle 3\rangle = \{\rho\langle 0\rangle, \rho\langle 2\rangle, \rho\langle 3\rangle\} \neq \{\rho\langle 2\rangle, \rho\langle 3\rangle\} = \{\rho\langle c\rangle \mid c = (b_0 + b_1) + b_2, b_0 = b_1 = 2, b_2 \in \{3, 4\}\}$. We also have $\rho\langle 2\rangle + (\rho\langle 2\rangle + \rho\langle 3\rangle) = \{\rho\langle 2\rangle, \rho\langle 3\rangle\}$, so the associativity holds only in weak manner for $(\mathbb{Z}_5/\rho, +)$.

Yet, some identities, like those which characterize the commutativity of an operation in an algebra $\mathfrak{B}$, hold in a strong manner in the multialgebra $\mathfrak{B}/\rho$.

Let us see what is the factor multialgebra in the case of semigroups, (Abelian) groups and rings.

The case of semigroups. Let $(S,\cdot)$ be a semigroup and let ρ be an equivalence relation on S. According to Remark 1.5.8 the hypergroupoid $(S/\rho,\cdot)$ verifies the associativity in a weak manner, so it is an H_v-semigroup.

The case of groups. Let $(G,\cdot)$ be a group and let ρ be a equivalence relation on G. The existence and the uniqueness of the solution for each of the equations $a = xb$ and $a = by$ allows us to define the operations $a/b = \{x \in G \mid a = xb\}$, $b\backslash a = \{y \in G \mid a = by\}$ on G and to identify the group G with a universal algebra $(G,\cdot,/,\backslash)$ (with $G \neq \emptyset$) which verifies the following identities: $(\mathbf{x}_0\cdot\mathbf{x}_1)\cdot\mathbf{x}_2 = \mathbf{x}_0\cdot(\mathbf{x}_1\cdot\mathbf{x}_2)$, $\mathbf{x}_1 = \mathbf{x}_0\cdot(\mathbf{x}_0\backslash\mathbf{x}_1)$, $\mathbf{x}_1 = (\mathbf{x}_1/\mathbf{x}_0)\cdot\mathbf{x}_0$, $\mathbf{x}_1 = \mathbf{x}_0\backslash(\mathbf{x}_0\cdot\mathbf{x}_1)$, $\mathbf{x}_1 = (\mathbf{x}_1\cdot\mathbf{x}_0)/\mathbf{x}_0$. We will obtain a multialgebra $(G/\rho,\cdot,/,\backslash)$ on G/ρ which satisfies the above identities in a weak manner. So, $(G/\rho,\cdot)$ is an H_v-group. Moreover, the class $\rho\langle 1\rangle \in G/\rho$ of the unit 1 of G verifies the condition $\rho\langle a\rangle \in \rho\langle a\rangle\cdot\rho\langle 1\rangle \cap \rho\langle a\rangle\cdot\rho\langle 1\rangle$, for any $a \in G$, so $\rho\langle 1\rangle$ is a unit in G/ρ. Also, any class $\rho\langle a\rangle \in G/\rho$ has an inverse since, if we consider the inverse a^{-1} of a in G, we have: $\rho\langle 1\rangle \in \rho\langle a^{-1}\rangle\cdot\rho\langle a\rangle \cap \rho\langle a\rangle\cdot\rho\langle a^{-1}\rangle$. If the group G is Abelian then the H_v-group G/ρ is commutative.

The case of rings. A hyperstructure $(R,+,\cdot)$ is called *H_v-ring* if $(R,+)$ is an H_v-group, $(R,\cdot)$ is an H_v-semigroup and for any $a,b,c \in R$ we have $a(b+c)\cap(ab+ac) \neq \emptyset$, $(b+c)a \cap (ba+ca) \neq \emptyset$. It is easy to observe that the factor multialgebra of a ring is an H_v-ring whose first multioperation is commutative.

Remark 1.5.11. In general, the factor multialgebra of a lattice is not a hyperlattice because the absorption (which appears in the definition of a hyperlattice — see [1, 2.1, Lemma 4] or [32]) is satisfied only in a weak manner in the factor multialgebra.

Example 1.5.12. Let us consider the lattice $(\mathbb{N},\wedge,\vee)$, where $\mathbb{N}$ is the set of the nonnegative integers, $a\wedge b = \gcd(a,b)$ and $a \vee b = \operatorname{lcm}(a,b)$. We denote by $\mathbb{P}$ the set of the prime numbers and we consider the relation $\rho = \mathbb{P}\times\mathbb{P} \cup \{(a,a) \mid a \in \mathbb{N}\setminus\mathbb{P}\}$. Clearly, ρ is an equivalence relation on $\mathbb{N}$ and we have $\rho\langle 2\rangle \in \rho\langle 2\rangle \vee (\rho\langle 2\rangle \wedge \rho\langle 6\rangle) = \rho\langle 2\rangle \vee \{\rho\langle 1\rangle, \rho\langle 2\rangle\} = \{\rho\langle 2\rangle\} \cup \{\rho\langle pq\rangle \mid p,q \in \mathbb{P},\ p \neq q\}$, so, the absorption holds only in a weak manner on $\mathbb{N}/\rho$.

The fact that an identity of an algebra $\mathfrak{B}$ is verified in weak or strong manner on the factor multialgebra $\mathfrak{B}/\rho$ also depends on the equivalence relation ρ. If ρ is a congruence relation on $\mathfrak{B}$ then the factor algebra $\mathfrak{B}/\rho$ verifies the identities of the algebra $\mathfrak{B}$ in a strong manner. This is not the only example in this respect.

Example 1.5.13. Let us see a finite group G as an universal algebra $(G,\cdot,1)$, and let us consider an equivalence relation ρ on G and an identity $\mathbf{1}\cdot\mathbf{x}_0 = \mathbf{x}_0$. If the relation $\rho = \sim$ is the conjugation on G then the multioperation $\rho\langle x\rangle\cdot\rho\langle y\rangle = \{\rho\langle z\rangle \mid z = x'y',\ x \sim x',\ y \sim y'\}$ organize G/ρ as a canonical hypergroup (see [72]). So, the above identity is satisfied on $(G/\rho,\cdot,\rho\langle 1\rangle)$ in a strong manner.

Things are different in a factor multialgebra obtained from G as in Example 1.1.3. Let us consider the symmetric group S_3 (seen as a universal algebra $(S_3,\circ,(1))$) and the equivalence relation (to the left) determined by the subgroup H generated by the transposition $(1,2)$. In the hypergroup S_3/H, $H\circ((1,3)\circ H) = \{(1,3)\circ H,\ (2,3)\circ H\}$, thus the identity $\mathbf{1}\cdot\mathbf{x}_0 = \mathbf{x}_0$ is not satisfied in a strong manner on $(S_3/H,\circ,H)$.

1.6 A class of ideal equivalences. The fundamental relation of a multialgebra

Let A be a set and let $P^*(A)$ be the set of its nonempty subsets. Let ρ be an equivalence relation on A and let $\overline{\overline{\rho}}$ be the relation defined on $P^*(A)$ by $A\overline{\overline{\rho}}B \Leftrightarrow a\rho b$, for any $a \in A,\ b \in B$. The relation $\overline{\overline{\rho}}$ is symmetric, transitive, but is not always reflexive (for example, if δ_A is the equality relation on A and $|A| \geq 2$, then $\overline{\overline{\delta_A}}$ is not reflexive).

Proposition 1.6.1. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and let ρ be an equivalence relation on A. The following statements are equivalent:*
(a) $\mathfrak{A}/\rho$ is a universal algebra;
(b) for any $\gamma < o(\tau)$, any $a, b, x_i \in A$ $(i \in \{0, \ldots, n_\gamma - 1\})$ such that $a\rho b$ and any $i \in \{0, \ldots, n_\gamma - 1\}$ we have $f_\gamma(x_0, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_\gamma-1})\overline{\overline{\rho}}f_\gamma(x_0, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_\gamma-1})$;
(c) for any $\gamma < o(\tau)$ and for any $x_i, y_i \in A$ such that $x_i\rho y_i$ for any $i \in \{0, \ldots, n_\gamma - 1\}$ we have $f_\gamma(x_0, \ldots, x_{n_\gamma-1})\overline{\overline{\rho}}f_\gamma(y_0, \ldots, y_{n_\gamma-1})$;
(d) for any $n \in \mathbb{N}$, any $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and any $x_i, y_i \in A$ such that $x_i\rho y_i$ for all $i \in \{0, \ldots, n-1\}$, we have $p(x_0, \ldots, x_{n-1})\overline{\overline{\rho}}p(y_0, \ldots, y_{n-1})$.

