Multialgebras, universal algebras and identities

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A recipe:

The first example of multialgebra (hypergroup) was introduced by Marty at The 8th Congress of the Scandinavian Mathematicians (Stockholm, 1934) as follows:

Let \((G, \cdot)\) be a group, \(H \leq G\) and

\[ G/H = \{xH \mid x \in G\}. \]

The equality

\[(1) \quad (xH)(yH) = \{zH \mid z = x'y', \; x' \in xH, \; y' \in yH\}. \]

defines an operation on \(G/H\) if and only if \(H \leq G\).

Otherwise, (1) defines a function

\[ G/H \times G/H \to P^*(G/H) \]

called binary multioperation (on \(G/H\)).

A multialgebra \(\mathcal{A} = (A, (f_{\gamma})_{\gamma < o(\tau)})\) consists in a set \(A\) and a family of multioperations

\[ f_{\gamma} : A^{n_{\gamma}} \to P^*(A), \; \gamma < o(\tau) \]

\((f_{\gamma} \text{ is a multioperation of arity } n_{\gamma} \text{ that corresponds to a symbol } f_{\gamma}).\)
The recipe:

Later, in ‘A representation theorem for multi-algebras’, *Arch. Math.*, 3 1962, 452–456, Grätzer proved that:

Any multialgebra $\mathfrak{A}$ results from a universal algebra $\mathfrak{B}$ and an appropriate equivalence on $B$ as before, i.e. by considering

$$f_\gamma(\rho\langle a_0 \rangle, \ldots, \rho\langle a_{n\gamma - 1} \rangle) = \{\rho\langle b \rangle \mid b \in f_\gamma(b_0, \ldots, b_{n\gamma - 1}), \ a_i \rho b_i, \ i \in \{0, \ldots, n\gamma - 1\}\}.$$

⇒ the importance of factor multialgebras in multialgebra theory.
The start:


⇒ among the equivalence relations of (semi)hypergroups, a great importance have those equivalence relations for which the factor hypergroup(oid) is a group (strongly regular equivalences)
Let \((H, \cdot)\) be a (semi)hypergroup.

- the smallest strongly regular equivalence of \((H, \cdot)\) =
  the fundamental relation of \((H, \cdot)\) =
  the transitive closure \(\beta^*\) of the relation

\[
x \beta y \iff \exists n \in \mathbb{N}^*, \exists a_1, \ldots, a_n \in H : x, y \in a_1 \cdots a_n
\]

(if \((H, \cdot)\) is a hypergroup, then \(\beta^* = \beta\))

- the smallest strongly regular equivalence of \((H, \cdot)\) for
  which the factor hypergroup is a commutative group =
  the transitive closure \(\gamma^*\) of the relation

\[
\gamma = \bigcup_{n \in \mathbb{N}^*} \gamma_n,
\]

where \(\gamma_1 = \delta_H\) and, for any \(n > 1\), \(\gamma_n\) is defined by

\[
x \gamma_n y \iff \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x = \prod_{i=1}^{n} z_i, y = \prod_{i=1}^{n} z_{\sigma(i)}
\]

(if \((H, \cdot)\) is a hypergroup, then \(\gamma^* = \gamma\))
The problems:

1. Define the fundamental relation $\alpha^*$ for a (general) multialgebra and prove that

$$\left( B/\rho \right)/\alpha^* \cong B/\theta(\rho),$$

where $B$ is a universal algebra, $\rho \in E(B)$, and $\theta(\rho)$ is the smallest congruence relation on $B$ which contains $\rho$.

2. Given $q, r \in P^{(n)}(\tau) \ (n \in \mathbb{N})$, determine the smallest equivalence relation $\alpha^{*\text{qr}}$ of a (general) multialgebra for which the factor multialgebra is a universal algebra satisfying the identity

$$q = r$$

and prove that

$$\left( B/\rho \right)/\alpha^{*\text{qr}} \cong B/\theta(\rho_{\text{qr}}),$$

where $B$ is a universal algebra, $\rho \in E(B)$, $\rho_{\text{qr}}$ is the smallest equivalence on $B$ containing $\rho$ and all the pairs

$$(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})), \ b_0, \ldots, b_{n-1} \in B,$$

$\theta(\rho_{\text{qr}})$ is the smallest congruence relation on $B$ which contains $\rho$ and gives (after factorization) a universal algebra satisfying the identity $q = r$ (i.e. the smallest congruence relation on $B$ containing

$$\rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}).$$
The tools:

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < \omega(\tau)})$ be a multialgebra.

