Term functions and polynomial functions in hyperstructure theory

Ioan Purdea and Cosmin Pelea

Multialgebra

Let $\tau = (n_\gamma)_{\gamma < o(\tau)}$ be a sequence of nonnegative integers ($o(\tau)$ is an ordinal) and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation.

A *multialgebra* $\mathfrak{A}$ *of type* τ consists in a set A and a family of multioperations $(f_\gamma)_{\gamma < o(\tau)}$, where

$$f_\gamma : A^{n_\gamma} \to P^*(A)$$

is the n_γ-ary multioperation which corresponds to the symbol $\mathbf{f}_\gamma$.

Denote by

$$\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma < o(\tau)})$$

the algebra of the n-ary terms (of type τ*).*

- A multialgebra $\mathfrak{A} = (f_\gamma)_{\gamma<o(\tau)}$ can be seen as a relational system $(A, (r_\gamma)_{\gamma<o(\tau)})$, where r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \ldots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow \ a_{n_\gamma} \in f_\gamma(a_0, \ldots, a_{n_\gamma-1}).$$

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- *The algebra of the nonempty subsets of a multialgebra* (H. E. Pickett):

  A multialgebra $\mathfrak{A}$ determines a universal algebra $\mathfrak{P}^*(\mathfrak{A})$ on $P^*(A)$ defining for any $A_0, \ldots, A_{n_\gamma-1} \in P^*(A)$,

$$f_\gamma(A_0, ..., A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, ..., a_{n_\gamma-1}) \mid a_i \in A_i, i = \overline{0, n_\gamma - 1}\}.$$

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is satisfied on $\mathfrak{A}$ if

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(q and r denotes the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ on $\mathfrak{P}^*(\mathfrak{A})$.)

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- The algebra $\mathfrak{P}^{(n)}_{A}(\mathfrak{P}^*(\mathfrak{A}))$ which is the subalgebra of $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \dots, n-1\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are defined by

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- The algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$, which is the subalgebra of $\mathfrak{P}^{(n)}_{A}(\mathfrak{P}^*(\mathfrak{A}))$ generated by

$$\{e_i^n \mid i \in \{0, \ldots, n-1\}\}.$$

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  - the fundamental relation of $\mathfrak{A}$;
- direct products of multialgebras;
- direct limits of direct systems of multialgebras.

In an overwhelming number of situations we apply term functions or polynomial functions from $P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ to one element sets.

$\Rightarrow$ **Question:** Is it necessary to redefine term functions and polynomial functions as some sort of special multioperations for these special situations?

Example

For a hypergroupoid $(H, \circ)$ and $a, b, c \in H$, the nonempty subsets

$$a \circ (b \circ c),\ (a \circ b) \circ (a \circ c)$$

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$\Rightarrow$ **Question:** Is it necessary to redefine term functions and polynomial functions as some sort of special multioperations for these special situations?

Our opinion: No, since from a certain step of their construction we operate sets, not elements. (Of course, they are multioperations, but ignoring the fact that we are working in $\mathfrak{P}^*(\mathfrak{A})$ can be dangerous.)

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Submultialgebras

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra.

A subset $B \subseteq A$ is a *submultialgebra* of $\mathfrak{A}$ if

$$f_\gamma(b_0, \ldots, b_{n_\gamma - 1}) \subseteq B,\ \forall \gamma < o(\tau),\ \forall b_0, \ldots, b_{n_\gamma - 1} \in B.$$

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$\Rightarrow$ If B is a submultialgebra of $\mathfrak{A}$, $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $b_0, \ldots, b_{n-1} \in B$ then

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$$p(b_0, \ldots, b_{n-1}) \subseteq B.$$
- $\Rightarrow$ (Breaz, Pelea) If $X \subseteq A$ then $a \in \langle X \rangle$ if and only if there exist $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $x_0, \ldots, x_{n-1} \in X$ such that
$$a \in p(x_0, \ldots, x_{n-1}).$$

Homomorphisms

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ and $\mathfrak{B} = (B, (f_\gamma)_{\gamma < o(\tau)})$ be multialgebras of type τ and $h : A \to B$.

- h is a *multialgebra homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ if for any $\gamma < o(\tau)$ and $a_0, \ldots, a_{n_\gamma - 1} \in A$,

$$h(f_\gamma(a_0, \ldots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \ldots, h(a_{n_\gamma-1})).$$

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- the *isomorphisms* are the bijective ideal homomorphisms.

The mapping h induces the mapping

$$h_* : P^*(A) \to P^*(B),\ \ h_*(X) = h(X) = \{h(x) \mid x \in X\}.$$

Theorem (Pickett)

h is a multialgebra ideal homomorphism if and only if h_* is a universal algebra homomorphism.