We easily observe that any equivalence ρ on A for which $\mathfrak{A}/\rho$ is a universal algebra is ideal.

Remark 1.6.3. If an equivalence ρ satisfies one of the equivalent conditions from Proposition 1.6.1 then the operations of the algebra $\mathfrak{A}/\rho$ are defined by $f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \rho\langle b\rangle$, for any $a_0, \ldots, a_{n_\gamma-1} \in A$ and $b \in f_\gamma(a_0, \ldots, a_{n_\gamma-1})$.

Remark 1.6.5. Let $(H, \circ)$ be a hypergroupoid. An equivalence relation ρ on H such that $(H/\rho, \circ)$ is a groupoid is called *strongly regular* (see [10, Definition 8]). If ρ is a strongly regular equivalence on H and $(H, \circ)$ is a semihypergroup then $(H/\rho, \circ)$ is a semigroup. If $(H, \circ)$ is a hypergroup then $(H/\rho, \circ)$ is a group ([10, Theorem 31]). Moreover, if $(H, \circ)$ is a hypergroup and we see it as a multialgebra $(H, \circ, /, \backslash)$ as in the paragraph 1.1 then the multioperations $/$ and $\backslash$ become on H/ρ the binary operations which associate to each pair $(\rho\langle a\rangle, \rho\langle b\rangle) \in H/\rho \times H/\rho$ the (unique) solution from H/ρ of the equation $\rho\langle a\rangle = x \circ \rho\langle b\rangle$ and of the equation $\rho\langle a\rangle = \rho\langle b\rangle \circ x$, respectively.

Notation. We denote by $E_{ua}(\mathfrak{A})$ the set of those equivalence relations ρ of $\mathfrak{A}$ for which $\mathfrak{A}/\rho$ is a universal algebra.

Proposition 1.6.7. *The set $E_{ua}(\mathfrak{A})$ forms an algebraic closure system on $A \times A$.*

Corollary 1.6.9. *If $R \subseteq A \times A$ then $\alpha(R) = \bigcap\{\rho \in E_{ua}(\mathfrak{A}) \mid R \subseteq \rho\}$ is the smallest equivalence from $E_{ua}(\mathfrak{A})$ which contains R.*

If the multialgebra $\mathfrak{A}$ is not an universal algebra then $\delta_A \notin E_{ua}(\mathfrak{A})$.

Definition 1.6.1. Let $\mathfrak{A}$ be a multialgebra. The smallest equivalence from $E_{ua}(\mathfrak{A})$ is called *the fundamental relation* of $\mathfrak{A}$.

Let $\alpha_{\mathfrak{A}}$ be the relation defined on A as follows: if $x, y \in A$ then $x\alpha_{\mathfrak{A}}y$ if and only if $x, y \in p(a_0, \ldots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \ldots, a_{n-1} \in A$.

Remark 1.6.11. If $x, y \in A$ then $x\alpha_{\mathfrak{A}}y$ if and only if $x, y \in p(a_0, \ldots, a_{n-1})$ for some $n \in \mathbb{N}$, some term function $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and some elements $a_0, \ldots, a_{n-1} \in A$.

The relation $\alpha_{\mathfrak{A}}$ is symmetric, reflexive, but is not always transitive (see [10, Remark 82]). Let $\alpha^*_{\mathfrak{A}}$ be the transitive closure of $\alpha_{\mathfrak{A}}$.

Theorem 1.6.13. *The relation $\alpha^*_{\mathfrak{A}}$ is the fundamental relation of $\mathfrak{A}$.*

Definition 1.6.2. Let $\mathfrak{A}$ be a multialgebra and let $\alpha^*_{\mathfrak{A}}$ be its fundamental relation. The universal algebra $\mathfrak{A}/\alpha^*_{\mathfrak{A}}$ is called the *fundamental algebra* of $\mathfrak{A}$.

Remark 1.6.14. The canonical projection $\varphi_A : A \to A/\alpha^*_{\mathfrak{A}}$ is an ideal homomorphism.

Example 1.6.15. The *fundamental relation of a semihypergroup* $(H, \circ)$ is the transitive closure β^* of the relation $\beta = \bigcup_{n\in\mathbb{N}^*} \beta_n$ where $x\beta_n y \Leftrightarrow \exists a_1, \ldots, a_n \in H : \; x, y \in a_1 \circ \cdots \circ a_n$.

Example 1.6.16 If $(H, \circ)$ is a hypergroup then β is transitive and it is the fundamental relation of the multialgebra $(H, \circ, /, \backslash)$ obtained as in the paragraph 1.1.

Example 1.6.17. Let $(R, +, \cdot)$ be a hyperring. The *fundamental relation of the hyperring* R is the transitive closure γ^* of the relation γ from [83, Definition 1]:

$$x\gamma y \Leftrightarrow \exists l, k_j \in \mathbb{N}^*,\ \exists a_{ij} \in R,\ j \in \{0, \ldots, l-1\},\ i \in \{0, \ldots, k_j - 1\} : \; x, y \in \sum_{j=0}^{l-1} \left(\prod_{i=0}^{k_j-1} a_{ij} \right).$$

Theorem 1.6.18 *Let $\mathfrak{A}$, $\mathfrak{B}$ be multialgebras of type τ and $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*_{\mathfrak{A}}$, $\overline{\mathfrak{B}} = \mathfrak{B}/\alpha^*_{\mathfrak{B}}$ their corresponding fundamental algebras. If $h : A \to B$ is a multialgebra homomorphism then there exists a unique algebra homomorphism $\overline{h} : \overline{A} \to \overline{B}$ such that*

$$\varphi_B \circ h = \overline{h} \circ \varphi_A. \tag{1.6.1}$$

Corollary 1.6.19. *If $\mathfrak{A}$ is a multialgebra then $\overline{1_A} = 1_{\overline{A}}$.*

Corollary 1.6.20. *If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of type τ and $h : A \to B$, $g : B \to C$ are multialgebra homomorphisms then $\overline{g \circ h} = \overline{g} \circ \overline{h}$.*

Remark 1.6.21. The factorization modulo fundamental relation determines a covariant functor F from $\mathbf{Malg}(\tau)$ into $\mathbf{Alg}(\tau)$ defined by: $F(\mathfrak{A}) = \overline{\mathfrak{A}}$ for any multialgebra $\mathfrak{A}$ and $F(h) = \overline{h}$ (from (1.6.1)) for any homomorphism h between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$.

1.7 Identities on multialgebras. Complete multialgebras

Proposition 1.7.1. *Let $\mathfrak{A}$ be a multialgebra, $n \in \mathbb{N}$, and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{A}$ then $\mathbf{q} = \mathbf{r}$ is satisfied on $\overline{\mathfrak{A}}$.*

Corollary 1.7.2. *Let $\mathfrak{A}$ be a multialgebra, $n \in \mathbb{N}$, and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{A}$ then $\mathbf{q} = \mathbf{r}$ is satisfied on $\overline{\mathfrak{A}}$.*

Remark 1.7.3. A result similar to Proposition 1.7.1 holds for any relation from $E_{ua}(\mathfrak{A})$.

Remark 1.7.5. The fundamental algebra of a hypergroup is a group. Consequently, we can define, as in Remark 1.6.21, a covariant functor F from the category **HG** of hypergroups into the category **Grp** of groups.

Among the multialgebras which have the same fundamental algebra (supposing that this algebra has more than one element) we can find multialgebras which verify the identities of their fundamental algebra, some of them in a weak manner, others, in a strong manner.

Proposition 1.7.6. *Let* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *be a multialgebra and let* $\overline{\mathfrak{A}} = (\overline{A}, (f_\gamma)_{\gamma<o(\tau)})$ *be its fundamental algebra. If* $|\overline{A}| \neq 1$, $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ *and* $\mathbf{q} = \mathbf{r}$ *is satisfied on* $\overline{\mathfrak{A}}$ *then there exists a structure of multialgebra* $\mathfrak{A}'$ *on* A *with the multioperation* f'_γ, $\gamma < o(\tau)$, *such that* $\overline{\mathfrak{A}'} = \overline{\mathfrak{A}}$ *and* $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ *is satisfied on* $\mathfrak{A}'$.

If $|\overline{A}| > 1$ then the multioperations f'_γ can be defined by:

$$f'_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \{a \in A \mid \overline{a} = f_\gamma(\overline{a_0}, \ldots, \overline{a_{n_\gamma-1}})\}. \tag{1.7.2}$$

The properties of the multialgebra $\mathfrak{A}'$ lead us to a particular class of multialgebras.

