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Multialgebras, universal algebras and identities

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C. Pelea, I. Purdea, 'Multialgebras, universal algebras and identities', *J. Aust. Math. Soc*, **81**, 2006, 121–139

A recipe:

The first example of multialgebra (hypergroup) was introduced by Marty at The 8th Congress of the Scandinavian Mathematicians (Stockholm, 1934) as follows:

Let $(G, \cdot)$ be a group, $H \leq G$ and

$$G/H = \{xH \mid x \in G\}.$$

The equality

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}. \tag{1}$$

defines an operation on G/H if and only if $H \trianglelefteq G$.

Otherwise, (1) defines a function

$$G/H \times G/H \to P^*(G/H)$$

called binary multioperation (on G/H).

A multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ consists in a set A and a family of multioperations

$$f_\gamma : A^{n_\gamma} \to P^*(A),\ \gamma < o(\tau)$$

(f_γ is a multioperation of arity n_γ that corresponds to a symbol $\mathbf{f}_\gamma$).

The recipe:

Later, in ‘A representation theorem for multi-algebras’, *Arch. Math.*, **3** 1962, 452–456, Grätzer proved that:

Any multialgebra $\mathfrak{A}$ results from a universal algebra $\mathfrak{B}$ and an appropriate equivalence on B as before, i.e. by considering

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma - 1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma - 1}),\ a_i \rho b_i,$$

$$i \in \{0, \ldots, n_\gamma - 1\}\}.$$

$\Rightarrow$ the importance of factor multialgebras in multialgebra theory.

The start:

M. Dresher, O. Ore, 'Theory of multigroups', *Amer. J. Math.*, **60** 1938, 705–733.

T. Vougiouklis, 'Representations of hypergroups by generalized permutations', *Algebra Universalis*, **29** 1992, 172–183.

D. Freni, 'A new characterization of the derived hypergroup via strongly regular equivalences', *Comm. Algebra*, **30** 2002, 3977–3989.

$\Rightarrow$ among the equivalence relations of (semi)hypergroups, a great importance have those equivalence relations for which the factor hypergroup(oid) is a group (strongly regular equivalences)

Let $(H, \cdot)$ be a (semi)hypergroup.

• the smallest strongly regular equivalence of $(H, \cdot)$ = the fundamental relation of $(H, \cdot)$ = the transitive closure β^* of the relation

$$x \beta y \Leftrightarrow \exists n \in \mathbb{N}^*, \ \exists a_1, \ldots, a_n \in H : \ x, y \in a_1 \cdots a_n$$

(if $(H, \cdot)$ is a hypergroup, then $\beta^* = \beta$)

• the smallest strongly regular equivalence of $(H, \cdot)$ for which the factor hypergroup is a commutative group = the transitive closure γ^* of the relation

$$\gamma = \bigcup_{n \in \mathbb{N}^*} \gamma_n,$$

where $\gamma_1 = \delta_H$ and, for any $n > 1$, γ_n is defined by

$$x \gamma_n y \Leftrightarrow \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, \ y \in \prod_{i=1}^{n} z_{\sigma(i)}$$

(if $(H, \cdot)$ is a hypergroup, then $\gamma^* = \gamma$)

The problems:

1. Define the fundamental relation α^* for a (general) multialgebra and prove that

$$(\mathfrak{B}/\rho)/\alpha^* \cong \mathfrak{B}/\theta(\rho),$$

where $\mathfrak{B}$ is a universal algebra, $\rho \in E(B)$, and $\theta(\rho)$ is the smallest congruence relation on $\mathfrak{B}$ which contains ρ.

2. Given $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ $(n \in \mathbb{N})$, determine the smallest equivalence relation $\alpha^*_{\mathbf{qr}}$ of a (general) multialgebra for which the factor multialgebra is a universal algebra satisfying the identity

$$\mathbf{q} = \mathbf{r}$$

and prove that

$$(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}} \cong \mathfrak{B}/\theta(\rho_{\mathbf{qr}}),$$

where $\mathfrak{B}$ is a universal algebra, $\rho \in E(B)$, $\rho_{\mathbf{qr}}$ is the smallest equivalence on B containing ρ and all the pairs

$$(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})),\ b_0, \ldots, b_{n-1} \in B,$$

$\theta(\rho_{\mathbf{qr}})$ is the smallest congruence relation on $\mathfrak{B}$ which contains ρ and gives (after factorization) a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$ (i.e. the smallest congruence relation on $\mathfrak{B}$ containing

$$\rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}).$$

The tools:

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra.