$\Rightarrow$ If $h : A \to B$ is an ideal homomorphism, $n \in \mathbb{N}$, $p \in \mathbf{P}^{(n)}(\tau)$, and $a_0, \ldots, a_{n-1} \in A$ then

$$h(p(a_0, \ldots, a_{n-1})) = p(h(a_0), \ldots, h(a_{n-1})).$$

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- The mappings
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- **QUESTION 1:** Can we use these mappings or this functor to establish other connections between universal algebras and hyperstructure theory which can provide interesting and useful results for both theories, especially for hyperstructure theory?

Factor multialgebras

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ and let ρ be an equivalence relation on A. Let $\rho\langle x\rangle$ be the class of x modulo ρ, and

$$A/\rho = \{\rho\langle x\rangle \mid x \in A\}.$$

Defining for each $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma - 1}\rangle) = \{\rho\langle b\rangle | b \in f_\gamma(b_0, ..., b_{n_\gamma - 1}), a_i \rho b_i, i = \overline{0, n_\gamma - 1}\},$$

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- G. Grätzer proved that *any multialgebra is a factor of a universal algebra modulo an equivalence relation.*

Since the canonical map $A \to A/\rho$ is a multialgebra homomorphism, for any $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$,

$$\{\rho\langle a\rangle \mid a \in (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}(a_0, \ldots, a_{n-1})\} \subseteq (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle)$$

This inclusion holds if we replace $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}$ and $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$ by $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $p' \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho))$ respectively, where the polynomial function p' corresponding to p is given as follows:

(i) if $p = c_a^n$, then $p' = c_{\rho\langle a\rangle}^n$;

So, we can write

$$\{\rho\langle a\rangle \mid a \in p(a_0, \ldots, a_{n-1})\} \subseteq p'(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle).$$

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(ii) if $p = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A})}$, then $p' = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$;

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(iii) if $p = f_\gamma(p_0, \ldots, p_{n_\gamma-1})$ and $p'_0, \ldots, p'_{n_\gamma-1} \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho))$ are the polynomial functions which correspond to $p_0, \ldots, p_{n_\gamma-1} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ respectively, then

$$p' = f_\gamma(p'_0, \ldots, p'_{n_\gamma-1}).$$

So, we can write

$$\{\rho\langle a\rangle \mid a \in p(a_0, \ldots, a_{n-1})\} \subseteq p'(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle).$$

• Any identity of $\mathfrak{A}$ is weakly satisfied on $\mathfrak{A}/\rho$.

• If $\mathfrak{A}$ is a universal algebra the multioperations from $\mathfrak{A}/\rho$ are defined by the equalities

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle | b = f_\gamma(b_0, ..., b_{n_\gamma-1}), a_i\rho b_i, i = \overline{0, n_\gamma - 1}\},$$

and taking an n-ary term **p** we have

$$p(\rho\langle a_0\rangle, ..., \rho\langle a_{n-1}\rangle) \supseteq \{\rho\langle b\rangle | b = p(b_0, ..., b_{n-1}), a_i\rho b_i, i = \overline{0, n-1}\}.$$

In general, this inclusion is not an equality and the identities of $\mathfrak{A}$ are only weakly preserved by $\mathfrak{A}/\rho$.

Example

If $(\mathbb{Z}_5, +)$ is the cyclic group of order 5 and we take the equivalence

$$\rho = \{0,1\} \times \{0,1\} \cup \{2\} \times \{2\} \cup \{3,4\} \times \{3,4\}.$$

$(\mathbb{Z}_5/\rho, +)$ is a hypergroupoid on $\mathbb{Z}_5/\rho = \{\{0,1\}, \{2\}, \{3,4\}\}$ given by

$+$	$\{0,1\}$	$\{2\}$	$\{3,4\}$
$\{0,1\}$	$\{0,1\}, \{2\}$	$\{2\}, \{3,4\}$	$\{0,1\}, \{3,4\}$
$\{2\}$	$\{2\}, \{3,4\}$	$\{3,4\}$	$\{0,1\}$
$\{3,4\}$	$\{0,1\}, \{3,4\}$	$\{0,1\}$	$\{0,1\}, \{2\}, \{3,4\}$

$$\{\rho\langle c\rangle \mid c = (b_0 + b_1) + b_2, b_0 = b_1 = 2, b_2 \in \{3,4\}\} = \{\rho\langle 2\rangle, \rho\langle 3\rangle\}$$

$$(\rho\langle 2\rangle + \rho\langle 2\rangle) + \rho\langle 3\rangle = \rho\langle 3\rangle + \rho\langle 3\rangle = \{\rho\langle 0\rangle, \rho\langle 2\rangle, \rho\langle 3\rangle\},$$

and the associativity is only weakly satisfied on $(\mathbb{Z}_5/\rho, +)$ since

$$\rho\langle 2\rangle + (\rho\langle 2\rangle + \rho\langle 3\rangle) = \rho\langle 2\rangle + \rho\langle 0\rangle = \{\rho\langle 2\rangle, \rho\langle 3\rangle\}.$$