Proposition 1.7.11. *For a multialgebra* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *the following statements are equivalent:*
(i) for any $\gamma < o(\tau)$ *and* $a_0, \ldots, a_{n_\gamma-1} \in A$, $a \in f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \Rightarrow \overline{a} = f_\gamma(a_0, \ldots, a_{n_\gamma-1})$;
(ii) for any $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ *and* $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in A$, *from* $q(a_0, \ldots, a_{n-1}) \cap r(b_0, \ldots, b_{n-1}) \neq \emptyset$ *it results that* $q(a_0, \ldots, a_{n-1}) = r(b_0, \ldots, b_{n-1})$.

Example 1.7.13. For a semihypergroup $(H, \circ)$, the condition (ii) from the previous proposition is the following: for any $m, n \in \mathbb{N}$, $m, n \geq 2$ and any $a_1, \ldots, a_m, b_1, \ldots, b_n \in H$, from $a_1 \circ \cdots \circ a_m \cap b_1 \circ \cdots \circ b_n \neq \emptyset$ it results that $a_1 \circ \cdots \circ a_m = b_1 \circ \cdots \circ b_n$. This condition defines the *complete semihypergroups* (see [10, Definition 137]).

These facts suggest the following definition:

Definition 1.7.1. A multialgebra $\mathfrak{A}$ which verifies the equivalent conditions from Proposition 1.7.11 is called *complete multialgebra.*

Remark 1.7.17. The multialgebra $\mathfrak{A}'$ from the proof of Proposition 1.7.6 is complete.

Remark 1.7.18. For any complete multialgebra $\mathfrak{A}$ the relation $\alpha_{\mathfrak{A}}$ is transitive.

Remark 1.7.19. The complete multialgebras of type τ form a full subcategory $\mathbf{CMalg}(\tau)$ of $\mathbf{Malg}(\tau)$.

Proposition 1.7.20. *Let* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *be a multialgebra of type* τ. *The multialgebra* $\mathfrak{A}$ *is complete if and only if there exist a universal algebra* $\mathfrak{B} = (B, (f'_\gamma)_{\gamma<o(\tau)})$ *and a partition* $\{A_b \mid b \in B\}$ *of* A *such that* $A_{b_1} \cap A_{b_2} = \emptyset$ *for any* $b_1 \neq b_2$ *from* B *and for any* $\gamma < o(\tau)$ *and* $a_0, \ldots, a_{n_\gamma-1} \in A$ *with* $a_i \in A_{b_i}$ $(i \in \{0, \ldots, n_\gamma - 1\})$, *we have* $f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = A_{f_\gamma(b_0, \ldots, b_{n_\gamma-1})}$.

1.8 Identities and algebras obtained as factor multialgebras

We proved that if $\mathfrak{A}$ is a multialgebra and $\rho \in E_{ua}(\mathfrak{A})$ then any identity (weak or strong) satisfied on $\mathfrak{A}$, is also satisfied on the algebra $\mathfrak{A}/\rho$.

Remark 1.8.1. Let $\mathbf{q}, \mathbf{r}$ be two n-ary terms and let $R_{\mathbf{qr}} = \{(x, y) \in A \times A \mid x \in q(a_0, \ldots, a_{n-1}), y \in r(a_0, \ldots, a_{n-1}), a_0, \ldots, a_{n-1} \in A\}$. The smallest relation from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra which verifies the identity $\mathbf{q} = \mathbf{r}$ is $\alpha(R_{\mathbf{qr}}) = \bigcap\{\rho \in E_{ua}(\mathfrak{A}) \mid R_{\mathbf{qr}} \subseteq \rho\}$.

Notation. We will denote the relation $\alpha(R_{\mathbf{qr}})$ by $\alpha^*_{\mathbf{qr}}$.

Theorem 1.8.3. *The relation $\alpha^*_{\mathbf{qr}}$ is the transitive closure of the relation $\alpha_{\mathbf{qr}} \subseteq A \times A$ defined by $x\alpha_{\mathbf{qr}}y$ if and only if $x \in p(q(a_0,\ldots,a_{n-1}))$, $y \in p(r(a_0,\ldots,a_{n-1}))$ or $y \in p(q(a_0,\ldots,a_{n-1}))$, $x \in p(r(a_0,\ldots,a_{n-1}))$ for some $p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0,\ldots,a_{n-1} \in A$.*

Example 1.8.4. In [26] is presented a characterization for the smallest strongly regular equivalence on a semihypergroup $(H,\circ)$ for which the factor multialgebra is a commutative semigroup. This relation, denoted by γ^*, is the transitive closure of the relation $\gamma = \bigcup_{n\in\mathbb{N}^*}\gamma_n$ where $\gamma_1 = \delta_H$ and, for any $n > 1$, the relation γ_n is defined by $x\gamma_n y \Leftrightarrow \exists z_1,\ldots,z_n \in H,\ \exists\sigma \in S_n :\ x \in z_1 \circ\cdots\circ z_n,\ y \in z_{\sigma(1)} \circ\cdots\circ z_{\sigma(n)}$ (S_n denotes the set of the permutations of $\{1,\ldots,n\}$). Since the set $\{(1,2),(2,3),\ldots,(n-1,n)\}$ generates the group S_n it follows that γ^* is the transitive closure of the relation $\gamma' = \bigcup_{n\in\mathbb{N}^*}\gamma'_n$, where $\gamma'_1 = \delta_H$ and for $n > 1$, $x\gamma'_n y$ if and only if

$$\exists z_1,\ldots,z_n \in H,\ \exists i \in \{1,\ldots,n-1\} : x \in z_1 \circ\cdots\circ z_{i-1} \circ (z_i \circ z_{i+1}) \circ z_{i+2} \circ\cdots\circ z_n,$$
$$y \in z_1 \circ\cdots\circ z_{i-1} \circ (z_{i+1} \circ z_i) \circ z_{i+2} \circ\cdots\circ z_n.$$

Clearly, $\gamma' = \alpha_{\mathbf{qr}}$ where $\mathbf{q} = \mathbf{x}_0 \circ \mathbf{x}_1$ and $\mathbf{r} = \mathbf{x}_1 \circ \mathbf{x}_0$. In [26] it is also proved that if $(H,\circ)$ is a hypergroup then γ is transitive and $\gamma^* = \gamma$ is the smallest equivalence on H such that H/γ^* is a commutative group.

Corollary 1.8.6. *The fundamental relation α^* of $\mathfrak{A}$ is the transitive closure of the relation $\alpha' \subseteq A \times A$ defined by $x\alpha' y$ if and only if $x, y \in p(a)$ for some $p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a \in A$.*

Let $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)$, let $\mathfrak{B}$ be a universal algebra of type τ and let ρ be an equivalence relation on B. We denote by $\rho_{\mathbf{qr}}$ the smallest equivalence relation on B which contains ρ and the pairs $(q(b_0,\ldots,b_{n-1}), r(b_0,\ldots,b_{n-1}))$ with $b_0,\ldots,b_{n-1} \in B$. Obviously, the smallest congruence relation on $\mathfrak{B}$ which contains $\rho_{\mathbf{qr}}$, denoted here by $\theta(\rho_{\mathbf{qr}})$, is the smallest congruence on $\mathfrak{B}$ which contains $\rho \cup \{(q(b_0,\ldots,b_{n-1}), r(b_0,\ldots,b_{n-1})) \mid b_0,\ldots,b_{n-1} \in B\}$.

Theorem 1.8.9. $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}} \cong \mathfrak{B}/\theta(\rho_{\mathbf{qr}})$.

Example 1.8.10. Let $(G,\cdot)$ be a group, H a subgroup of G and let $(G/H,\cdot)$ be the hypergroup obtained as in Example 1.1.3. Let γ be the smallest strongly regular equivalence on G/H such that the group obtained as a factor of this hypergroup is commutative. If G' is the derived subgroup of G then Theorem 1.8.9, leads us to the group isomorphism $h : (G/H)/\gamma \to G/(G'H),\ h(\gamma\langle xH\rangle) = x(G'H)$. The *derived (sub)hypergroup* $D(K)$ of a hypergroup $(K,\cdot)$ is $\varphi_K^{-1}(1_{K/\gamma})$, where $\varphi_K : K \to K/\gamma$ is the canonical projection and $1_{K/\gamma}$ is the unit of the group $(K/\gamma,\cdot)$ (see [26, Theorem 3.1]). Let $\pi_H : G \to G/H$ and $\varphi_{G/H} : G/H \to (G/H)/\gamma$ be also the canonical projections. It follows that $D(G/H) = (h \circ \varphi_{G/H})^{-1}(G'H) = \{xH \mid x \in G'H\} = (G'H)/H = \pi_H(G')$.