• the (universal) algebra $\mathfrak{P}^*(\mathfrak{A})$ of the nonempty subsets of $\mathfrak{A}$, defined by

$$f_\gamma(A_0, \dots, A_{n_\gamma - 1}) = \\ \bigcup \{ f_\gamma(a_0, \dots, a_{n_\gamma - 1}) \mid \ a_i \in A_i,\ i \in \{0, \dots, n_\gamma - 1\}\};$$

• the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$;

• the algebra $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary polynomial functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$;

• the subalgebra $\mathfrak{P}^{(n)}_{A}(\mathfrak{P}^*(\mathfrak{A}))$ of $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \dots, n-1\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are given by

$$c^n_a(A_0, \dots, A_{n-1}) = \{a\} \text{ and } e^n_i(A_0, \dots, A_{n-1}) = A_i;$$

• the set $E_{ua}(\mathfrak{A})$ of the equivalence relations ρ on A for which $\mathfrak{A}/\rho$ is a universal algebra (which proves to be an algebraic closure system on $A \times A$).

The results:

• the fundamental relation α^* of the multialgebra $\mathfrak{A}$ = the smallest element from $E_{ua}(\mathfrak{A})$ = the transitive closure of the relation α defined by

$$x\alpha y \Leftrightarrow x, y \in p(a_0, \ldots, a_{n-1})$$

for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \ldots, a_{n-1} \in A$;

• the smallest equivalence relation $\alpha^*_{\mathbf{qr}}$ from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$ is the transitive closure of the relation

$$\begin{aligned} x\alpha_{\mathbf{qr}}y \ \Leftrightarrow\ & \exists p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists\ a_0, \ldots, a_{n-1} \in A: \\ & x \in p(q(a_0, \ldots, a_{n-1})),\ y \in p(r(a_0, \ldots, a_{n-1})) \text{ or} \\ & y \in p(q(a_0, \ldots, a_{n-1})),\ x \in p(r(a_0, \ldots, a_{n-1})); \end{aligned}$$

$\Rightarrow$ the fundamental relation α^* of a multialgebra $\mathfrak{A}$ is, also, the transitive closure of the relation

$$x\alpha' y \Leftrightarrow \exists p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})), \exists a \in A : x, y \in p(a);$$

• the correspondence

$$(B/\rho)/\alpha^*_{\mathbf{qr}} \to B/\theta(\rho_{\mathbf{qr}}),\ \ \alpha^*_{\mathbf{qr}}\langle\rho\langle a\rangle\rangle \mapsto \theta(\rho_{\mathbf{qr}})\langle a\rangle$$

is a universal algebra isomorphism.

A return to hypergroups:

• Let $(H, \cdot)$ be a hypergroup. The strongly regular relation $\gamma^* = \gamma$ is the transitive closure of

$$\gamma' = \alpha_{\mathbf{x_0 x_1, x_1 x_0}}$$

i.e.

$$\gamma' = \bigcup_{n \in \mathbb{N}^*} \gamma'_n,$$

where $\gamma'_1 = \delta_H$ and for $n > 1$,

$$\begin{aligned} x \gamma'_n y \Leftrightarrow \exists z_1, \ldots, z_n \in H,\ \exists i \in \{1, \ldots, n-1\} : \\ x \in z_1 \cdots z_{i-1}(z_i z_{i+1}) z_{i+2} \cdots z_n, \\ y \in z_1 \cdots z_{i-1}(z_{i+1} z_i) z_{i+2} \cdots z_n, \end{aligned}$$

which is, clearly, (the transitive closure of) the relation

$$\gamma = \bigcup_{n \in \mathbb{N}^*} \gamma_n,$$

with $\gamma_1 = \delta_H$ and, for any $n > 1$,

$$x \gamma_n y \Leftrightarrow \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

since the cycles $(1,2), (2,3), \ldots, (n-1, n)$ generate the symmetric group $(S_n, \circ)$.

The derived subhypergroup of a hypergroup $(K, \cdot)$ is

$$D(K) = \varphi_K^{-1}(1_{K/\gamma}),$$

where $\varphi_K : K \to K/\gamma$ is the canonical projection and $1_{K/\gamma}$ is the identity of the group $(K/\gamma, \cdot)$.

• Let $(G, \cdot)$ be a group, G' its derived subgroup, $H \leq G$ and $(G/H, \cdot)$ the hypergroup defined by

(1) $\quad (xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$

Let $\pi_H : G \to G/H$ and $\varphi_{G/H} : G/H \to (G/H)/\gamma$ be the canonical projections. The group isomorphism

$$h : (G/H)/\gamma \to G/(G'H),\ h(\gamma\langle xH\rangle) = x(G'H)$$

helped us to establish a connection between the derived subhypergroup of G/H and the derived subgroup of G

$$D(G/H) = (h \circ \varphi_{G/H})^{-1}(G'H) = (G'H)/H = \pi_H(G').$$

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