- Yet, there are identities, like

$$\mathbf{f}_\gamma(\mathbf{x}_0, \ldots, \mathbf{x}_{n_\gamma-1}) = \mathbf{f}_\gamma(\mathbf{x}_{\sigma(0)}, \ldots, \mathbf{x}_{\sigma(n_\gamma-1)}) \quad (\sigma \in S_{n_\gamma}),$$

  which characterize the commutativity of an n_γ-ary operation of an algebra, which always hold in a strong manner in the factor multialgebra.

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  which characterize the commutativity of an n_γ-ary operation of an algebra, which always hold in a strong manner in the factor multialgebra.
- **QUESTION 2:** Can we determine/characterize the identities of a (multi)algebra which are always strongly preserved by the factor multialgebra?

Example

(Roth) If $(G, \cdot)$ is a finite group and ρ is the conjugacy relation on G then G/ρ is the set of the conjugacy classes of G, and the hypergroupoid $(G/\rho, \cdot)$ is a canonical hypergroup having the identity element $\rho\langle 1\rangle = \{1\}$, and for each conjugacy class C from G, the inverse element is the class C^{-1} which consists in the inverses of the elements of C. Let 1 be the nullary operation which points out the identity element, and $^{-1}$ the unary operation which associates to each element its inverse. Then $\rho\langle 1\rangle$ and $^{-1}$ are operations on G/ρ, too, and both multialgebras $(G, \cdot, 1, ^{-1})$ and $(G/\rho, \cdot, \rho\langle 1\rangle, ^{-1})$ satisfy the (strong) identities:

$$(\mathbf{x}_0 \cdot \mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_0 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_2),\ \mathbf{x}_0 \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{x}_0 = \mathbf{x}_0.$$

QUESTION 3: Can we determine/identify classes of equivalence relations in (multi)algebra which determine factor multialgebra which strongly preserves certain (sets of) identities?

Ideal equivalences

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra. An equivalence relation ρ on A determines on $P^*(A)$ the relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ defined by:

$$X\overline{\rho}Y \Leftrightarrow \forall x \in X,\ \exists y \in Y :\ x\rho y \text{ and } \forall y \in Y,\ \exists x \in X :\ x\rho y;$$

$$X\overline{\overline{\rho}}Y \ \Leftrightarrow\ x\rho y,\ \forall x \in X,\ \forall y \in Y \ \Leftrightarrow\ X \times Y \subseteq \rho.$$

The relation ρ is an *ideal equivalence* on $\mathfrak{A}$ if for any $\gamma < o(\tau)$ and any $x_i, y_i \in A$ for which $x_i \rho y_i$ for all $i \in \{0, \ldots, n_\gamma - 1\}$ we have

$$a \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) \Rightarrow \exists b \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1}) :\ a\rho b.$$

Theorem (Pickett)

If ρ is an ideal equivalence on $\mathfrak{A}$ then the canonical projection

$$\pi_\rho : A \to A/\rho,\ \pi_\rho(a) = \rho\langle a\rangle$$

is an ideal homomorphism. Conversely, if $h : A \to B$ is an ideal homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, its kernel

$$\ker h = \{(x, y) \in A \times A \mid h(x) = h(y)\}$$

is an ideal equivalence on $\mathfrak{A}$. Moreover, the mapping

$$h(a) \mapsto \pi_{\ker h}(a)$$

is an multialgebra isomorphism between $h(\mathfrak{A})$ and $\mathfrak{A}/\ker h$.

- (Breaz, Pelea) An equivalence relation ρ of the multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ is ideal if and only if $\overline{\rho}$ is a congruence relation on $\mathfrak{P}^*(\mathfrak{A})$.

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- The equivalence relations ρ for which the factor multialgebra $\mathfrak{A}/\rho$ is a universal algebra are particular ideal equivalence relations.

Proposition

(Breaz, Pelea) Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ *be a multialgebra and let* ρ *be an equivalence relation on* A*. The following conditions are equivalent:*

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(b) *for any* $\gamma < o(\tau)$*, and any* $x_i, y_i \in A$ *for which* $x_i \rho y_i$ *for all* $i \in \{0, \ldots, n_\gamma - 1\}$ *we have*

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(d) *for any $n \in \mathbb{N}$, any $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$, and any $x_i, y_i \in A$ with $x_i \rho y_i$ ($i \in \{0, \dots, n-1\}$) we have*

$$p(x_0, \dots, x_{n-1}) \overline{\rho} p(y_0, \dots, y_{n-1}).$$

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$\Rightarrow$ any identity (weak or strong) which holds on $\mathfrak{A}$ is also satisfied in its fundamental algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*$.