Corollary 1.8.11. $\overline{\mathfrak{B}/\rho} \cong \mathfrak{B}/\theta(\rho)$.

Example 1.8.12. If $(G,\cdot)$ is a group, H is a subgroup of G, $\overline{H}$ is the smallest normal subgroup containing H then the fundamental group $\overline{G/H}$ is isomorphic to the factor group $G/\overline{H}$.

Corollary 1.8.13. $\overline{\mathfrak{B}/\rho_{\mathbf{qr}}} \cong \mathfrak{B}/\theta(\rho_{\mathbf{qr}})$.

Corollary 1.8.14. $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}} \cong \overline{\mathfrak{B}/\rho_{\mathbf{qr}}}$.

Chapter 2

Constructions of multialgebras

2.1 The direct product of multialgebras

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. The Cartesian product $\prod_{i\in I} A_i$ is a multialgebra of type τ with the multioperations $f_\gamma((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma-1})$. This multialgebra is called the *direct product* of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. The canonical projections of the product, $e_j^I : \prod_{i\in I} A_i \to A_j$, $e_j^I((a_i)_{i\in I}) = a_j$ $(j \in I)$, are multialgebra homomorphisms.

Proposition 2.1.1. *The multialgebra $\prod_{i\in I} \mathfrak{A}_i$, with the canonical projections e_i^I, $i \in I$, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category $\mathbf{Malg}(\tau)$.*

Lemma 2.1.2. *For any $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$ we have $p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1})$.*

Proposition 2.1.3. *Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras and let $\mathbf{q}, \mathbf{r}$ be n-ary terms. If the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\prod_{i\in I} \mathfrak{A}_i$.*

Proposition 2.1.4. *Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras and let $\mathbf{q}, \mathbf{r}$ be n-ary terms. If the identity $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ then the $\mathbf{q} = \mathbf{r}$ is satisfied on $\prod_{i\in I} \mathfrak{A}_i$.*

Proposition 2.1.5. *A direct product of hypergroups is a hypergroup.*

Corollary 2.1.6. *The category $\mathbf{HG}$ of hypergroups is isomorphic to a subcategory closed under products of the category $\mathbf{Malg}((2,2,2))$.*

Proposition 2.1.7. *A direct product of complete multialgebras is a complete multialgebra.*

Corollary 2.1.8. $\mathbf{CMalg}(\tau)$ *is a subcategory closed under products of the category $\mathbf{Malg}(\tau)$.*

2.2 The fundamental algebra of a direct product of multialgebras

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras and let $(\overline{\mathfrak{A}_i} \mid i \in I)$ be the family of the corresponding fundamental algebras. Let us consider the universal algebra $\prod_{i\in I} \overline{\mathfrak{A}_i}$ and the canonical projections

$\pi_j : \prod_{i\in I} \overline{A_i} \to \overline{A_j}$ $(j \in I)$. There exists a unique homomorphism φ of universal algebra such that $\overline{e_j^I} = \pi_j \circ \varphi$ for any $j \in I$. The homomorphism φ is given by $\varphi(\overline{(a_i)_{i\in I}}) = (\overline{a_i})_{i\in I}$ and it is surjective. So, the universal algebra $\overline{\prod_{i\in I} \mathfrak{A}_i}$, with $(\overline{e_i^I} \mid i \in I)$, is the product of the algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$ (in $\mathbf{Alg}(\tau)$) if and only if φ is also injective. This does not always happen, as it follows from:

Example 2.2.1. Let $(H_1 = \{a, b, c\}, \circ)$, $(H_2 = \{x, y, z\}, \circ)$ be the hypergroupoids given bellow:

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

$\circ$	x	y	z
x	x	$y,\ z$	$y,\ z$
y	$y,\ z$	$y,\ z$	$y,\ z$
z	$y,\ z$	$y,\ z$	$y,\ z$

$\overline{H_1 \times H_2}$ has 8 elements, while $\overline{H_1} \times \overline{H_2}$ has only 6 elements.

Proposition 2.2.2. *Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. Suppose that I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for each $i \in I$. The homomorphism φ is injective if and only if for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ and any $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$ there exist $m \in \mathbb{N}^*$, $k_j \in \mathbb{N}$, $\mathbf{q}^j \in \mathbf{P}^{(k_j)}(\tau)$ and $(b_i^{0,j})_{i\in I}, \dots, (b_i^{k_j-1,j})_{i\in I} \in \prod_{i\in I} A_i$, $j \in \{0, \dots, m-1\}$ such that*

$$(x_i)_{i\in I} \in q^0((b_i^{0,0})_{i\in I}, \dots, (b_i^{k_0-1,0})_{i\in I}),\ (y_i)_{i\in I} \in q^{m-1}((b_i^{0,m-1})_{i\in I}, \dots, (b_i^{k_{m-1}-1,m-1})_{i\in I})$$

and for each $j \in \{1, \dots, m-1\}$,

$$q^{j-1}((b_i^{0,j-1})_{i\in I}, \dots, (b_i^{k_{j-1}-1,j-1})_{i\in I}) \cap q^j((b_i^{0,j})_{i\in I}, \dots, (b_i^{k_j-1,j})_{i\in I}) \neq \emptyset. \tag{2.2.1}$$

Corollary 2.2.3. *Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. Suppose that I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for each $i \in I$. If for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that*

$$\prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) \subseteq q((b_i^0)_{i\in I}, \dots, (b_i^{n-1})_{i\in I}) \tag{2.2.2}$$

then the homomorphism φ is injective.

Let $\mathcal{C}$ be a subcategory of $\mathbf{Malg}(\tau)$, let $U : \mathcal{C} \longrightarrow \mathbf{Malg}(\tau)$ be the inclusion functor and let F be the functor from Remark 1.6.21. In the following propositions we will refer to $F \circ U$ as F.

Proposition 2.2.5. *Let $\mathcal{C}$ be a subcategory of $\mathbf{Malg}(\tau)$ closed under finite products. Assume that for any finite set I, for any family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras from $\mathcal{C}$ and for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (2.2.2) holds. Then the functor $F : \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ preserves the finite products.*

Proposition 2.2.6. *Let $\mathcal{C}$ be a subcategory of $\mathbf{Malg}(\tau)$ closed under products and let us consider that $\alpha_{\mathfrak{A}}$ is transitive for each $\mathfrak{A} \in \mathcal{C}$. Assume that for any set I, for any family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras from $\mathcal{C}$ and for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (2.2.2) holds. Then the functor $F : \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ preserves the products.*

The case of hypergroups

Proposition 2.2.7. *$F : \mathbf{HG} \longrightarrow \mathbf{Grp}$ (from Remark 1.7.5) preserves the finite products.*

Yet, F does not preserve the arbitrary products of hypergroups.

Example 2.2.8. Let us consider the hypergroupoid $(\mathbb{Z}, \circ)$ on the set of integers $\mathbb{Z}$, given by $x \circ y = \{x + y, x + y + 1\}$ for any $x, y \in \mathbb{Z}$. $(\mathbb{Z}, \circ)$ is a hypergroup with the fundamental relation $\beta = \mathbb{Z} \times \mathbb{Z}$. The fundamental group of the hypergroup $(\mathbb{Z}^{\mathbb{N}}, \circ)$ has at least two elements since $f, g : \mathbb{N} \to \mathbb{Z}$, $f(n) = 0$, $g(n) = n$ $(n \in \mathbb{N})$ are not in the same equivalence class of the fundamental relation of the hypergroup $(\mathbb{Z}^{\mathbb{N}}, \circ)$.

