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Direct limits of direct systems of hypergroupoids associated to binary relations

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Rosenberg, I. G., Hypergroups and join spaces determined by relations, *Riv. Math. Pura e Appl.* **4** (1998) 93–101.

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Let H be a set and let R be a binary relation on H and let $\overset{-1}{R}$ be the inverse of the relation R.

$$\overset{-1}{R}(X) = D(R) = \{x \in H \mid \exists y \in H : xRy\}$$

is called the domain of the relation R.

If $x, x_1, \dots, x_n \in H$, $X \subseteq H$ we denote

$$R\langle x \rangle = \{y \in H \mid xRy\},\ R(X) = \{y \in H \mid \exists x \in X : xRy\},$$

$$R(x_1, \dots, x_n) = R(\{x_1, \dots, x_n\}).$$

An element $x \in H$ is an *outer element* of (the relation) R if there exists $h \in H$ such that $(h, x) \notin R^2$.

Rosenberg associated to the binary relation $R \subseteq H \times H$ the partial hypergroupoid $H_R = (H, \circ)$ defined by

$$x \circ y = R(x, y).$$

Obviously, $x^2 = x \circ x = R\langle x \rangle$ and $x \circ y = x^2 \cup y^2$.

Lemma 1. *Let H be a set and let R be a binary relation on H. The partial hypergroupoid $H_R = (H, \circ)$ is a hypergroupoid if and only if the domain of R is H.*

Proposition 1. *Let R be a binary relation on H with full domain. The hypergroupoid H_R is a semihypergroup if and only if $R \subseteq R^2$ and*

$$(a, x) \in R^2 \Rightarrow (a, x) \in R$$

whenever x is an outer element of R.

Proposition 2. *Let $H \neq \emptyset$ and let R be a binary relation on H. The hypergroupoid H_R is a hypergroup if and only if the following conditions hold:*

1) $\overset{-1}{R}(H) = H$;
2) $R(H) = H$;
3) $R \subseteq R^2$;
4) whenever x is an outer element of R we have

$$(a, x) \in R^2 \Rightarrow (a, x) \in R.$$

Proposition 3. *Let* $(H, *)$ *be a semihypergroup. There exists a binary relation* R *on* H *such that* $(H, *) = H_R$ *if and only if the following conditions are satisfied for any* $x, y \in H$*:*
a) $x * y = x^2 \cup y^2$;
b) $x^2 \subseteq (x^2)^2$;
c) $(x^2)^2 \cap (H \setminus (y^2)^2) \subseteq x^2$.

The binary relation $R \subseteq H \times H$ from Rosenberg's proof is defined by

$$xRy \Leftrightarrow y \in x * x$$

and $\overset{-1}{R}(H) = H$. The above proposition can be restated:

Proposition 4. *Let* $(H, *)$ *be a hypergroupoid. There exists a binary relation* R *on* H *such that* $(H, *) = H_R$ *if and only if*

$$(*) \qquad x * y = x^2 \cup y^2, \ \forall x, y \in H.$$

A hypergroupoid $(H, *)$ *which satisfies the condition* $(*)$ *is a semihypergroup if and only if it verifies the conditions b) and c) from previous Proposition.*

Remark 1. A hypergroupoid $(H, *)$ which satisfies the condition $(*)$ is a hypergroup if and only if it verifies the above conditions b), c), and $\bigcup_{x \in H} x^2 = H$.

Corsini proved that:

Theorem 1. *If* $((H_i, R_i) \mid i \in I)$ *is a direct system of relational systems,* $(H, R) = \varinjlim_{i \in I}(H_i, R_i)$*, and if for any* $i \in I$ *there exists* $k \in I$*,* $i \leq k$ *such that* $(H_k)_{R_k}$ *is a hypergroup then* H_R *is a hypergroup.*

If R is a binary relation on the set H and $D(R) = H$, we can identify (H, R) with the multialgebra (H, f) with one unary multioperation $f : H \to P^*(H)$ defined by

$$xRy \Leftrightarrow y \in f(x).$$

If (H', R') is also a relational system with $D(R') = H'$, (H', f') is the corresponding monounary multialgebra and $h : H \to H'$ is a relational homomorphisms between (H, R) and (H', R') then h is a relational homomorphism between (H, R) and (H', R') if and only if h is a homomorphism between the multialgebras (H, f) and (H', f').