- Even if $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is not satisfied on $\mathfrak{A}$ we can obtain a factor multialgebras of $\mathfrak{A}$ which are universal algebras satisfying $\mathbf{q} = \mathbf{r}$ by factorizing it modulo relations from $E_{ua}(\mathfrak{A})$ which contain

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- (Pelea, Purdea) $\alpha^*_{\mathbf{qr}}$ is the transitive closure of the relation $\alpha_{\mathbf{qr}}$ defined by

$$\begin{aligned} x \alpha_{\mathbf{qr}} y \ \Leftrightarrow\ & \exists p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists\ a_0, \dots, a_{n-1} \in A : \\ & x \in p(q(a_0, \dots, a_{n-1})),\ y \in p(r(a_0, \dots, a_{n-1})) \text{ or} \\ & y \in p(q(a_0, \dots, a_{n-1})),\ x \in p(r(a_0, \dots, a_{n-1})). \end{aligned}$$

The fundamental relation of a multialgebra

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- For any polynomial function $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and any $a_0, \ldots, a_{n-1} \in A$, there exist $m \in \mathbb{N}$, $m \geq n$, $b_0, \ldots, b_{m-1} \in A$ and a term function $p' \in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

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- (Pelea, Purdea) α^* is also the transitive closure of the relation

$$x\alpha' y \;\Leftrightarrow\; \exists p\in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists a\in A:\ x,y\in p(a).$$

Let $\mathfrak{B}$ be a universal algebra and ρ an equivalence relation on B. Denote by $\theta(\rho)$ the smallest congruence relation on $\mathfrak{B}$ containing ρ.

- (Pelea, Purdea) For $n \in \mathbb{N}$, $p \in P^{(n)}_{B/\rho}(\mathfrak{P}^*(\mathfrak{B}/\rho))$ and $x, y, z_0, \ldots, z_{n-1} \in B$ we have

$$\rho\langle x\rangle, \rho\langle y\rangle \in p(\rho\langle z_0\rangle, \ldots, \rho\langle z_{n-1}\rangle) \;\Rightarrow\; x\theta(\rho)y.$$

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$\Rightarrow$ (Pelea, Purdea)

$$\overline{\mathfrak{B}/\rho} \cong \mathfrak{B}/\theta(\rho).$$

Example

Let $(G, \cdot)$ be a group, H a subgroup of G,

$$G/H = \{xH \mid x \in G\}$$

and

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

The hypergroupoid $(G/H, \cdot)$ is a hypergroup (Marty).

If $\overline{G/H}$ is the fundamental group of the hypergroup G/H and $\overline{H}$ is the smallest normal subgroup of G which contains H then

$$\overline{G/H} \cong G/\overline{H}.$$

Direct products of multialgebras

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. The Cartesian product $\prod_{i \in I} A_i$ with the multioperations

$$f_\gamma((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I}) = \prod_{i \in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}),$$

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- If $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$ then

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- The direct product of a family of multialgebras which satisfy a certain identity (weak or strong) satisfies the same identity.

Direct limits of direct systems of multialgebras

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let A_∞ be the direct limit of the direct system of their supporting sets.

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- $(I, \leq)$ is a directed preordered set;
- the set A_∞ is the factor of the disjoint union A of the sets A_i modulo the equivalence relation $\equiv$ defined as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and

$$x \equiv y \ \Leftrightarrow\ \exists k \in I,\ i \leq k,\ j \leq k:\ \varphi_{ik}(x) = \varphi_{jk}(y).$$

We define the multioperations f_γ on $A_\infty = \{\widehat{x} \mid x \in A\}$ as follows: if $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ and for any $j \in \{0, ..., n_\gamma - 1\}$ we consider that $x_j \in A_{i_j}$ $(i_j \in I)$ then

$$f_\gamma(\widehat{a_0}, \ldots, \widehat{a_{n_\gamma - 1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I, i_0 \leq m, \ldots, i_{n_\gamma - 1} \leq m,\\ a \in f_\gamma(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n_\gamma - 1} m}(a_{n_\gamma - 1}))\}.$$

The multialgebra $\mathfrak{A}_\infty = (A_\infty, (f_\gamma)_{\gamma < o(\tau)})$ is called *the direct limit of the direct system* $\mathcal{A}$.

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- If $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$, $a_0, \ldots, a_{n-1} \in A$ and $i_0, \ldots, i_{n-1} \in I$ are such that $a_j \in A_{i_j}$ for all $j \in \{0, \ldots, n-1\}$ then

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- The direct limit of a direct system of multialgebras which satisfy a certain identity (weak or strong) satisfies the same identity.

For details, see

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