For the arbitrary products of hypergroups we have:

Theorem 2.2.9. *Let us consider the hypergroups H_i, $i \in I$, with the fundamental relations β^{H_i}. The group $\overline{\prod_{i\in I} H_i}$, with the homomorphisms $(\overline{e_i^I} \mid i \in I)$, is the product of the groups $(\overline{H_i} \mid i \in I)$ if and only if there exists $n \in \mathbb{N}^*$ such that $\beta^{H_i} \subseteq \beta_n^{H_i}$, for all i from I except for a finite number.*

Corollary 2.2.10 *Let $n \in \mathbb{N}$. If $\mathcal{C}_n$ is the class of those hypergroups which satisfy the condition $\beta = \beta_n$ then $\mathcal{C}_n$ is closed under the formation of the direct products and the functor $F : \mathcal{C}_n \longrightarrow \mathbf{Grp}$ determined by the factorization with the fundamental relation preserves the products.*

Corollary 2.2.11. *The functor F preserves the products of complete hypergroups.*

The case of complete multialgebras

Theorem 2.2.12. *Let $(\mathfrak{A}_i \mid i \in I)$ be a family of complete multialgebras of type τ. The following statements are equivalent:*
i) $\overline{\prod_{i\in I} \mathfrak{A}_i}$ (with the homomorphisms $(\overline{e_i^I} \mid i \in I)$ is the product of the universal algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$;
ii) for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$ and $a_i^0, \dots, a_i^{n_i-1} \in A_i$, $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (2.2.2) *holds;*
iii) for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$ and $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ we have $\left|\prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})\right| = 1$ or there exist $\gamma < o(\tau)$, $b_i^0, \dots, b_i^{n_\gamma-1} \in A_i$ $(i \in I)$ such that

$$\prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) \subseteq f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}). \tag{2.2.3}$$

Remark 2.2.16. From Corollary 2.2.11 we deduce that the complete hypergroups are complete multialgebras such that for any family of such multialgebras, the fundamental algebra of the direct product is the direct product of the corresponding fundamental algebras.

2.3 The direct limit of a direct system of multialgebras

Let $((A_i \mid i \in I), (\varphi_{ij} : A_i \to A_j \mid i, j \in I,\ i \le j))$ be a a direct system of sets and let $A_\infty = A/_\equiv = \{\widehat{x} \mid x \in A\}$ be the direct limit of the given direct system of sets (see [29, Definition 22.2]).

If each A_i is the support of a multialgebra $\mathfrak{A}_i$ of type τ and φ_{ij} are multialgebra homomorphisms then $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \le j))$ is a *direct system of multialgebras.* If $(I, \le)$ is well ordered then we will call the direct system $\mathcal{A}$, well ordered.

We will obtain a multialgebra $\mathfrak{A}_\infty$ of type τ on A_∞ if we consider for any $\gamma < o(\tau)$,

$$f_\gamma(\widehat{x_0},\ldots,\widehat{x_{n_\gamma-1}}) = \{\widehat{x'} \mid \exists m \in I,\ \forall j \in \{0,\ldots,n_\gamma-1\},\ \exists x'_j \in \widehat{x_j} \cap A_m :\ x' \in f_\gamma(x'_0,\ldots,x'_{n_\gamma-1})\}.$$

Example 2.3.3. For a direct system of semihypergroups $((H_i,\circ_i) \mid i \in I)$ the multioperation $\circ$ is defined on the direct limit direct of the sets H_i by $\widehat{z} \in \widehat{x} \circ \widehat{y}$ if and only if there exist $m \in I$, $x_m \in \widehat{x} \cap A_m$, $y_m \in \widehat{y} \cap A_m$ and $z_m \in \widehat{z} \cap A_m$ such that $z_m \in x_m \circ_m y_m$. In this way one obtains in [70] the direct limit $(H_\infty,\circ)$ of the direct system of semihypergroups $((H_i,\circ_i) \mid i \in I)$.

Lemma 2.3.4. *Let $\gamma < o(\tau)$ and $\widehat{x_0},\ldots,\widehat{x_{n_\gamma-1}} \in A_\infty$. If $i_0,\ldots,i_{n_\gamma-1} \in I$ are such that $x_0 \in A_{i_0}$,...,$x_{n_\gamma-1} \in A_{i_{n_\gamma-1}}$ then $f_\gamma(\widehat{x_0},\ldots,\widehat{x_{n_\gamma-1}}) = \{\widehat{x'} \in A_\infty \mid \exists m \in I,\ i_0 \le m,\ldots,i_{n_\gamma-1} \le m :\ x' \in f_\gamma(\varphi_{i_0m}(x_0),\ldots,\varphi_{i_{n_\gamma-1}m}(x_{n_\gamma-1}))\}$.*

Remark 2.3.5. If for $\gamma < o(\tau)$, f_γ is an operation in all the multialgebras $\mathfrak{A}_i$ then f_γ is an operation in $\mathfrak{A}_\infty$. As a matter of fact, in order that f_γ be an operation for an ordinal $\gamma < o(\tau)$ it is enough that for any two elements from I to exist an upper bound $m \in I$ such that in $\mathfrak{A}_m$, f_γ is an operation.

Remark 2.3.6. The maps $\varphi_{i\infty} : A_i \to A_\infty$, $\varphi_{i\infty}(x) = \widehat{x}$ are multialgebra homomorphisms.

Theorem 2.3.7. *Let us see the direct system of multialgebras $((\mathfrak{A}_i \mid i \in I),(\varphi_{ij} : A_i \to A_j \mid i,j \in I,\ i \le j))$ as a covariant functor $G : \mathcal{I} \longrightarrow$ **Malg**(τ). The multialgebra $\mathfrak{A}_\infty$ with the homomorphisms $(\varphi_{i\infty} \mid i \in I)$ is the direct limit of G.*

Definition 2.3.1. We will call the multialgebra $\mathfrak{A}_\infty$ the *direct limit of the direct system of multialgebras* $\mathcal{A}$ and we will denote it by $\varinjlim \mathcal{A}$ or by $\varinjlim_{i\in I}\mathfrak{A}_i$.

The next results are generalizations for some results known for universal algebras (see [29, §21]). Now, we will consider that $(I,\le)$ is a directed partially ordered set. Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I),(\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J,\le)$ is also a directed partially ordered set. We will denote by $\mathcal{A}_J$ the direct system consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J,\le)$ and the homomorphisms are $(\varphi_{ij} \mid i,j \in J,\ i \le j)$.

Proposition 2.3.8. *If the subset J is cofinal with $(I,\le)$ then the multialgebras $\varinjlim \mathcal{A}$ and $\varinjlim \mathcal{A}_J$ are isomorphic.*

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I),(\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of multialgebras with $I = \bigcup_{p\in P} I_p$, where $(I_p,\le)$ is a directed partially ordered subset of $(I,\le)$ for each $p \in P$ and $(P,\le)$ is also a directed partially ordered set such that $I_p \subseteq I_q$, whenever $p,q \in P,\ p \le q$. Denote $\varinjlim \mathcal{A} = \mathfrak{A}_\infty = (A_\infty,(f_\gamma)_{\gamma<o(\tau)})$ and $\varinjlim \mathcal{A}_{I_p} = \mathfrak{A}^p_\infty = (A^p_\infty,(f_\gamma)_{\gamma<o(\tau)})$ if $p \in P$. For any $p,q \in P,\ p \le q$ we can define the map $\psi_{pq} : A^p_\infty \to A^q_\infty$, $\psi_{pq}(\widehat{x}_{I_p}) = \widehat{x}_{I_q}$, (where $x \in A_i$, for some $i \in I_p$). This way we obtain a direct system of multialgebras $\mathcal{A}/P$ consisting of $(P,\le)$, the multialgebras $\mathfrak{A}^p_\infty$, and the homomorphisms ψ_{pq}.

Theorem 2.3.10. *The multialgebras $\varinjlim \mathcal{A}$ and $\varinjlim \mathcal{A}/P$ are isomorphic.*

As in the case of universal algebras, we will call algebraic class of multialgebras a class of multialgebras closed under the formation of the formation of isomorphic images.

Theorem 2.3.12. *An algebraic class of multialgebras is closed under the formation of the direct limits of arbitrary direct families if and only if it is closed under the formation of the direct limits of well ordered direct families.*

Applications to particular multialgebras

Lemma 2.3.13. *Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of multialgebras, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$. If $i_0, \ldots, i_{n-1} \in I$ are such that $a_j \in A_{i_j}$ for any $j \in \{0, \ldots, n-1\}$ then*

$$\begin{aligned} p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) =& \{\widehat{a} \mid \exists m \in I,\ \forall j \in \{0, \ldots, n-1\},\ \exists a'_j \in \widehat{a_j} \cap A_m :\ a \in p(a'_0, \ldots, a'_{n-1})\} \\ =& \{\widehat{a} \mid \exists m \in I,\ i_0, \ldots, i_{n-1} \leq m,\ a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}. \end{aligned}$$

Proposition 2.3.14. *Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of multialgebras and let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$ $(i \in I)$ then the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{A}_\infty$.*

Proposition 2.3.16. *let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of multialgebras and let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If the identity $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ $(i \in I)$ then the identity $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{A}_\infty$.*

Proposition 2.3.18. *The direct limit of a direct system of complete multialgebras is a complete multialgebra.*

The case of hypergroups. Let $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of semihypergroups and let us denote by $(H', \circ)$ its direct limit. From Proposition 2.3.16 we deduce:

Theorem 2.3.20. *[70, Theorem 3] $(H', \circ)$ is a semihypergroup.*

Using Proposition 2.3.8 and Proposition 2.3.14 we can obtain the following theorem from [70]:

Theorem 2.3.21. *[70, Theorem 4] If for any $i, j \in I$ there exists $k \in I$, $i \leq k$, $j \leq k$ such that $(H_k, \circ_k)$ is a hypergroup then $(H', \circ)$ is a hypergroup.*

The heart ω_H of a hypergroup $(H, \cdot)$ is the set of those $x \in H$ for which the class $\overline{x}$ of the fundamental group $(\overline{H}, \cdot)$ is the unit from $\overline{H}$. Using Proposition 2.3.8 we deduce the following result:

Theorem 2.3.23. *[40, Theorem 10] Let $((H_i, \circ_i) \mid i \in I)$ be a direct system of semihypergroups such that the following conditions hold:*
1) for any $i, j \in I$ there exist $k \in I$, $i \leq k$, $j \leq k$ such that H_k is a hypergroup;
2) $K = \{k \in I \mid H_k$ is a hypergroup$\}$ is such that $|K| < \aleph_0$.
If $s = \max\{k \mid k \in K\}$ then there exist $\widehat{a_j} \in H'$ $(j \in \{1, \ldots, n\})$ such that $\omega_{H'} = \widehat{a_1} \circ \cdots \circ \widehat{a_n}$ if and only if for all $j \in \{1, \ldots, n\}$ there exists $a_{s,j} \in \widehat{a_j}$ such that $\omega_{H_s} = a_{s,1} \circ \cdots \circ a_{s,n}$.

The case of SHR-semigroups. A semigroup $(S, \cdot)$ is called *SHR-semigroup* if we can enrich the set S with an element 0 such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$ (if we do not already have such an element in S) and we can define a multioperation $+$ on $S^0 = S \cup \{0\}$ such that $(S^0, +, \cdot, 0)$ is a Krasner hyperring. From Proposition 2.3.8, 2.3.16, Remark 1.5.3, 2.3.5 and Example 1.5.5, 1.5.6 follows the next theorem which is one of the main results in [46]:

Theorem 2.3.24. *[46, Theorem 3] Let $(((H_i, \circ_i) \mid i \in I), (f_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of semigroups, such that for all $i \in I$, there exists $k \in I$, $i \leq k$ for which $(H_k, \circ_k)$ is an SHR*

semigroup. Let $K = \{k \in I \mid (H_k, \circ_k)$ is an SHR semigroup$\}$. So, for each $k \in K$, there exists a multioperation $\oplus_k$ on H_k^0 such that $(H_k^0, \oplus_k, \circ_k, 0_k)$ is a hyperring. If for any $k, l \in K$, $k \leq l$, f_{kl} is a homomorphism of hyperrings, then the direct limit of the direct family of semihypergroups $((H_i, \circ_i) \mid i \in I)$ is an SHR semigroup.

On a subcategory of multialgebras

The properties presented in the first part of this paragraph hold for the subcategory of $\mathbf{Malg}(\tau)$ obtained by considering as morphisms the ideal homomorphisms. In other words, the results established here hold if we replace 'homomorphism' by 'ideal homomorphism'. In this case, we can define the multioperations on A_∞ as follows: for each $\gamma < o(\tau)$ and for any $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ with $x_0 \in A_{i_0}, \ldots, x_{n_\gamma - 1} \in A_{i_{n_\gamma - 1}}$ we consider $m \in I$, $i_0, \ldots, i_{n_\gamma - 1} \leq m$ and we set

$$f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}}) = \{\widehat{x} \mid x \in f_\gamma(\varphi_{i_0 m}(x_0), \ldots, \varphi_{i_{n_\gamma - 1} m}(x_{n_\gamma - 1}))\}. \tag{2.3.1}$$

Remark 2.3.32. Since the algebra homomorphisms are ideal homomorphisms, the definition of the operations from the the direct limit of a direct system of universal algebras is (2.3.1). We deduce that the direct limit of a direct system of multialgebras generalize the direct limit of a direct system of universal algebras.

2.4 The fundamental algebra of the direct limit of a direct system of multialgebras

Theorem 2.4.1. *The functor $F : \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$ (from Remark 1.6.21) is a left adjoint for the inclusion functor $U : \mathbf{Alg}(\tau) \longrightarrow \mathbf{Malg}(\tau)$.*

Corollary 2.4.2. *Let $(I, \leq)$ be a directed preordered set and let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras of type τ with the direct limit $\mathfrak{A}_\infty$. The universal algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$ and the homomorphisms $(\overline{\varphi_{ij}} \mid i, j \in I,\ i \leq j)$ form a direct system $\overline{\mathcal{A}}$ of universal algebras of type τ and the direct limit algebra of the direct system $\overline{\mathcal{A}}$ is isomorphic to the universal algebra $\overline{\mathfrak{A}_\infty}$.*

Remark 2.4.3. From Theorem 2.3.7 we obtain the isomorphism μ between $\varinjlim \overline{\mathcal{A}}$ and $\overline{\varinjlim \mathcal{A}}$ defined by $\mu(\widehat{\alpha^*_{\mathfrak{A}_i}\langle a\rangle}) = \alpha^*_{\mathfrak{A}_\infty}\langle \widehat{a}\rangle$, (where $a \in A_i$, $i \in I$).

Remark 2.4.4. Let $(H_i \mid i \in I)$ be a direct system of semihypergroups, let us consider that the semihypergroup H_i has the fundamental relation $\beta^*_{H_i}$ and let us denote by H' the direct limit of this direct system and by $\beta_{H'}$ its fundamental relation. In [70, Theorem 5] it is proved that *if $x, y \in H_i$, $x \beta^*_{H_i} y$ then $\widehat{x} \beta_{H'} \widehat{y}$*, and that *if $x, y \in H'$, $\widehat{x} \beta^*_{H'} \widehat{y}$ then there exist $i \in I$, $x_i \in \widehat{x} \cap H_i$, $y_i \in \widehat{y} \cap H_i$ such that $x_i \beta^*_{H_i} y_i$*. These statements also result from the fact that μ from Remark 2.4.3 is well defined and injective.

2.5 On the inverse limit of an inverse system of multialgebras

Let $((A_i \mid i \in I), (\varphi_j^k : A_j \to A_k \mid j,k \in I,\ j \leq k))$ be an inverse system of sets. Let us consider that each A_i is a support set for a multialgebra $\mathcal{A}_i$ of type τ and that each φ_j^k $(j,k \in I,\ j \leq k)$ is a homomorphism. This way we obtain an inverse system of multialgebras $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_j^k : A_j \to A_k \mid j,k \in I,\ j \leq k))$. If $(I, \leq)$ is a well ordered set then we say that $\mathcal{A}$ is a well ordered inverse system of multialgebras. We should remind that the inverse limit of the inverse system of sets $((A_i \mid i \in I), (\varphi_j^k : A_j \to A_k \mid j,k \in I,\ j \leq k))$ is the set $A^\infty = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i \mid \forall j,k \in I,\ j \leq k,\ \varphi_j^k(a_k) = a_j\}$, together with the maps $\varphi_j^\infty : A^\infty \to A_j,\ \varphi_j^\infty((a_i)_{i\in I}) = a_j$. We also remind that the inverse limit of an inverse system of nonempty sets can be empty (see [29, p.132]).

In [29] it is mentioned that the inverse limits for first order structures are defined the same way as for algebras, as suitable substructures of the direct product. If we see each n_γ-ary multioperation in each $\mathfrak{A}_i$ as an $(n_\gamma + 1)$-ary relation r_γ as in (1.1.3), we obtain the definitions for the relations on A^∞ : given $\gamma < o(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma - 1})_{i\in I}, (a_i)_{i\in I} \in A^\infty$ we have

$$((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I}, (a_i)_{i\in I}) \in r_\gamma \Leftrightarrow a_i \in f_\gamma(a_i^0, \ldots, a_i^{n_\gamma-1}),\ \forall i \in I.$$

Since we are dealing with multialgebras our question is whether the relational system obtained in this way is a multialgebra. If the answer were affirmative then, using again (1.1.3), it would follow that its multioperations would be defined by:

$$f_\gamma((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma-1}) \cap A^\infty, \tag{2.5.1}$$

for any $\gamma < o(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I} \in A^\infty$.

Remark 2.5.3. The inverse limit A^∞ of the inverse system of sets $((A_i \mid i \in I), (\varphi_j^k : A_j \to A_k \mid j,k \in I,\ j \leq k))$ is not, in general, a submultialgebra of $\prod_{i\in I} \mathfrak{A}_i$, thus the intersection with A^∞ cannot be omitted in (2.5.1).