Let $\mathcal{R}_2$ be the category of the relational systems with one binary relation, and let us denote by $\mathcal{R}'_2$ its full subcategory whose objects are the relational systems (H, R) for which $D(R) = H$. The category $\mathcal{R}'_2$ is isomorphic to the category $\mathbf{Malg}(1)$ of the monounary multialgebras.

$\Rightarrow$ we can translate the results of Rosenberg in terms of monounary multialgebras;

If $H \neq \emptyset$ and (H, f) is a monounary multialgebra then the hypergroupoid H_f is a hypergroup if and only if the following conditions hold:

i) $f(H) = H$;

ii) $f(x) \subseteq f(f(x)),\ \forall x \in H$;

iii) whenever x is an outer element we have

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

$\Rightarrow$ a hypergroupoid (or semihypergroup, or hypergroup) $(H, *)$ is determined by a unary multioperation f on H if and only if $(H, *)$ satisfies the condition $(*)$.

Besides $\mathcal{R}_2$, $\mathcal{R}'_2$ and $\mathbf{Malg}(1)$, the following categories drew our attention:

• the category $\mathbf{Malg}(2)$ of hypergroupoids: the morphisms are the hypergroupoid homomorphisms and the product of two morphisms is the usual composition of homomorphisms;

• the full subcategory of $\mathbf{Malg}(2)$ whose object are the hypergroupoids which satisfy $(*)$, denoted by $\mathbf{Malg}'(2)$;

• the full subcategory of $\mathbf{Malg}(2)$ whose object are the semihypergroups, denoted by $\mathbf{SHG}$;

• the full subcategory of $\mathbf{SHG}$ whose object are the semihypergroups which satisfy $(*)$, denoted by $\mathbf{SHG}'$;

• the category $\mathbf{HG}$ of hypergroups with hypergroup homomorphisms and the usual composition;

• the full subcategory of $\mathbf{HG}$ whose object are the hypergroups which satisfy $(*)$, denoted by $\mathbf{HG}'$;

• the full subcategory $\mathbf{Malg}'(1)$ of $\mathbf{Malg}(1)$ whose objects are the monounary multialgebras (H, f) which satisfy the conditions ii),iii);

• the full subcategory $\mathbf{Malg}''(1)$ of $\mathbf{Malg}(1)$ whose objects are the monounary multialgebras (H, f), $H \neq \emptyset$, which satisfy the conditions i), ii), iii).

Lemma 2. *Let* (H, f)*,* (H', f') *be two multialgebras from* $\mathbf{Malg}(1)$*, let* $H_f = (H, \circ)$*,* $H'_{f'} = (H', \circ')$ *be the hypergroupoids associated to the multioperations* f *and* f'*, respectively. The mapping* $h : H \to H'$ *is a homomorphism from* (H, f) *into* (H', f') *if and only if* h *is a homomorphism from* H_f *into* $H'_{f'}$*.*

$\Rightarrow$ the correspondence $(H, f) \mapsto H_f = (H, \circ)$ defines a covariant functor $F : \mathbf{Malg}(1) \to \mathbf{Malg}'(2)$.

