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Products of multialgebras and their fundamental algebras

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Marty, 1934: the first example of multialgebra (hypergroup): let $(G, \cdot)$ be a group, $H \leq G$ and

$$G/H = \{xH \mid x \in G\}.$$

The equality

$$(1) \qquad (xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

defines an operation on G/H if and only if $H \trianglelefteq G$.

Otherwise, (1) defines a function

$$G/H \times G/H \to P^*(G/H)$$

called binary multioperation (on G/H).

A multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ consists in a set A and a family of multioperations $f_\gamma : A^{n_\gamma} \to P^*(A),\ \gamma < o(\tau)$.

Grätzer, 1962: a representation theorem for multialgebras: any multialgebra $\mathfrak{A}$ results from a universal algebra $\mathfrak{B}$ and an appropriate equivalence on B as before, i.e. by considering

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma - 1}\rangle) = \{\rho\langle b\rangle \mid b = f_\gamma(b_0, \ldots, b_{n_\gamma - 1}),\ a_i \rho b_i,$$

$$i \in \{0, \ldots, n_\gamma - 1\}\}.$$

$\Rightarrow$ the importance of factor multialgebras in multialgebra theory

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$\Rightarrow$ the importance of studying the equivalence relations of a multialgebra $\mathfrak{A}$ for which the factor multialgebra is a universal algebra (congruences) and especially the smallest such equivalence (the fundamental relation of the multialgebra $\mathfrak{A}$).

Some facts on the fundamental relation of a multialgebra $\mathfrak{A}$:

• it can be characterized using the term functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$ of the nonempty subsets of $\mathfrak{A}$, defined by

$$f_\gamma(A_0,\ldots,A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0,\ldots,a_{n_\gamma-1}) \mid \ a_i \in A_i,\ i \in \{0,\ldots,n_\gamma-1\}\},$$

more exactly, the fundamental relation α^* of $\mathfrak{A}$ is the transitive closure of the relation α defined by

$$x\alpha y \Leftrightarrow x, y \in p(a_0,\ldots,a_{n-1}) \tag{2}$$

for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0,\ldots,a_{n-1} \in A$;

• (in particular:) the fundamental relation of a hypergroup $(H,\cdot)$ is $\beta = \bigcup_{n\in\mathbb{N}^*}\beta_n$, where

$$x\beta_n y \Leftrightarrow \exists a_1,\ldots,a_n \in H:\ x, y \in a_1\cdots a_n;$$

• the factorization modulo the fundamental relation provides a covariant functor F from the category of the multialgebras of a given type into the category of the universal algebra of the same type.

Problem: Study some preservation properties of F with respect to products.

Question 1: Is the fundamental algebra of the direct product of two multialgebras (isomorphic to) the direct product of the fundamental algebra of the given multialgebras?

Answer: No.

Example: Take the hypergroupoids $(H_1, \circ)$ and $(H_2, \circ)$ given by the following tables:

H_1	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

H_2	x	y	z
x	x	$y,\ z$	$y,\ z$
y	$y,\ z$	$y,\ z$	$y,\ z$
z	$y,\ z$	$y,\ z$	$y,\ z$

In $\overline{H_1} \times \overline{H_2}$, $(\bar{b}, \overline{y}) = (\bar{b}, \overline{z})$ but in $\overline{H_1 \times H_2}$ the supposition $\overline{(b, y)} = \overline{(b, z)}$ leads us to the fact that $y = z$, which is not true.

Question 2: When is the fundamental algebra of the direct product of multialgebras (isomorphic to) the direct product of the fundamental algebra of the given multialgebras?

The cases under study:

• we are dealing with finite families of multialgebras;

• for each multialgebra from the family we are dealing with (which is not necessarily finite) the relation α defined by (2) is (already) transitive.

The (very complicated) **common answer we have given:**

Lemma: If I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for any $i \in I$, then the homomorphism φ is injective if and only if for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$ $(i \in I)$ and for any

$$(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \ldots, a_i^{n_i-1})$$

there exist $m, k_j \in \mathbb{N}$, $\mathbf{q}^j \in \mathbf{P}^{(k_j)}(\tau)$ and

$$(b_i^0)_{i\in I}^j, \ldots, (b_i^{k_j-1})_{i\in I}^j \in \prod_{i\in I} A_i \;\; (j \in \{0, \ldots, m-1\})$$

such that

$$(x_i)_{i\in I} \in q^0((b_i^0)_{i\in I}^0, \ldots, (b_i^{k_0-1})_{i\in I}^0),$$

$$(y_i)_{i\in I} \in q^{m-1}((b_i^0)_{i\in I}^{m-1}, \ldots, (b_i^{k_{m-1}-1})_{i\in I}^{m-1}),$$

and for all $j \in \{1, \ldots, m-1\}$,

$$q^{j-1}((b_i^0)_{i\in I}^{j-1}, \ldots, (b_i^{k_{j-1}-1})_{i\in I}^{j-1}) \cap q^j((b_i^0)_{i\in I}^j, \ldots, (b_i^{k_j-1})_{i\in I}^j) \neq \emptyset.$$