Example 2.5.4. Let us consider $I = \{1, 2\}$ ordered by the relation $\leq$, induced by the usual ordering from $\mathbb{N}$. Let us also consider the inverse system consisting of the hypergroupoids $(H_1, \circ)$, $(H_2, \circ)$ defined on $H_1 = H_2 = \{x, y\}$ by $x \circ x = x \circ y = y \circ x = y \circ y = \{x, y\}$ and by the (ideal) homomorphisms $\varphi_1^1 = 1_{H_1},\ \varphi_2^2 = 1_{H_2}$ and $\varphi_1^2 : H_2 \to H_1,\ \varphi_1^2(x) = y,\ \varphi_1^2(y) = x$. Then $H^\infty = \{(x, y), (y, x)\}$ is not a subhypergroupoid of $H_1 \times H_2$.

Remark 2.5.5. The correspondences f_γ given by (2.5.1) are not always multioperations on A^∞. Even if $A^\infty \neq \emptyset$, the intersection in the second member of the equality can be the empty set. As a matter of fact, for any $\gamma < o(\tau)$, $(a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I} \in A^\infty$ and any $j,k \in I,\ j \leq k$, the sets $(f_\gamma(a_i^0, \ldots, a_i^{n_\gamma-1}) \mid i \in I)$ with the corresponding restrictions of the maps φ_k^j, form an inverse system of sets and the second member in (2.5.1) is the inverse limit of this system of sets. So, $f_\gamma((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma-1})_{i\in I})$ can be empty even if A^∞ is not.

Example 2.5.6. In [35], G. Higman and A. H. Stone present an example of inverse system of (countable) sets $(S_\alpha \mid \alpha < \omega_1)$, with surjective maps and empty inverse limit. It follows that the family $(S_\alpha \mid 1 \leq \alpha < \omega_1)$, with the corresponding maps, form an inverse system with empty limit. Let us consider for each $1 \leq \alpha < \omega_1$, $A_\alpha = S_\alpha \cup \{0_{E_\alpha}\}$, where $0_{E_\alpha} : E_\alpha \to \mathbb{R}$, $0_{E_\alpha}(\gamma) = 0$. Let us define the binary multioperation $\circ$ on A_α by $f \circ g = S_\alpha$, if $f = 0_{E_\alpha} = g$ or $f \neq 0_{E_\alpha} \neq g$ and $f \circ g = \{0_{E_\alpha}\}$, otherwise. The maps $\varphi_\alpha^\beta : A_\beta \to A_\alpha$, $\varphi_\alpha^\beta(f) = f|_{E_\alpha}$ $(\alpha < \beta)$ are (ideal) homomorphisms. This way we obtain an inverse system of hypergroupoids. We have $A^\infty = \{(0_{E_\alpha})_{1 \leq \alpha < \omega_1}\}$ and $(0_{E_\alpha})_{1 \leq \alpha < \omega_1} \circ (0_{E_\alpha})_{1 \leq \alpha < \omega_1} = \emptyset$.

Remark 2.5.7. In order to obtain a multialgebra $\mathfrak{A}^\infty$ on $A^\infty \neq \emptyset$ defined by (2.5.1) it would be required that for every $\gamma < o(\tau)$ and $(a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I} \in A^\infty$, $\underleftarrow{\lim}_{i \in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}) \neq \emptyset$. Such a case is given by the condition that for every $i \in I$, $\gamma < o(\tau)$ and $a_i^0, \ldots, a_i^{n_\gamma - 1} \in A_i$, the set $f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1})$ to be nonempty and finite (see [29, §21]).

Remark 2.5.8. Since $|f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1})| = 1$ for any universal universal algebra with nonempty support set, we deduce that the universal algebras satisfy the above conditions. It is also clear that if for $\gamma < o(\tau)$, f_γ is an operation in each multialgebra $\mathfrak{A}_i$ then f_γ is an operation in $\mathfrak{A}^\infty$. If this is the case, (2.5.1) can be rewritten as $f_\gamma((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I}) = (f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}))_{i \in I}$, for any $(a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I}$.

Remark 2.5.9. If $\mathfrak{A}^\infty$ is a multialgebra then the maps φ_j^∞ $(j \in I)$ are multialgebra homomorphisms.

Theorem 2.5.12. *The inverse system of multialgebras* $((\mathfrak{A}_i \mid i \in I), (\varphi_j^k \mid j, k \in I,\ j \leq k))$ *determines a contravariant functor* $G : \mathcal{I} \longrightarrow \mathbf{Malg}(\tau)$. *If* $\mathfrak{A}^\infty = (A^\infty, (f_\gamma)_{\gamma < o(\tau)})$ *is a multialgebra then, together with the homomorphisms* $(\varphi_j^\infty \mid j \in I)$, *is the inverse limit of* G.

The last three results of this section are generalizations for some results presented for universal algebras in [29, §21]. From now on we will consider that $(I, \leq)$ is a directed partially ordered set. Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_j^k \mid j, k \in I,\ j \leq k))$ be an inverse system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also a directed partially ordered set. We will denote by $\mathcal{A}_J$ the inverse system of multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J, \leq)$ and whose homomorphisms are φ_j^i, with $i, j \in J,\ i \leq j$.

Proposition 2.5.14. *If J is cofinal with $(I, \leq)$, then the relational system* $\underleftarrow{\lim}\mathcal{A}$ *is a multialgebra if and only if* $\underleftarrow{\lim}\mathcal{A}_J$ *is a multialgebra. If this is the case, the two multialgebras are isomorphic.*

Remark 2.5.15. The inverse limits from [16], [44] and [46] are inverse limits of inverse systems of (particular) multialgebras with the carrier $(I, \leq)$ directed ordered set with a maximum. From Proposition 2.5.14, it follows that such an inverse limit exists and it is isomorphic to the member of the system having this maximum as an index. It is clear that such an inverse limit exists and it has all the properties of this member.

Let us consider that the support set I of the carrier $(I, \leq)$ of the inverse system $\mathcal{A} = (\mathfrak{A}_i \mid i \in I)$ of multialgebras can be written as $I = \bigcup_{p \in P} I_p$, where $(P, \leq)$ and $(I_p, \leq)$ $(p \in P)$ are directed partially ordered sets such that $I_p \subseteq I_q$, whenever $p, q \in P,\ p \leq q$. We will denote

$$\underleftarrow{\lim}\mathcal{A} = \mathfrak{A}^\infty = (A^\infty, (f_\gamma)_{\gamma < o(\tau)}),\ \underleftarrow{\lim}\mathcal{A}_{I_p} = \mathfrak{A}_p^\infty = (A_p^\infty, (f_\gamma)_{\gamma < o(\tau)})\ (p \in P).$$

For any $p, q \in P,\ p \leq q$ we can define the map $\psi_p^q : A_q^\infty \to A_p^\infty$, $\psi_p^q((a_i)_{i \in I_q}) = (a_i)_{i \in I_p}$. In this way we obtain an inverse system of sets $\mathcal{A}/P$ consisting of $(P, \leq)$, the sets A_p^∞, and the maps ψ_p^q.

Theorem 2.5.16. *Assume that for each $p \in P$, $\mathfrak{A}_p^\infty$ is a multialgebra. Then $\mathcal{A}/P$ is an inverse system of multialgebras and $\varprojlim \mathcal{A}$ is a multialgebra if and only if $\varprojlim \mathcal{A}/P$ is a multialgebra. If this is the case, the two multialgebras are isomorphic.*

Theorem 2.5.17. *An algebraic class of multialgebras is closed under the formation of inverse limits of arbitrary inverse systems if and only if it is closed under the formation of inverse limits of well ordered inverse systems.*

2.6 On the fundamental algebra of the inverse limit of an inverse system of multialgebras

In general, the functor F from Remark 1.6.21 does not preserve the inverse limits of inverse families of multialgebras, even if they are multialgebras. This will result from Example 2.6.2.

Example 2.6.1. An useful example of inverse limit of multialgebras can be given as in [29, p.133]. So, let us consider a set I and a family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras of type τ. We can get an inverse system of multialgebras taking $(J, \subseteq)$ to be the set of all the finite nonempty subsets of the set I, ordered with the set inclusion, $\mathfrak{B}_j = \prod_{i \in j} \mathfrak{A}_i$, for any $j \in J$, and the canonical projections $\varphi_{j_1}^{j_0}$ from $\prod_{i \in j_0} \mathfrak{A}_i$ onto $\prod_{i \in j_1} \mathfrak{A}_i$, for any $j_0 \supseteq j_1$ from J. The inverse limit of this inverse system of multialgebras exists and it is isomorphic to $\prod_{i \in I} \mathfrak{A}_i$.