If in the previous Lemma we consider two multialgebras (H, f) and (H', f') from $\mathbf{Malg}'(1)$ then h is a morphism in $\mathbf{SHG}'$ between H_f and $H'_{f'}$. Hence, we obtain a functor

$$F' : \mathbf{Malg}'(1) \to \mathbf{SHG}'.$$

If in we consider two multialgebras (H, f) and (H', f') from $\mathbf{Malg}''(1)$ then h is a morphism in $\mathbf{HG}'$ between H_f and $H'_{f'}$. Hence, we obtain a covariant functor

$$F'' : \mathbf{Malg}''(1) \to \mathbf{HG}'.$$

If $(H, *)$ is a hypergroupoid and we consider

$$f_* : H \to P^*(H),\ \ f_*(x) = x * x$$

then $(H, f_*) \in \mathbf{Malg}(1)$, and if $h \in H_{\mathbf{Malg}(2)}((H,*),(H',*'))$ then $h \in H_{\mathbf{Malg}(1)}((H, f_*),(H', f_{*'}))$.

$\Rightarrow$ the correspondences

$$(H, *) \mapsto (H, f_*),\ \ h \mapsto h$$

define a covariant functor $\mathbf{Malg}(2) \to \mathbf{Malg}(1)$.

Compose this functor with the inclusion functor

$$\mathbf{Malg}'(2) \to \mathbf{Malg}(2)$$

and we obtain a covariant functor

$$G : \mathbf{Malg}'(2) \to \mathbf{Malg}(1).$$

Lemma 3. *F is an isomorphism between the categories $\mathbf{Malg}(1)$ and $\mathbf{Malg}'(2)$, and G is the inverse of F.*

Corollary 1. *F' is an isomorphism between $\mathbf{Malg}'(1)$ and $\mathbf{SHG}'$, and the inverse of F' is $G' : \mathbf{SHG}' \to \mathbf{Malg}'(1)$,*

$$G'(H, *) = (H, f_*),\ \ G'(h) = h.$$

Corollary 2. *F'' is an isomorphism between $\mathbf{Malg}''(1)$ and $\mathbf{HG}'$, and the inverse of F'' is $G'' : \mathbf{HG}' \to \mathbf{Malg}''(1)$,*

$$G''(H, *) = (H, f_*),\ \ G''(h) = h.$$

Let $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ be a direct system of multialgebras from $\mathbf{Malg}(1)$ and for each $i \in I$ let $F(H_i, f_i) = (H_i, \circ_i)$. Then

$$((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$$

is a direct system of hypergroupoids, denoted by $F(\mathcal{H})$.

Remember that $(I, \leq)$ is a directed preordered set and the homomorphisms φ_{ij} $(i, j \in I,\ i \leq j)$ are such that

$\varphi_{ii} = 1_{A_i},\ \forall i \in I$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik},\ \forall i, j, k \in I,\ i \leq j \leq k.$

The relation $\equiv$ defined on the disjoint union H of the sets H_i as follows: for any $x, y \in A$ there exist $i, j \in I$ such that $x \in H_i,\ y \in H_j$, and

$$x \equiv y \Leftrightarrow \exists k \in I,\ i \leq k,\ j \leq k:\ \varphi_{ik}(x) = \varphi_{jk}(y)$$

is an equivalence relation on H and the factor set

$$H_\infty = H/_\equiv = \{\widehat{x} \mid x \in H\}$$

is the direct limit of the direct system of sets

$$((H_i \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j)).$$

The direct limit $\varinjlim \mathcal{H}$ of the direct system of multialgebras $\mathcal{H}$ is the monounary multialgebra (H_∞, f) with f defined as follows: if $\widehat{x} \in A_\infty$ and $i \in I$ such that $x \in H_i$ then

$$f(\widehat{x}) = \{\widehat{y} \mid \exists m \in I,\ i \leq m,\ y \in f_m(\varphi_{im}(x))\}.$$