A consequence:

The (complicated) condition from the previous lemma is verified if there exist

$$n \in \mathbb{N},\ \mathbf{q} \in \mathbf{P}^{(n)}(\tau),\ b_i^0, \ldots, b_i^{n-1} \in A_i\ (i \in I)$$

such that

$$(3) \qquad \prod_{i \in I} q_i(a_i^0, \ldots, a_i^{n_i-1}) \subseteq q((b_i^0)_{i\in I}, \ldots, (b_i^{n-1})_{i\in I}).$$

Some applications:

1. *The case of hypergroups:*

Proposition: The functor $F : \mathbf{HG} \longrightarrow \mathbf{Grp}$ preserves the finite products.

Yet, if we consider on the set of the integers $\mathbb{Z}$, the hypergroup $(\mathbb{Z}, \circ)$, defined by

$$x \circ y = \{x + y, x + y + 1\},$$

its fundamental relation is $\beta = \mathbb{Z} \times \mathbb{Z}$. So, its fundamental group is a one-element group, but if we consider the product $(\mathbb{Z}^{\mathbb{N}}, \circ)$, the fundamental group of this hypergroup has more than one element.

Theorem: Let I be a set and consider the hypergroups H_i $(i \in I)$ with the fundamental relations β^{H_i}. The group $\overline{\prod_{i\in I} H_i}$, with the homomorphisms $(\overline{e_i^I} \mid i \in I)$, is the product of the family of groups $(\overline{H_i} \mid i \in I)$ if and only if there exists $n \in \mathbb{N}^*$ such that $\beta^{H_i} \subseteq \beta_n^{H_i}$, for all the elements i from I, except for a finite number of i's.

2. *The case of complete multialgebras:*

Theorem: For a family $(\mathfrak{A}_i \mid i \in I)$ of complete multialgebras of the same type τ, the following statements are equivalent:

i) $\overline{\prod_{i\in I}\mathfrak{A}_i}$ (together with the homomorphisms $\overline{e_i^I}$ $(i \in I)$) is the product of the family of the universal algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$;

ii) for any $n_i \in \mathbb{N},\ \mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau),\ a_i^0,\dots,a_i^{n_i-1} \in A_i,\ (i \in I)$ there exist $n \in \mathbb{N},\ \mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0,\dots,b_i^{n-1} \in A_i$ $(i \in I)$ such that

$$(3) \qquad \prod_{i\in I} q_i(a_i^0,\dots,a_i^{n_i-1}) \subseteq q((b_i^0)_{i\in I},\dots,(b_i^{n-1})_{i\in I});$$

iii) for any $n_i \in \mathbb{N},\ \mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau),\ a_i^0,\dots,a_i^{n_i-1} \in A_i$ $(i \in I)$ either

$$\left|\prod_{i\in I} q_i(a_i^0,\dots,a_i^{n_i-1})\right| = 1$$

or there exist $\gamma < o(\tau)$, $b_i^0,\dots,b_i^{n_\gamma-1} \in A_i$ $(i \in I)$ such that

$$(4) \qquad \prod_{i\in I} q_i(a_i^0,\dots,a_i^{n_i-1}) \subseteq f_\gamma((b_i^0)_{i\in I},\dots,(b_i^{n_\gamma-1})_{i\in I}).$$

Remarks:

1. For a family of complete multialgebras $(\mathfrak{A}_i \mid i \in I)$ the following conditions are equivalent:

a) there exist $n \in \mathbb{N}$ and $\mathbf{p} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, .., n-1\}\}$ such that for each $i \in I$ and for any $a_i \in A_i$ we have $a_i \in p(a_i^0, \ldots, a_i^{n-1})$ for some $a_i^0, \ldots, a_i^{n-1} \in A_i$;

b) there exists a $\gamma < o(\tau)$ such that for each $i \in I$ and for any $a_i \in A_i$ we have $a_i \in f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1})$ for some $a_i^0, \ldots, a_i^{n_\gamma - 1} \in A_i$.

2. If for a family of complete multialgebras one of the equivalent conditions $a)$ or $b)$ is satisfied, then the condition $i)$ from the previous theorem holds.

3. Either using the fact that for the complete hypergroups we have $\beta = \beta_2$ or using the above remarks we obtain that the functor F commutes with the products of complete hypergroups.

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