- the (universal) algebra $\mathfrak{P}^*(\mathfrak{A})$ of the nonempty subsets of $\mathfrak{A}$, defined by
  $$f_\gamma(A_0, \ldots, A_{n-1}) = \bigcup \{f_\gamma(a_0, \ldots, a_{n-1}) \mid a_i \in A_i, \ i \in \{0, \ldots, n_\gamma - 1\}\};$$

- the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the $n$-ary term functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$;

- the algebra $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ of the $n$-ary polynomial functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$;

- the subalgebra $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ of $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ generated by
  $$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \ldots, n - 1\}\},$$
  where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are given by
  $$c^n_a(A_0, \ldots, A_{n-1}) = \{a\} \text{ and } e^n_i(A_0, \ldots, A_{n-1}) = A_i;$$

- the set $E_{ua}(\mathfrak{A})$ of the equivalence relations $\rho$ on $A$ for which $\mathfrak{A}/\rho$ is a universal algebra (which proves to be an algebraic closure system on $A \times A$).
The results:

• the fundamental relation $\alpha^*$ of the multialgebra $\mathfrak{A} = \alpha$ is the smallest element from $E_{ua}(\mathfrak{A}) =$ the transitive closure of the relation $\alpha$ defined by

$$x \alpha y \iff x, y \in p(a_0, \ldots, a_{n-1})$$

for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \ldots, a_{n-1} \in A$;

• the smallest equivalence relation $\alpha^*_{qr}$ from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $q = r$ is the transitive closure of the relation

$$x \alpha_{qr} y \iff \exists p \in P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A})), \exists a_0, \ldots, a_{n-1} \in A:$$

$$x \in p(q(a_0, \ldots, a_{n-1})), \ y \in p(r(a_0, \ldots, a_{n-1})) \text{ or }$$

$$y \in p(q(a_0, \ldots, a_{n-1})), \ x \in p(r(a_0, \ldots, a_{n-1})) ;$$

$\Rightarrow$ the fundamental relation $\alpha^*$ of a multialgebra $\mathfrak{A}$ is, also, the transitive closure of the relation

$$x \alpha'y \iff \exists p \in P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A})), \exists a \in A : x, y \in p(a) ;$$

• the correspondence

$$(B/\rho)/\alpha^*_{qr} \to B/\theta(\rho_{qr}), \ a^*_{qr}\langle \rho\langle a \rangle \rangle \mapsto \theta(\rho_{qr})\langle a \rangle$$

is a universal algebra isomorphism.
A return to hypergroups:

• Let \((H, \cdot)\) be a hypergroup. The strongly regular relation \(\gamma^* = \gamma\) is the transitive closure of

\[
\gamma' = \alpha_{x_0 x_1 x_1 x_0}
\]
i.e.

\[
\gamma' = \bigcup_{n \in \mathbb{N}^*} \gamma'_n,
\]
where \(\gamma'_1 = \delta_H\) and for \(n > 1\),

\[
x \gamma'_n y \iff \exists z_1, \ldots, z_n \in H, \exists i \in \{1, \ldots, n - 1\}:
\]

\[
x \in z_1 \cdots z_{i-1}(z_i z_{i+1})z_{i+2} \cdots z_n,
\]

\[
y \in z_1 \cdots z_{i-1}(z_{i+1} z_i)z_{i+2} \cdots z_n,
\]

which is, clearly, (the transitive closure of) the relation

\[
\gamma = \bigcup_{n \in \mathbb{N}^*} \gamma_n,
\]
with \(\gamma'_1 = \delta_H\) and, for any \(n > 1\),

\[
x \gamma_n y \iff \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},
\]
since the cycles \((1, 2), (2, 3), \ldots, (n - 1, n)\) generate the symmetric group \((S_n, \circ)\).
The derived subhypergroup of a hypergroup \((K, \cdot)\) is
\[ D(K) = \varphi_K^{-1}(1_{K/\gamma}), \]
where \(\varphi_K : K \to K/\gamma\) is the canonical projection and \(1_{K/\gamma}\) is the identity of the group \((K/\gamma, \cdot)\).

Let \((G, \cdot)\) be a group, \(G'\) its derived subgroup, \(H \leq G\) and \((G/H, \cdot)\) the hypergroup defined by
\[
(xH)(yH) = \{zH \mid z = x'y', \ x' \in xH, \ y' \in yH\}. \tag{1}
\]
Let \(\pi_H : G \to G/H\) and \(\varphi_{G/H} : G/H \to (G/H)/\gamma\) be the canonical projections. The group isomorphism
\[ h : (G/H)/\gamma \to G/(G'H), \ h(\gamma\langle xH \rangle) = x(G'H) \]
helped us to establish a connection between the derived subhypergroup of \(G/H\) and the derived subgroup of \(G\)
\[ D(G/H) = (h \circ \varphi_{G/H})^{-1}(G'H) = (G'H)/H = \pi_H(G'). \]
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