The direct limit of the direct system of hypergroupoids $((H_i, \circ_i) \mid i \in I)$ (in $\mathbf{Malg}(2)$) is the hypergroupoid $(H_\infty, \circ)$ with $\circ$ defined as follows: if $\widehat{x_1}, \widehat{x_2} \in A_\infty$ and $i_1, i_2 \in I$ such that $x_1 \in H_{i_1}$, $x_2 \in H_{i_2}$ then

$$\widehat{x_1} \circ \widehat{x_2} = \{\widehat{y} \mid \exists m \in I, i_1 \leq m, i_2 \leq m, y \in \varphi_{i_1 m}(x_1) \circ_m \varphi_{i_2 m}(x_2)\}.$$

Theorem 2. *The hypergroupoid $(H_\infty, \circ)$ is the hypergroupoid determined by the multialgebra (H_∞, f).*

Corollary 3. *The subcategory $\mathbf{Malg}'(2)$ of $\mathbf{Malg}(2)$ is closed under direct limits of direct systems.*

Corollary 4. *If $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ is a direct system of multialgebras from $\mathbf{Malg}(1)$ and $F(\mathcal{H})$ is the direct system of hypergroupoids*

$$((F(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$$

then $F(\varinjlim \mathcal{H})$ is the direct limit of $F(\mathcal{H})$ in $\mathbf{Malg}(2)$.

Corollary 5. *If $((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ is a direct system of hypergroupoids from $\mathbf{Malg}'(2)$ then the direct limit of the direct system of multialgebras $((G(H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ from $\mathbf{Malg}(2)$ is the monounary multialgebra which determines the direct limit of the hypergroupoids $((H_i, \circ_i) \mid i \in I)$.*

Since the direct limit of a direct system of (semi)hypergroups is a (semi)hypergroup, we have:

Corollary 6. *The subcategory* $\mathbf{Malg}'(1)$ *of* $\mathbf{Malg}(1)$ *is closed under direct limits of direct systems. Moreover, if* $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ *is a direct system of multialgebras from* $\mathbf{Malg}'(1)$ *and* $F'(\mathcal{H})$ *is the direct system of semihypergroups*

$$((F'(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$$

then $F'(\varinjlim \mathcal{H})$ *is the direct limit of* $F'(\mathcal{H})$ *in* $\mathbf{Malg}(2)$.

Corollary 7. *The subcategory* $\mathbf{Malg}''(1)$ *of* $\mathbf{Malg}(1)$ *is closed under direct limits of direct systems. Moreover, if* $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ *is a direct system of multialgebras from* $\mathbf{Malg}''(1)$ *and* $F''(\mathcal{H})$ *is the direct system of semihypergroups*

$$((F''(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$$

then $F''(\varinjlim \mathcal{H})$ *is the direct limit of* $F''(\mathcal{H})$ *in* $\mathbf{Malg}(2)$.

Let $(I, \leq)$ be a directed partially ordered set and let

$$\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$$

be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also a directed partially ordered set. Denote by $\mathcal{A}_J$ the direct system consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J, \leq)$ and the homomorphisms are $(\varphi_{ij} \mid i, j \in J,\ i \leq j)$.

Proposition 5. *(Pelea) Let $\mathcal{A}$ be a direct system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. Then the multialgebras $\varinjlim \mathcal{A}$ and $\varinjlim \mathcal{A}_J$ are isomorphic.*

Corollary 8. *Let $(I, \leq)$ be a directed partially ordered set and $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. If $((H_i, f_i) \mid i \in I)$ is a direct system of monounary multialgebras and for any $i \in J$, (H_i, f_i) satisfies the conditions (i),) ii), iii) then the direct limit multialgebra $\varinjlim_{i \in I}(H_i, f_i)$ satisfies the conditions (i),) ii), iii). The hypergroupoid determined by the monounary multialgebra $\varinjlim_{i \in I}(H_i, f_i)$ is a (hypergroup) semihypergroup which is the direct limit of the (hypergroup) semihypergroups $((H_i)_{f_i} \mid i \in J)$ in $\mathbf{Malg}(2)$.*

$\Rightarrow$ the theorem of Corsini.

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