Example 2.6.2. In the previous example we take $I = \mathbb{N}$ and $\mathfrak{A}_i = (\mathbb{Z}, \circ)$ from Example 2.2.8 for each $i \in I$. It will result an inverse system consisting of the hypergroups $(H_j, \circ) = (\mathbb{Z}^j, \circ)$ with $j \subseteq \mathbb{N}$ finite. The fundamental group of each hypergroup H_i is a one element group. It follows that the inverse limit of the inverse system of the corresponding fundamental groups is the one element group. But the fundamental group of the inverse limit of the inverse system $((H_j, \circ) \mid j \subseteq \mathbb{N},\ j \text{ is finite})$ is isomorphic to the fundamental group of the direct power $\mathbb{Z}^{\mathbb{N}}$, and this has at least two elements.

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_j^k \mid j, k \in I,\ j \leq k))$ be an inverse system of multialgebras. We will denote by $\overline{\mathcal{A}}$ the inverse system of the fundamental algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$ of the multialgebras from $\mathcal{A}$, with the homomorphisms $(\overline{\varphi_j^i} \mid i, j \in I,\ i \geq j)$. So, if we see the inverse system $\mathcal{A}$ as a contravariant functor G then $\overline{\mathcal{A}}$ is the functor $F \circ G$. In this section, we will refer to the inverse limit $\varprojlim \mathcal{A}$ of the inverse system $\mathcal{A}$ as the inverse limit $(\mathfrak{A}^\infty,\ (\varphi_i^\infty \mid i \in I))$ of G. Clearly, $\varprojlim \overline{\mathcal{A}} = \varprojlim (FG)$. If we denote $(\overline{\mathfrak{A}^\infty},\ (\overline{\varphi_i^\infty} \mid i \in I))$ by $\overline{\varprojlim \mathcal{A}}$, we have:

Proposition 2.6.3. *Let $\mathcal{A}$ be an inverse system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ with $(J, \leq)$ a directed partially ordered set cofinal with $(I, \leq)$. Consider that $\varprojlim \mathcal{A}_J$ is a multialgebra. Under these conditions, $\overline{\varprojlim \mathcal{A}}$ is the inverse limit of the inverse system of universal algebras $\overline{\mathcal{A}}$ if and only if $\overline{\varprojlim \mathcal{A}_J}$ is the inverse limit of the inverse system of universal algebras $\overline{\mathcal{A}_J}$.*

Proposition 2.6.4. *Using the notations from the previous section, let us consider that $\mathfrak{A}_p^\infty$, $p \in P$, and $\mathfrak{A}^\infty$ are multialgebras. Let us also consider that for each $p \in P$, $\overline{\varprojlim \mathcal{A}_{I_p}}$ is the inverse limit of the inverse system $\overline{\mathcal{A}_{I_p}}$ of universal algebras. Then $\overline{\varprojlim \mathcal{A}}$ is the inverse limit of $\overline{\mathcal{A}}$ if and only if $\overline{\varprojlim \mathcal{A}/P}$ is the inverse limit of $\overline{\mathcal{A}/P}$.*

If K is a class of multialgebras of type τ then we can obtain a subcategory $\mathcal{K}$ of $\mathbf{Malg}(\tau)$ if we consider as morphisms only those homomorphisms which are defined between two multialgebras from K. Knowing the definition of U, in the next theorem we may use F instead of the composition $F \circ U$ of the functor F with the inclusion functor $U : \mathcal{K} \longrightarrow \mathbf{Malg}(\tau)$.

Theorem 2.6.5. *Let K be an algebraic class of multialgebras closed under the formation of inverse limits of well ordered inverse systems. Then F preserves the inverse limits of arbitrary inverse systems of multialgebras from K if and only if F preserves the inverse limits of well ordered inverse systems of multialgebras from K.*

Concluding remarks

Along the previous contributions in the theory of multialgebras, such as those of H. E. Pickett and G. Grätzer, our thesis confirms once again especially through the results obtained in the first chapter the fact that the theory of multialgebras is a natural extension of the theory of universal algebras. One can obtain important results concerning multialgebras as they are particular cases of relational systems and, at the same time, generalizations of the universal algebras. We also notice that the study of multialgebras and of the identities on multialgebras provide interesting information about some particular classes of multialgebras. Given the above affirmations and the genesis of this thesis, we will formulate a few problems which may be a continuation of this research.

We noticed that an important role in the study of multialgebras is played by the algebra the nonempty subsets of a multialgebra. This algebra was introduced starting form a multialgebra of type τ, $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$, defining the operations on the set $P^*(A)$ of the nonempty subsets of the set A through the equalities:

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\}\}.$$

A first problem could be *the characterization of the universal algebras that have the set $P^*(A)$ as support and which can be obtained as above from a multialgebra of support A.*

Looking at the characterization of the complete multialgebras given in Proposition 1.7.20, we observe that a complete multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ with the support set A results from a universal algebra $\mathfrak{A}' = (A, (f'_\gamma)_{\gamma<o(\tau)})$ on A (of the same type) and an equivalence relation ρ on A by considering for any $\gamma < o(\tau)$ and $a_0, \ldots, a_{n_\gamma-1} \in A$

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \rho\langle f'_\gamma(a_0, \ldots, a_{n_\gamma-1})\rangle.$$

Another problem could be *the characterization of the equivalence relations of the support set of a universal algebra which lead us to complete multialgebras by using the above procedure.*

Given the characterization theorem for multialgebras formulated by Grätzer, as well as the construction of the free algebra over a class of universal algebras, the following question arises: *can we generalize the construction of the free algebra in categories of multialgebras?*

We saw that the factor of a universal algebra $\mathfrak{B}$ modulo an equivalence ρ is a multialgebra which verifies in a weak manner the identities that are satisfied on $\mathfrak{B}$. Some identities on $\mathfrak{B}$, such as those that characterize the commutativity of an operation, are satisfied even in a strong manner on any factor multialgebra $\mathfrak{B}/\rho$. Also, owed to the equivalence used for the factorization the identities of the algebra $\mathfrak{B}$ can be satisfied in a strong manner on the factor multialgebra. *Can one characterize the identities of a universal algebra which are satisfied in a strong manner in any multialgebra obtained as a factor of the given universal algebra modulo an equivalence relation? What about the equivalences of a universal algebra for which an identity of the algebra holds in a strong manner on the factor multialgebra?*

The inverse limit of an inverse system of multialgebras that we studied is the limit in the category of the relational systems (of the same type). *Supposing that this limit is a multialgebra, what can be said about the identities verified on each multialgebra in the given inverse system?* Let us consider that the contravariant functor determined by an inverse system of multialgebras of the type τ. The following question arises: *when does the limit of this functor in the category* $\mathbf{Malg}(\tau)$ *exist?*

Of course, there also are other constructions of universal algebras that can be generalized to multialgebras, as well as constructions of relational systems that can be particularized to multialgebras. Among the constructions that are important both to the theory of universal algebras and the model theory, we mention ultraproducts. For a set I, a family of universal algebras $(\mathfrak{A}_i = (A_i, (f_\gamma)_{\gamma<o(\tau)}) \mid i \in I)$ and an ultrafilter U over I one considers the relation

$$\theta_U \subseteq \prod_{i\in I} A_i \times \prod_{i\in I} A_i,\ a\theta_U b \Leftrightarrow \{i \in I \mid a(i) = b(i)\} \in U$$

and it can be noticed that θ_U is a congruence on $\prod_{i\in I} \mathfrak{A}_i$. The factor algebra $\left(\prod_{i\in I} \mathfrak{A}_i\right)/\theta_U$ is called an ultraproduct of the family of algebras $(\mathfrak{A}_i \mid i \in I)$ (see [6, §6, Cap. IV]). Starting this construction by using a family of multialgebras $(\mathfrak{A}_i \mid i \in I)$ and given the fact that θ_U is an equivalence relation on $\prod_{i\in I} A_i$, $\prod_{i\in I} \mathfrak{A}_i/\theta_U$ is a multialgebra. From here results the following question: *when is the fundamental algebra of an ultraproduct of multialgebras isomorphic with an ultraproduct of the corresponding fundamental algebras?*

It would also be interesting to see *when* $\theta_U \in E_{ua}\left(\prod_{i\in I} \mathfrak{A}_i\right)$*?* Given the numerous applications of ultraproducts in the model theory and in the theory of universal algebras, the above construction opens up new possibilities of research in the field of multialgebras.

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